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A GENERAL MODEL OF COEXISTING HIDDEN ACTION AND HIDDEN INFORMATION

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Summary. We consider a general agency model with coexisting hidden action and hidden information. We prove that, with minor technical qualifications, independence of the production technology from the consumer type is necessary and sufficient for welfare irrelevance of hidden action. Our result clarifies and confirms the main conclusion drawn in the existing literature on mixed models, that if the parties are risk neutral and the production technology is not correlated with private information, then hidden action is irrelevant. However it makes it clear that even under risk neutrality this conclusion does not extend to the correlated case, which in practice occurs quite frequently. We illustrate it with a realistic example where neither hidden action nor hidden information on their own lead to welfare losses, while their combination does.

Keywords and Phrases: hidden action, hidden information, Fredholm integral equations of the first type.

JEL classification numbers: C6, D8.

1 Introduction

Agency relationships form an important part of economic life. Among the most common examples are managers acting on behalf of an owner, workers supplying labor to a firm, and customers buying coverage from an insurance company. The common feature of all these examples is that *unobservable* actions undertaken by one party have payoff relevant consequences for another. This creates a *moral hazard problem*. The main trade-off the contracting parties face in moral hazard situations is a trade-off between risk and incentives.[1] Hence, if both parties are risk neutral moral hazard will not create welfare losses.

Often a moral hazard situation is complicated by the existence of hidden information. For example, managers or workers may have different costs of effort, or customers buying a medical insurance may have different health conditions. Below, for concreteness, we will call the unobservable action *effort* and the hidden information the *type* of the agent. This type of models were pioneered by Laffont and Tirole (1986) and later developed by Picard (1987), Rogerson (1988), Guesnerie, Picard, and Rey (1989), Melumad and Reichelstein (1989), and Caillaud, Guesnerie, and Ray (1992). The common

[1] For a review of standard principal-agent problem, see Grossman and Hart (1983).

assumption in these papers is that the noise in the production technology is independent of the agent's type, that is, the models are just noisy hidden information models. The main result of this literature is that, if both parties are risk neutral, then in most such models the principal can reach the same utility as in the absence of noise.

To understand this result let us as a first stage assume that the effort is contractible and solve the adverse selection problem. The result will be a wage schedule conditional on the effort level. To implement the same effort at the same cost when the effort is not contractible, the principal has to find a wage schedule, which depends only on the observable signal, such that the expectation of this schedule conditional on effort gives the schedule found at stage one. This problem can be reduced to solving a Fredholm integral equation of the first type. If the density of the noise is sufficiently well behaved (does not have interior singularities) this equation always possesses a solution.

One can, however, easily come up with examples of economically interesting situations, where the production technology is type dependent. For instance, assume that different research institutions compete for the government's grants. Each institution has a research project which is characterized by the potential success probability θ. The actual success probability depends on both θ, which can be interpreted as the quality of the project, and the effort level. In that case the expected payoff to the project for a given effort will still depend on the private information of the institutions, and it is not clear whether the principal can reach the same utility as in the absence of noise.

In this paper we start by formulating a general model with hidden action and hidden information. We provide the first order characterization of the solution and use it to prove that the principal can achieve the same utility under the hidden action as in the case of the observable effort *if and only if* the production technology is independent on the agent's type and some mild regularity conditions on the noise density are satisfied. When the latter is the case we find explicit solutions in the cases when the optimal compensation schedule in the pure hidden action model is analytical and the production noise is either additive and is normally distributed or multiplicative and is exponentially distributed.

The paper is organized in the following way. In Section 2 we introduce the main model and derive the first order characterization of the solution. In Section 3 we derive the necessary and sufficient conditions for the irrelevance of moral hazard and solve some examples. These examples allow the reader to get the better understanding how to apply the theorems of the paper and also provide useful explicit solutions. In Section 4 we solve an example for which both hidden action and hidden information are relevant in determining the welfare of the parties. The striking feature of that example is that in the situation it describes neither hidden action nor hidden information on their own entail any welfare losses, while their combination does. Section 5 concludes.

2 The model

Consider a risk neutral principal and risk neutral agent who are engaged in a following type of a transaction. An agent undertakes an effort z that generates the distribution of profits $f(x; z, \theta)$ for the principal and entails cost $c(z, \theta)$ for an agent. Variable θ is privately observed by the agent and can be interpreted as her type. Neither the type of the agent nor the effort are observed by the principal. The profits, on the other hand are observable and verifiable. Upon the profit realization, x, the agent receives wage $w(x)$ according to the in-advance-agreed-upon wage schedule, $w(\cdot)$.

