ISSN 0819-2642
ISBN 0 7340 2564 5

THE UNIVERSITY OF MELBOURNE

DEPARTMENT OF ECONOMICS

RESEARCH PAPER NUMBER 908

JULY 2004

STOCHASTIC OPTIMAL GROWTH WHEN THE DISCOUNT RATE VANISHES

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STOCHASTIC OPTIMAL GROWTH WHEN THE DISCOUNT RATE VANISHES

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ABSTRACT. It has been shown that long-run optimality of the limit of discounted optima when the discount rate vanishes is implied by a condition on the value function of the optimal program. We suggest a new method to verify this condition in the context of one-sector optimal growth. The idea should be more widely applicable.

1. INTRODUCTION

Discounted dynamic programming is a standard paradigm for analyzing economic outcomes when expectations are rational and information is perfect. (For dynamics in imperfect information economies see, for example, Chiarella and Szidarovzky [3] and references.) An established theory exists, along with practical methods of numerical computation. However, optimal behavior when the future is not discounted has also been studied, perhaps most famously in the classic paper of Ramsey [8].[1] Another well-known example is the no-discounting paper by Brock and Mirman [2], albeit much less so than its famous discounting cousin [1].

A number of no-discounting criteria exist for optimality. In the mathematical literature on stochastic dynamic programming, however, no-discounting research is now mainly focused on long-run average reward (LAR) optimality, which maximizes the average of the undiscounted

Date: July 9, 2004.

Key words and phrases. Dynamic programming, Long-run optimality.

[1] According to Ramsey, "discount[ing] later enjoyments in comparison with earlier ones [is] ethically indefensible, and arises merely from the weakness of the imagination" [8, p. 543].

sequence of period rewards.[2] For example, LAR-optimality is the standard criterion for on-line computer task scheduling and network routing.

It is of great practical interest to identify relationships between discounted reward (DR) optimal policies and LAR-optimal policies. (One reason is that contraction mapping techniques allow many optimal growth problems to be solved efficiently for a DR-optimal policy.) For example, if π_ϱ is a DR-optimal policy for discount factor $\varrho \in (0,1)$, and if π_ϱ converges to a limit π_1 when $\varrho \uparrow 1$, it seems likely that π_1 will be—at least in some sense—long-run optimal.

An important contribution to our understanding of the relationship between DR- and LAR-optimality is the study of Dutta [5]. In this short paper we use a condition established by Dutta [5, Theorem 3] to verify the conjecture that π_1 defined above is LAR-optimal for a neoclassical stochastic optimal growth model with unbounded state. Our proof is based on a "coupling" technique.

In his study Dutta [5] previously gave several useful applications to neoclassical growth. Our technique extends these ideas to state spaces which do not have a largest element. Such spaces are often encountered in applications (see, for example, the dynamic stochastic general equilibrium models estimated in macroeconomics). For these models too Dutta's condition is seen to be readily applicable.[3]

2. Formulation of the Problem

Consider the neoclassical infinite horizon economy of Brock and Mirman [1]. Depreciation is assumed total in each period for simplicity, so that current savings, investment and the capital stock K_t can all be identified. At time t income Y_t is observed, a savings decision K_t is made, the current shock ξ_t is then revealed to the agent, and production takes place, realizing at the start $t+1$ random output $Y_{t+1} = f(K_t)\,\xi_t$. The process then repeats.

[2] If $(r_t)_{t \geq 0}$ is a bounded sequence of rewards, then the average is usually defined to be $\underline{\lim}_{t\to\infty}(1/t)\sum_{s=0}^{t-1} r_s$.

[3] As well as Dutta [5], a number of our ideas draw on the coupling techniques used to study ergodicity in Rosenthal [9].

Preferences are specified by period utility function u and discount factor $\varrho \in (0,1)$.

Assumption 2.1. Both the utility function u and the production function f are strictly increasing, continuously differentiable, bounded and strictly concave. Also, $u(0) = f(0) = 0$ and $u'(0) = f'(0) = \infty$.

Assumption 2.2. The sequence $(\xi_t)_{t=0}^{\infty}$ is independent collection of random variables on probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Each ξ_t has identical distribution function G on $[0,\infty)$. Also, $\mathbb{E}(\xi) < \infty$, $\mathbb{E}(1/\xi) < \infty$, $G(0) = 0$ and $0 < G(x) < 1$ for all $x > 0$.