The probability density $f(\cdot; z, \theta)$ is assumed to be continuously differentiable and strictly positive on its support, while the cost, $c(\cdot; \theta)$ is assumed to be increasing and convex. Moreover, $c(\cdot, \cdot)$ is twice differentiable and satisfies the Spence-Mirrlees condition, i. e. $c_{z\theta} < 0$. While θ is private information, we assume that the principal believes it comes from a distribution with a continuously differentiable and strictly positive on its support density $g(\cdot)$. The support of the distribution is assumed to be a segment $[\underline{\theta}, \overline{\theta}]$, where $0 \leq \underline{\theta} < \overline{\theta} \leq \infty$. Denote by $G(\cdot)$ is the corresponding cumulative distribution function. Let

$$\pi(z, \theta) = \int x f(x; z, \theta) dx \tag{1}$$

be the expected payoff for the principal if the agent of type θ chooses effort level z. Finally, let us assume that the function $V(z, \theta)$ defined by

$$V(z, \theta) = \pi(z, \theta) - c(z, \theta) + \frac{1 - G(\theta)}{g(\theta)} c_\theta(z, \theta) \tag{2}$$

is strictly concave in z and supermodular in (z, θ).

2.1 The case of the observable effort

In this subsection we will concentrate on the case of observable effort, i. e. the principal faces a pure hidden information problem. In that case the analysis is standard and the assumptions guarantee that the solutions to the relaxed and complete problems coincide. Let $z(\theta)$ be the optimal effort schedule in the case when the effort is observable. It solves

$$z(\theta) = \arg\max V(z, \theta). \tag{3}$$

For a discussion, see Mussa and Rosen (1978). Define the consumer surplus, $\xi(\cdot)$ as the unique solution to the Cauchy problem

$$\begin{cases} \xi_\theta = -c_\theta(z(\theta), \theta) \\ \xi(\underline{\theta}) = 0 \end{cases}. \tag{4}$$

Then the wage schedule that implements effort levels $z(\theta)$ can be found as

$$v(z) = \min_\theta (c(z, \theta) + \xi(\theta)). \tag{5}$$

Intuitively, assume that the principal has to compensate the agent for the cost of effort and leave her information rents $\xi(\theta)$. If she wants to induce level of effort z, she selects the type for which the total cost of inducing this effort is minimal.

2.2 The case of the unobservable effort

Let us return to our model with unobservable effort. The principal's problem is to find a wage schedule $w(\cdot)$ to solve:

$$\begin{gathered} \max \iint (\pi(z,\theta) - w(x))f(x;z,\theta)g(\theta)dxd\theta \\ s.t.\ z \in \arg\max_z(\int w(x))f(x;z,\theta)dx - c(z,\theta)) \ \cdot \\ \max_z(\int w(x))f(x;z,\theta)dx - c(z,\theta)) \geq 0 \end{gathered} \tag{6}$$

Let us introduce the agent's surplus by

$$s(\theta) = \max_z(\int w(x)f(x;z,\theta)dx - c(z,\theta)). \tag{7}$$

Then the relaxed problem for the principal is

$$\max \int_{\underline{\theta}}^{\overline{\theta}} (\pi(z,\theta) - c(z,\theta) - s(\theta))g(\theta)d\theta$$
$$s.t.\ s_\theta(\theta) = \int w(x) f_\theta(x; z, \theta)dx - c_\theta(z,\theta)) \quad . \tag{8}$$
$$s(\theta) = \int w(x) f(x; z, \theta)dx - c(z,\theta)$$
$$s(\theta) \geq 0$$

The Hamiltonian for the problem (8) is:

$$H = (\pi(z,\theta) - c(z,\theta) - s(\theta))g(\theta) + \lambda(\theta)(\int w(x) f_\theta(x; z, \theta)dx - c_\theta(z,\theta)) \tag{9}$$
$$\mu(\theta)(s(\theta) - \int w(x) f(x; z, \theta)dx - c(z,\theta)). \tag{10}$$

Our next objective is to prove that $\mu(\theta) = 0$ a.e. with respect to the Lebesgue measure. We do it in a sequence of the following two lemmata.