Assumption 2.2 is satisfied by the lognormal and other standard distributions used in empirical modeling.[4]

Define Π to be the set of all feasible savings policies, which are Borel functions π from the positive reals to itself satisfying $\pi(y) \leq y$ for all y. Each $\pi \in \Pi$ determines a Markov process for income $(Y_t)_{t=0}^{\infty}$ via

$$\text{(MAR)} \qquad Y_{t+1} = f(\pi \circ Y_t)\,\xi_t, \quad Y_0 \equiv y_0 \text{ given.}$$

Of course $\pi \circ Y_t$ is the composition of π and the random variable Y_t.

The optimal investment problem is then to solve

$$\text{(DR-}\varrho\text{)} \qquad \max_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(Y_t - \pi \circ Y_t)\right], \quad (Y_t)_{t=0}^{\infty} \text{ given by (MAR).}$$

A policy is called DR-ϱ-optimal if it is feasible and solves (DR-ϱ). The value function v_ϱ is defined at y as the supremum of (DR-ϱ) over Π when $Y_0 \equiv y$. The next result is very well-known.

Theorem 2.1 (Mirman and Zilcha [6])**.** *For each $\varrho \in (0,1)$, there is a unique $\pi_\varrho \in \Pi$ which attains the maximum in* (DR-ϱ). *The value function v_ϱ is increasing, concave and differentiable, with $v'_\varrho(y) = u'(y - \pi_\varrho(y))$. The DR-$\varrho$-optimal policy π_ϱ is increasing, continuous and interior, as is the consumption function $y \mapsto y - \pi_\varrho(y)$.*

[4] The restrictions on $\mathbb{E}(\xi)$ and $\mathbb{E}(1/\xi)$ bound the right and left hand tails respectively, and can be interpreted as generalizations of the common assumption that the shock has compact support [7]. Also note that our main results still hold without $G(0) = 0$. In fact it is easy to show that when $G(0) > 0$ the distributions of the state variables converge to the distribution concentrated at zero geometrically in total variation norm. But this behavior is in some sense trivial so we avoid it.

The other optimality criterion we consider is LAR-optimality. A policy is called LAR-optimal if it solves

$$\text{(LAR)} \qquad \max_{\pi \in \Pi} \varliminf_{t \to \infty} \mathbb{E} \left[\frac{1}{t} \sum_{s=0}^{t-1} u(Y_s - \pi \circ Y_s) \right],$$

where again π determines the process $(Y_t)_{t=0}^{\infty}$ via (MAR).[5]

3. Results

It has been observed [4, Theorem 5.1] that for the stochastic growth model, the DR-ϱ-optimal policy π_ϱ is pointwise increasing in ϱ. In other words, agents who discount the future more slowly invest more in all states. Given this monotonicity, we can always define $\pi_1 := \lim_{\varrho \uparrow 1} \pi_\varrho$. It is natural to then conjecture that π_1 is LAR-optimal.

One of the most readily applicable conditions for linking DR- and LAR-optimality is *value boundedness* [5].

Condition 3.1 (Value Boundedness)**.** There exists a $z \in (0, \infty)$, a constant M and a real function $y \mapsto M(y)$ such that

$$-\infty < M(y) \leq v_\varrho(y) - v_\varrho(z) \leq M < \infty, \quad \forall y \in (0, \infty), \ \varrho \in (0, 1).$$

It is immediate from Dutta [5, Theorem 3] that

Theorem 3.1. *If value boundedness holds, then $\pi_1 := \lim_{\varrho \uparrow 1} \pi_\varrho$ is LAR-optimal for the stochastic neoclassical growth model defined above, where π_ϱ is the DR-ϱ-optimal policy for each $\varrho \in (0, 1)$.*[6]

The main result of this paper is

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[5]The meaning of the average reward criterion is clearest when $(Y_t)_{t=0}^{\infty}$ is ergodic. In that case the sequence $\mathbb{E}\, u(Y_s - \pi \circ Y_s)$ and in fact the average $\left[\frac{1}{t} \sum_{s=0}^{t-1} \mathbb{E}\, u(Y_s - \pi \circ Y_s)\right]$ converge to $\int u(y - \pi(y)) F_\pi^*(dy)$, where F_π^* is the ergodic distribution corresponding to π. Then LAR-optimality becomes equivalent to maximizing expected utility of consumption at the stochastic steady state—a generalization of the Phelps–Solow golden rule.

[6]Dutta requires that the control space is compact. In this case we can take the savings *rate* to be the control, rather than savings, and then define savings as the rate multiplied by current income. Obviously the two controls are equivalent.

Theorem 3.2. *Under Assumptions 2.1 and 2.2, the stochastic neoclassical growth model is value bounded. As a result, the pointwise limit as* $\varrho \uparrow 1$ *of the sequence of DR-*ϱ*-optimal policies is LAR-optimal.*

4. Proof

We begin by outlining our overall strategy for proving Theorem 3.2. In all of what follows let $z \in (0, \infty)$ be fixed. Note for starters that $|v_\varrho(y) - v_\varrho(z)| \leq 2K/(1-\varrho)$, where $K := \sup u$. Therefore when establishing value boundedness we can and do assume (in all of what follows) that $\varrho \in [\hat{\varrho}, 1)$ for some fixed $\hat{\varrho} \in (0,1)$. So now fix any $\varrho \in [\hat{\varrho}, 1)$, and any initial condition $y_0 \in (0, \infty)$.

Suppose to begin with that $y_0 \geq z$, in which case $v_\varrho(y_0) - v_\varrho(z) \geq 0$. To verify value boundedness, then, we need only bound this number from *above* independent of $y_0 \in [z, \infty)$ and $\varrho \in [\hat{\varrho}, 1)$.

We use a coupling approach: Consider two economies with identical structure (u, f, G), which we call Country A and Country B. Both discount future utility according to ϱ. Country A is perturbed by the sequence of shocks $(\xi_t)_{t=0}^\infty$ as above, with $(Y_t^a)_{t=0}^\infty$ defined by (MAR) using the unique DR-ϱ-optimal policy π_ϱ defined in Theorem 2.1.

Country B is exactly the same, except that it is perturbed by a different and *independent* sequence of shocks $(\xi_t')_{t=0}^\infty$—also temporally independent and identically distributed by G. The output series for this economy $(Y_t^b)_{t=0}^\infty$ is then defined recursively by (MAR) with identical DR-ϱ-optimal policy π_ϱ.

The other difference between the two countries is initial income. Country A (resp., Country B) has initial income $Y_0^a \equiv y_0$ (resp., $Y_0^b \equiv z$), where y_0 and z are chosen above. Also, the sequences $(\xi_t)_{t=0}^\infty$ and $(\xi_t')_{t=0}^\infty$ are again taken to be defined on $(\Omega, \mathscr{F}, \mathbf{P})$—clearly it can be so constructed.

We now show that the term $v_\varrho(y_0) - v_\varrho(z)$ that we seek to bound from above will become arbitrarily large as $\varrho \to 1$ when Country A, *starting with higher income level* y_0, continues to enjoy higher income into the distant future relative to that of Country B, which starts at z. To understand this, first define $r(y) := u(y - \pi_\varrho(y))$, the value of period utility under policy π_ϱ when income equals y. Note that r is always increasing in y. It follows that if A's income stream is expected to remain larger than B, then each value of the term $\mathbb{E}(r \circ Y_t^a - r \circ Y_t^b)$ will be large. Finally, note that we can write (this is the coupling step)

$$\sum_{t=0}^{\infty} \varrho^t \mathbb{E}(r \circ Y_t^a - r \circ Y_t^b) = \mathbb{E} \sum_{t=0}^{\infty} \varrho^t \, r \circ Y_t^a - \mathbb{E} \sum_{t=0}^{\infty} \varrho^t \, r \circ Y_t^b$$
$$= v_\varrho(y_0) - v_\varrho(z).$$

This suggests the following decomposition of the probability space. Define the random variable $\tau \colon \Omega \to \mathbb{N} \cup \{\infty\}$ by

$$\tau := \inf\{t \in \mathbb{N} : Y_t^a \leq Y_t^b\}, \tag{1}$$

with the usual convention $\inf \emptyset := \infty$. That is, τ is the first time that relative incomes reverse, and Country A becomes poorer than Country B. On that subset of Ω where $\tau \leq t$, we would imagine that Country A has *lower* expected time t utility than Country B. In other words, integrating over only these outcomes would lead to $\mathbb{E}(r \circ Y_t^a - r \circ Y_t^b)$ being *negative*:

Lemma 4.1. $\mathbb{E}[\,(r \circ Y_t^a - r \circ Y_t^b)\mathbb{1}\{\tau \leq t\}\,] \leq 0$ *holds for all* t.[7]

$$\begin{aligned} \therefore \quad \mathbb{E}(r \circ Y_t^a - r \circ Y_t^b) &= \mathbb{E}[\,(r \circ Y_t^a - r \circ Y_t^b)(\mathbb{1}\{\tau \leq t\} + \mathbb{1}\{\tau > t\})\,] \\ &\leq \mathbb{E}[\,(r \circ Y_t^a - r \circ Y_t^b)\mathbb{1}\{\tau > t\}\,] \\ &\leq K\mathbf{P}\{\tau > t\} \qquad\qquad (\because\ r \leq K := \sup u). \end{aligned}$$

$$\therefore \quad v_\varrho(y_0) - v_\varrho(z) \leq K \sum_{t=0}^{\infty} \varrho^t \mathbf{P}\{\tau > t\} \leq K \sum_{t=0}^{\infty} \mathbf{P}\{\tau > t\}. \tag{2}$$