Lemma 1 *Assume*

$$\int_{\underline{\theta}}^{\overline{\theta}} f_\theta(x; z, \theta)dx \tag{11}$$

converges uniformly in θ. Let $(z(\theta), s(\theta), w(x))$ solve the optimal control problem (8) and let $\lambda(\theta)$ and $\mu(\theta)$ be the Lagrange multipliers for the first and second constraint respectively. Then $\mu(\theta) \leq 0$ a. e. with respect to the

Lebesgue measure.

Proof. Consider the optimal control problem:

$$\begin{gathered}
\max \int_{\underline{\theta}}^{\overline{\theta}} (\pi(z,\theta) - c(z,\theta) - s(\theta))g(\theta)d\theta \\
s.t.\ s_\theta(\theta) = \int w(x) f_\theta(x; z,\theta)dx - c_\theta(z,\theta)) \\
s(\theta) \leq \int w(x) f(x; z,\theta)dx - c(z,\theta) \\
s(\theta) \geq 0
\end{gathered} \quad . \tag{12}$$

If $(z(\theta), s(\theta), w(x))$ solve this optimal control problem and let $\lambda(\theta)$ and $\mu(\theta)$ be the Lagrange multipliers, than the Kuhn-Tucker necessary conditions insure that $\mu(\theta) \leq 0$ a. e. with respect to the Lebesgue measure. To complete the proof we have to argue that the second constraint binds. Indeed, assuming the constraint is slack one can increase $w(x)$ by sufficiently small ε for all profit realizations. Then this constraint will still hold. But such a change does not affect the first constraint, since

$$\int_{\underline{\theta}}^{\overline{\theta}} f(x; z,\theta)dx = 1 \tag{13}$$

for all θ and assumption that

$$\int\limits_{\underline{\theta}}^{\overline{\theta}} f_\theta(x; z, \theta)dx \tag{14}$$

converges uniformly assures that the last integral is zero.

Q. E. D.

Lemma 2 *Under assumptions of Lemma 1* $\mu(\theta) = 0$ *a. e. with respect to the Lebesgue measure.*

Proof. Assume that $(z(\theta), s(\theta); \lambda(\theta), \mu(\theta))$ are defined as in Lemma 1. Then the wage schedule $w(\cdot)$ solves:

$$\begin{array}{c} \max[\lambda(\theta)(\int w(x)f_\theta(x; z, \theta) - \mu(\theta)w(x)f(x; z, \theta)dx)] \\ s.t. \int w(x)f_\theta(x; z, \theta) - c(z, \theta) \geq 0 \end{array} . \tag{15}$$

Taking into account the envelope condition one obtains:

$$\begin{array}{c} \max[\lambda(\theta)s_\theta(\theta) - \mu(\theta) \int w(x)f(x; z, \theta)dx] \\ s.t. \int w(x)f_\theta(x; z, \theta) - c(z, \theta) \geq 0. \end{array} \tag{16}$$

If $\mu(\theta) < 0$ on a set of positive measure one can always increase $w(x)$, which will increase the value of the objective function on a set of positive measure.

Q. E. D.

These two lemmata allow us to exclude the last term from the Hamiltonian and write the first order conditions in a form:

$$\begin{cases} \lambda_\theta = g(\theta), \ \lambda(\overline{\theta}) = 0. \\ (\pi_z(z,\theta) - c_z(z,\theta))g(\theta) + \lambda(\theta)(\int w(x) f_{\theta z}(x;z,\theta)dx - c_{\theta z}(z,\theta)) = 0 \end{cases} . \tag{17}$$

This equations together with the constraints of problem (8) determine the solution. One can eliminate surplus from these constraints and solve the first of the first order conditions for λ to obtain the following system of equations for $z(\cdot)$ and $w(\cdot)$:

$$\begin{cases} \pi_z(z,\theta) - c_z(z,\theta) = \frac{1-G(\theta)}{g(\theta)}(\int w(x) f_{\theta z}(x;z,\theta)dx - c_{\theta z}(z,\theta)) \\ \int w(x) f_z(x;z,\theta)dx = c_z(z,\theta). \end{cases} . \tag{18}$$

Note that the second of these equations is simply the first order condition with respect to the effort for the agent who faces the wage schedule $w(\cdot)$.

3 When the hidden information is irrelevant?

In this Section we are going to address the question: Under which conditions will the solution to the problem with hidden effort implies the same effort and same expected surplus for all types as in the case when the effort is observable? To begin let us prove the following lemma.