Inequality (2) contains the essential idea of our paper. If $\mathbf{P}\{\tau > t\}$, the probability that Country A is *always richer* than Country B in the period up until t, diminishes sufficiently quickly with t, then value boundedness will hold.

That it does diminish sufficiently quickly for the neoclassical optimal growth model is proved as follows. If the income of initially poorer

[7]The proofs of this and all other lemmata are deferred to the end of the paper.

Country B converges to zero in probability then $\mathbf{P}\{\tau > t\}$ might conceivably be large, even for large t. If, however, Y_t^b always returns to some region such as (c, ∞), where $c > 0$, then on those occasions income rank reverses with positive probability at least $\varepsilon > 0$, to be calculated below. The fundamental stability of the Brock–Mirman model implies that Y_t^b does indeed always return to a region (c, ∞), where c depends on the parameters of the model. Together these facts imply that $\mathbf{P}\{\tau > t\} \to 0$ relatively quickly.

Let $\mathscr{F}_t$ be the σ-algebra generated by (Y_s^a, Y_s^b), $s \leq t$.[8] From (MAR) it is intuitively clear (and can easily be proved) that if w is any bounded or nonnegative real function then our time t prediction of the value $w \circ Y_{t+1}^b$ satisfies

$$\mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] = \int w[f(\pi_\varrho \circ Y_t^b) z] G(dz) \quad \mathbf{P}\text{-a.s.} \tag{3}$$

An identical relation holds for Country A.

The next step of the proof is to bound the far right hand term in (2) independent of $y_0 \in [\,z, \infty)$ and $\varrho \in [\,\hat{\varrho}, 1)$. To do so we need the following lemma, which is a simple consequence of the Euler equation.

Lemma 4.2. *There are positive constants λ, β and a decreasing, real valued function w on $(0, \infty)$, all independent of ϱ, y_0 and z, such that* (i) $w \geq 1$, (ii) $w(x) \to \infty$ *as* $x \to 0$, (iii) $\lambda < 1$, *and*

$$\mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot \leq \lambda \cdot w \circ Y_t^b + \beta \quad \mathbf{P}\text{-a.s.} \tag{4}$$

Corollary 4.1. *There is a constant $c > 0$ and an $\alpha \in (0, 1)$, both independent of ϱ, y_0 and z, such that*

$$\mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot \mathbb{1}\{Y_t^b \leq c\} \leq \alpha \cdot w \circ Y_t^b \cdot \mathbb{1}\{Y_t^b \leq c\}. \tag{5}$$

Proof. By (ii) there is a $c > 0$ such that $w(c) > \beta(1 - \lambda)^{-1}$. Since w is decreasing, $w(x) \geq w(c)$ for all $x \in (0, c]$. Define

$$\alpha := \lambda + \frac{\beta}{w(c)},$$

so that $\lambda < \alpha < 1$. By Lemma 4.2, then,

$$\mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot \mathbb{1}\{Y_t^b \leq c\} \leq (\lambda \cdot w \circ Y_t^b + \beta) \cdot \mathbb{1}\{Y_t^b \leq c\}.$$

[8] Since $\mathscr{F}_t$ represents information at time t, it does not contain ξ_t or ξ_t', otherwise it would contain Y_{t+1}^a and Y_{t+1}^b in light of (MAR).

$$\therefore \quad \frac{\mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot \mathbb{1}\{Y_t^b \leq c\}}{w \circ Y_t^b} \leq \left(\lambda + \frac{\beta}{w \circ Y_t^b}\right) \mathbb{1}\{Y_t^b \leq c\}$$
$$\leq \alpha \mathbb{1}\{Y_t^b \leq c\}.$$

□

Now define $N_t := \sum_{i=0}^t \mathbb{1}\{Y_i^b > c\}$, so that N_t is the number of times that Country B has income exceeding c in the period $0, \ldots, t$. We have

$$\mathbf{P}\{\tau > t\} = \mathbf{P}\{\tau > t\} \cap \{N_t > j\} + \mathbf{P}\{\tau > t\} \cap \{N_t \leq j\}. \tag{6}$$

The two terms on the right hand side need to be bounded. The first term has the following simple bound. The intuition is that whenever $Y_t^b > c$ the income ranking reverses with independent probability at least ε.