Lemma 3 *Let the effort level and the expected surplus be the same for all types as in the case of the observable effort. Then the expected payment of the principal conditional on effort, z, is $v(z)$, i. e. it is the same as in the case of the observable effort.*

Proof. Let $z(\cdot)$, and $s(\cdot)$ be defined by (3)-(4). From the definition of the agent's surplus one obtains:

$$s(\theta) + c(z, \theta) = \int w(x) f(x; z, \theta) dx. \tag{19}$$

According to equation (4) the left hand side of (19) does not depend on θ. Moreover, according to (5) it equals $v(z)$. To complete the proof note that the right had side of (19) is the expected payment of the principal conditional on effort.

Q. E. D.

Using Lemma 3 we will prove the following result.

Lemma 4 *The principal will choose to implement the same effort levels for all types and make the same expected payment conditional on effort as in the case of the observable type if and only if there exists function $w : R_+ \to R_+$ such that for any $\theta \in [\underline{\theta}, \overline{\theta}]$*

$$\int w(x)f(x; z, \theta)dx = v(z). \tag{20}$$

Proof. The necessity follows from Lemma 3. To prove sufficiency note that equation (20) implies that

$$\begin{aligned} \int w(x)f_\theta(x; z, \theta)dx &= 0. & (21) \\ \int w(x)f_{\theta z}(x; z, \theta)dx &= 0. & (22) \end{aligned}$$

Therefore, $\lambda(\theta) = G(\theta) - 1$ and $z = z(\theta)$, where $z(\theta)$ is defined by (3) solve system (17). Moreover, the Hamiltonian becomes

$$H = g(\theta)V(z, \theta). \tag{23}$$

Under our assumptions on function $V(\cdot, \cdot)$ the effort schedule $z(\theta)$ is implementable and maximizes the Hamiltonian. Therefore, it is optimal.

Q. E. D.

Lemma 4 reduces the task of analyzing the conditions for the irrelevance of the hidden action to task of studying the conditions of the existence of a solution of equation (20).

Lemma 5 *Let $w(x)$ solves equation (20). Then it also solves equation*

$$\int w(x)(\int_{\Omega} f(x;z,\theta)g(\theta)d\theta)dx = v(z). \qquad (24)$$

Proof. A straightforward calculation changing the order of integration proves our assertion.

Q. E. D.

Note that the right hand side of equation (20) does not depend on θ. This suggests that for a solution to exist it is necessary that the output density be independent on θ as well.

To formulate it precisely let us every function $h(x,z)$ such that $h(x,\cdot)$,

$h(\cdot, z) \in L^2(R_+)$ let us introduce the following symmetric functions:

$$
\begin{aligned}
k^h(x,z) &= \int\limits_0^\infty h(x,y)h(z,y)dy \\
k_1^h(x,z) &= \int\limits_0^\infty h(y,x)h(y,z)dy.
\end{aligned}
\tag{25}
$$

We will refer to this function below as symmetrized kernels. We will say that the kernel $k_1(\cdot,\cdot)$ is closed if

$$
\int\limits_0^\infty k_1^h(x,z)\phi(z)dz = 0 \tag{26}
$$

if and only if $\phi(z) = 0$ almost everywhere. Now we are ready to formulate our next result.

Theorem 1 *Assume that $v(\cdot)$ is different from zero at a set of a positive measure, there exists $\theta \in \Omega$ such that the kernel $k_1^{f_\theta}(x,z;\theta)$ is closed and for any $w(\cdot)$ which satisfies equation (24) the integral*

$$
\int f_\theta(x,z,\theta)w(x)dx \tag{27}
$$

converges uniformly in θ. Then equation (20) has no solutions.

Note that assumption that $K(x,z)$ is closed in particular implies that

$f_\theta(x, z, \theta)$ should differ from zero on a set of a positive measure.

Proof. Let equation (20) possess a solution. Denote it by $w(x)$. Note that $w(\cdot)$ cannot be identically zero almost everywhere. According to Lemma 5, function $w(\cdot)$ satisfies equation (24). Differentiating equation (20) with respect to θ one obtains

$$\int f_\theta(x, z, \theta) w(x) dx = 0. \tag{28}$$

The differentiation under the sign of integral is legitimate because of our assumption of the uniform convergence. If equation (28) has a non-trivial solution, kernel $k_1^{f_\theta}(x, z; \theta)$ should be *not* closed for all θ (see, Pogorzelski, 1966), which proves the Theorem.