Lemma 4.3. *There is an $\varepsilon > 0$ independent of ϱ, z and y_0 such that*

$$\mathbf{P}\{\tau > t\} \cap \{N_t > j\} \leq (1 - \varepsilon)^j.$$

It remains to bound the second term in (6). For this purpose, let $B := \alpha^{-1} \int w[f(\pi_{\hat{\varrho}}(c))z]G(dz)$, which can be shown to be finite using (4). Next, let $M_t := \alpha^{-t} B^{-N_{t-1}} \, w \circ Y_t^b$, where $N_{-1} := 0$, so $M_0 = w \circ Y_0^b \equiv w(z)$.

Lemma 4.4. *The sequence $(M_t)_{t=0}^\infty$ is a supermartingale with respect to the filtration $(\mathscr{F}_t)_{t=0}^\infty$.*

It follows in particular that $\mathbb{E} M_t \leq \mathbb{E}\, M_0 = w(z)$, whence

$$\begin{aligned}
\mathbf{P}\{\tau > t\} \cap \{N_t \leq j\} &\leq \mathbf{P}\{N_{t-1} \leq j\} \\
&= \mathbf{P}\{B^{-N_{t-1}} \geq B^{-j}\} \quad (\because B \geq 1) \\
&\leq B^j \, \mathbb{E}\, B^{-N_{t-1}} \quad (\because \text{ Chebychev's ineq.}) \\
&\leq \alpha^t B^j \, \mathbb{E}\, M_t \quad (\because \ w \geq 1) \\
&\leq \alpha^t B^j \, w(z).
\end{aligned}$$

Recall that all of these terms in the final bound are independent of ϱ, y_0 and z. Choose $n \in \mathbb{N}$ such that $\delta := \alpha^n B < 1$, and set $j = t/n$, so that $\alpha^t B^j = \delta^{t/n}$. Combining this bound with (2), (6) and Lemma 4.3 gives

$$v_\varrho(y_0) - v_\varrho(z) \le K \sum_{t=0}^{\infty} [(1-\varepsilon)^{t/n} + \delta^{t/n} w(z)].$$

That is, $v_\varrho(y_0) - v_\varrho(z) \le Q + R \cdot w(z)$, where constants Q and R are independent of ϱ, z and y_0. Thus $v_\varrho(y_0) - v_\varrho(z)$ is indeed bounded from above independent of $y_0 \in [z, \infty)$ and $\varrho \in [\hat{\varrho}, 1)$, as was to be shown.

It remains to consider the case $y_0 \in (0, z)$. We need to check that $-\infty < -M(y_0) \le v_\varrho(y_0) - v_\varrho(z)$, or, in other words, $v_\varrho(z) - v_\varrho(y_0) \le M(y_0)$ for all $\varrho \in [\hat{\varrho}, 1)$. Since the problem is entirely symmetric, by repeating all of the above argument exactly, but swapping "Country A" with "Country B" and "Y_t^a" with "Y_t^b" gives $v_\varrho(z) - v_\varrho(y_0) \le Q + R \cdot w(y_0)$ for all $\varrho \in [\hat{\varrho}, 1)$. This completes the proof.

Appendix A

Remaining proofs are now given. In what follows, $\mathscr{B}$ is the Borel sets on $(0, \infty)$. Also $b\mathscr{B}$ is the bounded Borel functions on $(0, \infty)$, and $ib\mathscr{B}$ is those functions in $b\mathscr{B}$ which are nondecreasing. Let $\mathscr{P}$ be the probabilities on $((0, \infty), \mathscr{B})$. For $\mu \in \mathscr{P}$ and $h \in b\mathscr{B}$ we sometimes use the notation $\langle h, \mu \rangle$ for $\int h d\mu$. The symbol $\le_s$ denotes the stochastic dominance ordering on $\mathscr{P}$. That is, $\mu \le_s \mu'$ iff $\langle \mu, h \rangle \le \langle \mu', h \rangle$ for all $h \in ib\mathscr{B}$.

Proof of Lemma 4.1. Unfortunately some new concepts and notations are necessary. To begin, a transition probability function [10, p. 212] is a function $\mathbf{Q} \colon (0, \infty) \times \mathscr{B} \to [0, 1]$ such that $\mathbf{Q}(y, \cdot) \in \mathscr{P}$ for each $y \in (0, \infty)$ and $\mathbf{Q}(\cdot, B) \in b\mathscr{B}$ for each $B \in \mathscr{B}$. Define also the iterates of $\mathbf{Q}$:

$$\mathbf{Q}^t(y, B) := \int \mathbf{Q}(y, dy') \mathbf{Q}^{t-1}(y', B), \quad \mathbf{Q}^1 := \mathbf{Q}.$$

(All of these iterates are themselves transition probability functions.)

For $t = 0$ let $\mathbf{Q}^t$ be the identity map. We define using $\mathbf{Q}$ two operators. One acts on functions to the right, mapping $b\mathscr{B}$ into itself, and is defined at $h \in b\mathscr{B}$ by $(\mathbf{Q}h)(y) := \int \mathbf{Q}(y, dy') h(y')$. The other acts on measures to the left, maps $\mathscr{P}$ into itself, and is defined at $\mu \in \mathscr{P}$ by $(\mu \mathbf{Q})(B) := \int \mathbf{Q}(y, B) \mu(dy)$. It is well-known and easy to check that these two operators are adjoint in the sense that $\langle \mathbf{Q}h, \mu \rangle = \langle h, \mu\mathbf{Q} \rangle$ for all $h \in b\mathscr{B}$, $\mu \in \mathscr{P}$.

Corresponding to each first order stochastic difference equation there is a transition probability function $\mathbf{Q}$ whereby $\mathbf{Q}(y, B)$ is the probability that the state is in set B next period given that currently it is y. For (MAR) this representation is given by

$$\mathbf{Q}(y, B) = \int \mathbb{1}\{f(\pi_\varrho(y))z \in B\} G(dz). \tag{7}$$

In this case the real number $\mathbf{Q}^t h(y)$ can be thought of as the expectation of $h \circ Y_t$ when $Y_0 \equiv y$ and the state evolves according to (MAR). More generally, the Markov property states that for any $h \in b\mathscr{B}$ and any $s, t \in \mathbb{N}$ with $s \leq t$,

$$\mathbb{E}[h \circ Y_t \,|\, \mathscr{F}_s] = \mathbf{Q}^{t-s} h \circ Y_s \quad \mathbf{P}\text{-a.s.} \tag{8}$$

It is also well-known [10] that in the case of optimal growth, the operator $\mu \mapsto \mu\mathbf{Q}$ is monotone, which is to say that whenever $\mu, \mu' \in \mathscr{P}$ and $\mu \leq_s \mu'$ we have $\mu\mathbf{Q} \leq_s \mu'\mathbf{Q}$. (In fact this is easy to verify from (7) and monotonicity of $f \circ \pi_\varrho$.) Monotonicity clearly extends from $\mathbf{Q}$ to $\mathbf{Q}^j$ for any $j \geq 0$.

As is standard, $\mathscr{F}_\tau$ will be the collection of all $E \in \mathscr{F}$ such that $E \cap \{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Fix $t \in \mathbb{N}$. Evidently

$$\mathbb{E}[(r \circ Y_t^a - r \circ Y_t^b)\mathbb{1}\{\tau \leq t\}] = \mathbb{E}[\mathbb{E}[(r \circ Y_t^a - r \circ Y_t^b)\mathbb{1}\{\tau \leq t\} \,|\, \mathscr{F}_\tau]].$$

Let c stand for either a or b. A simple decomposition and the Markov property yield

$$\begin{aligned} \mathbb{E}[r \circ Y_t^c \mathbb{1}\{\tau \leq t\} \,|\, \mathscr{F}_\tau] &= \sum_{i=0}^{t} \mathbb{E}[r \circ Y_t^c \,|\, \mathscr{F}_\tau] \mathbb{1}\{\tau = i\} \\ &= \sum_{i=0}^{t} \mathbb{E}[r \circ Y_t^c \,|\, \mathscr{F}_i] \mathbb{1}\{\tau = i\}, \end{aligned}$$

where the second equality is a straightforward exercise in measure theory. Also, by the Markov property,

$$\begin{aligned} \mathbb{E}[r \circ Y_t^c \,|\, \mathscr{F}_i] \mathbb{1}\{\tau = i\} &= \mathbf{Q}^{t-i} r \circ Y_i^c \mathbb{1}\{\tau = i\} \\ &= \mathbf{Q}^{t-\tau} r \circ Y_\tau^c \mathbb{1}\{\tau = i\}. \end{aligned}$$

Reversing the decomposition gives

$$\mathbb{E}[r \circ Y_t^c \mathbb{1}\{\tau \le t\} \,|\, \mathscr{F}_\tau] = \mathbf{Q}^{t-\tau} r \circ Y_\tau^c \mathbb{1}\{\tau \le t\}.$$

Since conditional expectation is linear, it follows that

$$\mathbb{E}[(r \circ Y_t^a - r \circ Y_t^b)\mathbb{1}\{\tau \le t\} \,|\, \mathscr{F}_\tau] = (\mathbf{Q}^{t-\tau} r \circ Y_\tau^a - \mathbf{Q}^{t-\tau} r \circ Y_\tau^b)\, \mathbb{1}\{\tau \le t\}.$$

But on $\{\tau \le t\}$ the definition of τ implies that $Y_\tau^a \le Y_\tau^b$, and, since $r \in ib\mathscr{B}$ (Theorem 2.1) and $\mathbf{Q}^j$ is $\le_s$-monotone for any $j \ge 0$, we have

$$(\mathbf{Q}^{t-\tau} r \circ Y_\tau^a - \mathbf{Q}^{t-\tau} r \circ Y_\tau^b)\, \mathbb{1}\{\tau \le t\} \le 0 \quad \mathbf{P}\text{-a.s.}$$

The conclusion of the Lemma is now clear. □

Proof of Lemma 4.2. By a simple manipulation of the Euler equation, Nishimura and Stachurski [7, Proposition 4.2] show under assumptions weaker than Assumptions 2.1 and 2.2 that

$$\int w[f(\pi_{\hat\varrho}(y))z] G(dz) \le \left[\frac{\mathbb{E}(1/\varepsilon)}{\hat\varrho f'(\pi_{\hat\varrho}(y))} \right]^{1/2} w(y)$$

holds for all $y \in (0, \infty)$, where $w(y) := \sqrt{u'(y - \pi_{\hat\varrho}(y))}$. Of course here $\pi_{\hat\varrho}$ is the optimal policy when the discount factor is equal to $\hat\varrho$. By Assumptions 2.1 and 2.2 we have $\mathbb{E}(1/\varepsilon) < \infty$ and $\hat\varrho f'(\pi_{\hat\varrho}(y)) \to \infty$ as $y \to 0$. Evidently, then, when $\lambda \in (0,1)$ is taken as fixed,

$$\exists\, \delta_\lambda \text{ s.t. } y < \delta_\lambda \implies \int w[f(\pi_{\hat\varrho}(y))z] G(dz) \le \lambda w(y).$$

Note that the constants λ and δ_λ and the function w are independent of ϱ, y_0 and z.

Since the optimal savings policy is pointwise increasing in the discount factor we have $\pi_\varrho \ge \pi_{\hat\varrho}$, and since w is clearly decreasing, it then follows that

$$y < \delta_\lambda \implies \int w[f(\pi_\varrho(y))z] G(dz) \le \lambda w(y). \tag{9}$$

By the same rationale,

$$y \ge \delta_\lambda \implies \int w[f(\pi_\varrho(y))z] G(dz) \le \beta \tag{10}$$

when $\beta := \int w[f(\pi_{\hat\varrho}(\delta_\lambda))z] G(dz)$. Once again, the constant β is independent of ϱ, y_0 and z. Combining (9) and (10) gives

$$\int w[f(\pi_\varrho(y))z] G(dz) \le \lambda w(y) + \beta, \quad \forall y \in (0, \infty). \tag{11}$$

From (3) and (11) we get (4).

The only claims of Lemma 4.2 we have not verified are that $w(x) \to \infty$ as $x \to 0$ and $w \geq 1$. The first is obvious given the definition of w. The second is not necessarily true, but if the bound (4) holds for w, λ and β then clearly it also holds for $\hat{w} := w + 1$, $\hat{\lambda} := \lambda < 1$ and $\hat{\beta} := \beta + 1 < \infty$. □

Proof of Lemma 4.3. Let σ_i be the time of the i-th visit of Y_t^b to (c, ∞), so that $N_t = j$ if and only if $\sigma_j \leq t$ and $\sigma_{j+1} > t$. In order to prove Lemma 4.3 we first show that

Lemma A.1. *Let $S := \sup f$ and $T := f(\pi_{\hat{\varrho}}(c))$. If $\tau > t$ and $N_t > j$, then $S \cdot \xi_{\sigma_i} > T \cdot \xi'_{\sigma_i}$ for all $i = 1, \ldots, j$.*

Proof. Suppose instead that $S\xi_{\sigma_i} \leq T\xi'_{\sigma_i}$ for some i with $1 \leq i \leq j$. Then $f(\pi_\varrho \circ Y^a_{\sigma_i})\xi_{\sigma_i} \leq S\xi_{\sigma_i} \leq T\xi'_{\sigma_i} \leq f(\pi_\varrho(c))\xi'_{\sigma_i} \leq f(\pi_\varrho \circ Y^b_{\sigma_i})\xi'_{\sigma_i}$, since $\pi_\varrho \geq \pi_{\hat{\varrho}}$ and $Y^b_{\sigma_i} \geq c$. In other words, $Y^a_{\sigma_i+1} \leq Y^b_{\sigma_i+1}$, so that $\tau \leq \sigma_i + 1$. Also, we know that $N_t \geq j + 1$, so $\sigma_{j+1} \leq t$, and hence $\sigma_i + 1 \leq \sigma_j + 1 \leq \sigma_{j+1} \leq t$. This is a contradiction, because $\tau > t$. □

To continue with the proof of Lemma 4.3, note by Lemma A.1 that

$$\begin{aligned}
\mathbf{P}\{\tau > t\} \cap \{N_t > j\} &\leq \mathbf{P} \bigcap_{i=1}^{j} \{S \cdot \xi_{\sigma_i} > T \cdot \xi'_{\sigma_i}\} \\
&= \prod_{i=1}^{j} \mathbf{P}\{S \cdot \xi_{\sigma_i} > T \cdot \xi'_{\sigma_i}\} \\
&= \prod_{i=1}^{j} [1 - \mathbf{P}\{S \cdot \xi_{\sigma_i} \leq T \cdot \xi'_{\sigma_i}\}].
\end{aligned}$$

Now pick any $b > 0$. Clearly

$$\{S \cdot \xi_{\sigma_i} \leq T \cdot \xi'_{\sigma_i}\} \supset \{\xi'_{\sigma_i} \geq b\} \cap \{S \cdot \xi_{\sigma_i} \leq T \cdot b\}. \tag{12}$$

Letting $\varepsilon := \mathbf{P}\{\xi'_{\sigma_i} \geq b\}\mathbf{P}\{S \cdot \xi_{\sigma_i} \leq T \cdot b\}$ we have $\mathbf{P}\{S \cdot \xi_{\sigma_i} \leq T \cdot \xi'_{\sigma_i}\} \geq \varepsilon$ by (12) and the independence of ξ_{σ_i} and ξ'_{σ_i}. That $\varepsilon > 0$ follows from Assumption 2.2. Evidently it is independent of y_0, z and ϱ. The conclusion of Lemma 4.3 follows. □

Proof of Lemma 4.4. Clearly M_t is $\mathscr{F}_t$-measurable. It will be integrable provided that we can verify the key supermartingale property

$\mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \leq M_t$. To this end, let $F := \mathbb{1}\{Y_t^b > c\}$ and $F^c := 1 - F = \mathbb{1}\{Y_t^b \leq c\}$, so that

$$\mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] = \mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F + \mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F^c.$$

Consider the first term. On F we have $N_t = N_{t-1} + 1$, so

$$\begin{aligned}
\mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F &= \alpha^{-(t+1)} B^{-N_{t-1}} B^{-1} \mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot F \\
&= \alpha^{-(t+1)} B^{-N_{t-1}} B^{-1} \int w(f(\pi_\varrho \circ Y_t^b) z) G(dz) \cdot F \\
&\leq \alpha^{-(t+1)} B^{-N_{t-1}} B^{-1} \int w(f(\pi_{\hat{\varrho}}(c)) z) G(dz) \cdot F \\
&\leq \alpha^{-t} B^{-N_{t-1}} F.
\end{aligned}$$

Using this bound and $w \geq 1$ gives $\mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F \leq M_t \cdot F$. Also, on the set F^c we have $N_t = N_{t-1}$, and Corollary 4.1 applies. Hence,

$$\begin{aligned}
\mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F^c &= \alpha^{-t} B^{-N_{t-1}} \alpha^{-1} \mathbb{E}[w \circ Y_{t+1}^b \,|\, \mathscr{F}_t] \cdot F^c \\
&\leq \alpha^{-t} B^{-N_{t-1}} w \circ Y_{t+1}^b \cdot F^c.
\end{aligned}$$

$$\therefore \quad \mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \cdot F^c \leq M_t \cdot F^c.$$

$$\therefore \quad \mathbb{E}[M_{t+1} \,|\, \mathscr{F}_t] \leq M_t.$$

□

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