Q. E. D.

4 Type-independent technologies: a closer look

In the previous section we showed that independence of technology on the private information of agents is the basic economic assumption, which is necessary for hidden action to be irrelevant. Let us under when it is also sufficient. In order words, what are the conditions for the equation:

$$\int w(x)f(x,z)dx = v(z). \tag{29}$$

to possess a solution.

The classical theorem in this area is the Picard's Theorem (see, Pogorzelski, 1966). It was first applied to a problem in mechanism design by Melumad and Reichelstein (1989). One, however, need to adapt it slightly for this case. Indeed, to apply the Theorem the right hand side should be a square-integrable function. In our application, however, the right hand side is the effort-wage schedule, which should be increasing. Therefore, it could not be square integrable unless the effort has compact support. Fortunately, this problem can be easily circumvented if one multiplies both sides of equation (29) on a such function $g(\cdot)$ that it has full support and the right hand side becomes square intergrable For example, if $v(\cdot)$ is differentiable and not equal to zero anywhere, one can choose $g(\cdot)$ such that:

$$g^2(z) = \frac{2v'(z)}{v(z)} \exp(-v^2(z)). \tag{30}$$

The following theorem provides the necessary and sufficient conditions for the existence of the solution of equation (29).

Theorem 2 *Let function $g(\cdot) : R_+ \rightarrow R_+$ with a full support be such that:function $vg(\cdot) \in L^2(R_+)$. Equation (29) has a solution if and only if:*

1. *Integral operator $T : H^1(R_+) \rightarrow H^1(R_+)$ defined by:*

$$Th = \int_{R_+} k^{fg}(x,z)h(z)dz \tag{31}$$

has a discrete spectrum;

2. *The series*

$$\sum_{n=1}^{\infty} |\lambda_n v_n|^2 \tag{32}$$

converges, where λ_n are eigenvalues of T, i. e. they satisfy

$$T\psi_n = \lambda_n \psi_n \tag{33}$$

for some ψ_n with a unitary L^2-norm, and v_n are defined as

$$v_n = \int_{R_+} v(z)g(z)\psi_n(z)dz. \tag{34}$$

Moreover, if kernel $k^{fg}(x,z)$ is closed the solution is unique.

Proof. The theorem is a direct consequence of the Picard's theorem applied to the equation:

$$\int w(x)g(z)f(x,z)dx = v(z)g(z). \tag{35}$$

Q. E. D.

An easy corollary of this theorem is the following.

Corollary 1 *If operator T defined by equation (31) is compact then there exists a solution of equation (29).*

Proof. The spectrum of a compact operator is discrete and bounded, i. e. there exists $K > 0$ such that all eigenvalues satisfy

$$|\lambda_n| \leq K. \tag{36}$$

Therefore,

$$\sum_{n=1}^{\infty} |\lambda_n v_n|^2 \leq K \sum_{n=1}^{\infty} |v_n|^2 = K \int_{R_+} |v(z)g(z)|^2 < \infty. \tag{37}$$

The equality here is the Parceval's equality ((see, Pogorzelski, 1966).

Q. E. D.

After developing the general theory let us consider a few examples.

Example 1 *Let us assume that the production technology is multiplicative, i. e.*

$$x = z\varepsilon. \tag{38}$$

Let us also assume that ε is exponentially distributed with a unit mean. Then equation (29) becomes

$$\frac{1}{z}\int w(x)\exp(-\frac{x}{z})dx = v(z). \tag{39}$$

Let us first prove that if $v(\cdot)$ grows no faster than a polynomial equation (39) possesses a solution. Indeed, assume this is the case. Then function $z^2 v(z)\exp(-z) \in L^2(R_+)$ and

$$k^{fg}(x,z) = \frac{x^2 z^2}{x+z}\exp(-x-z). \tag{40}$$

To prove that operator T defined by (31) is compact we have to prove that for any family of functions H such that

$$\|h\|_{H^1(R_+)} \leq K \textit{ for } \forall h \in H \tag{41}$$

the family of functions

$$\{Th\}_{h\in H} \tag{42}$$

is uniformly bounded and equicontinuos. Uniform boundness follows from the following sequence of inequalities:

$$\|Th\|_{H^1(R_+)} = \sqrt{\int\limits_{R_+} \tfrac{x^2z^2}{x+z}\exp(-x-z)h(z)dz)^2dx} + \sqrt{\int\limits_{R_+} \tfrac{2xz^3}{x+z}\exp(-x-z)h(z)dz)^2dx} \leq$$
$$\sqrt{\int\limits_{R_+} h^2(z)dz(}\sqrt{\int\limits_{R_+} (\tfrac{2xz^3}{x+z})^2\exp(-2(x+z)dzdx} + \sqrt{\int\limits_{R_+} \tfrac{x^2z^2}{x+z}\exp(-2(x+z)dzdx)} < 4K, \tag{43}$$

where the last estimate comes from

$$\sqrt{\int\limits_{R_+} (\tfrac{2xz^3}{x+z})^2\exp(-2(x+z)dzdx} + \sqrt{\int\limits_{R_+} \tfrac{x^2z^2}{x+z}\exp(-2(x+z)dzdx} \leq$$
$$\sqrt{4\int\limits_{R_+} z^6\exp(-2z)dz\int\limits_{R_+}\exp(-2x)dx} + \int\limits_{R_+} x^2\exp(-2x)dx < 4. \tag{44}$$

To prove equicontinuity it is sufficient to prove that the derivatives of all *functions in the family are uniformly bounded. This follows from:*

$$
\begin{gathered}
Th'(x) = \int\limits_{R_+} \tfrac{2xz^3}{x+z} \exp(-x-z)h(z)dz \leq \\
\sqrt{\int\limits_{R_+} h^2(z)dz} \sqrt{\int\limits_{R_+} (\tfrac{2xz^3}{x+z})^2 \exp(-2(x+z))dz} \leq \\
2K \sqrt{\int\limits_{R_+} z^6 \exp(-2z)dz} < 6K.
\end{gathered} \tag{45}
$$

The existence of the solution follows know from the Corollary 1. Let us actually find a solution to equation (39) in the case

$$
v(z) = z^a, \ a > 0. \tag{46}
$$

Let us look for a solution in a form

$$
w(x) = cx^a, \tag{47}
$$

where c is some constant to be found. Then equation (39) becomes:

$$
c \int\limits_0^\infty x^a \exp(-\frac{x}{z})dx = z^{a+1}. \tag{48}
$$

Making a substitution $x = zt$ one obtains:

$$cz^{a+1}\int_0^\infty t^a \exp(-t)dt = z^{a+1}. \tag{49}$$

The solution is

$$c = \frac{1}{\Gamma(a+1)}, \tag{50}$$

where $\Gamma(\cdot)$ is Euler's gamma function defined by:

$$\Gamma(y) = \int_0^\infty t^{y-1} \exp(-t)dt. \tag{51}$$

In particular, for $a = n \in N$ one obtains

$$c = \frac{1}{n!}. \tag{52}$$

Note that for a convex function $v(\cdot)$ (which is usually the case in screening models), $w(\cdot)$ is flatter than $v(\cdot)$. The unobservability of effort leads to lower powered incentives despite the fact that the agents are risk neutral. Assume that function $v(\cdot)$ is analytical at zero, i. e. in some neighborhood of $z = 0$

it can be represented as a sum of a convergent series:

$$v(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{53}$$

Then according to formula (52) and the superposition principle for linear equations:

$$w(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n. \tag{54}$$

Note that the radius of convergence of series (54) is at least as big as that of series (53).

The choice of function $g(\cdot)$ in our example is rather instructive. The term z^2 was selected to kill the singularity at zero, while the exponent guaranteed that the symmetrized kernel will behave well at infinity. The key property that allowed as to do it was that function $f(\cdot, \cdot)$ did not have interior singularities. More precisely, the following result holds.

Theorem 3 *Let function $g(\cdot) : R_+ \to R_+$ with a full support be such that:function $vg(\cdot) \in L^2(R_+)$. Assume further that*

$$(k^{fg}(x, y))^2 \leq (k^{fg}(x, x))^2 + (k^{fg}(y, y))^2. \tag{55}$$

Then there exists such function $\phi(z)$ that the operator Q defined by:

$$Qh = \int_0^\infty k^{fg\phi}(x,z)h(z)dz \tag{56}$$

is a Hilbert-Schmidt operator and therefore, is compact.

Proof. An operator is a Hilbert-Schmidt operator if and only if its kernel is square intergrable (see, Pugachev and Sinitsyn, 1999). Therefore, one has to select $\phi(\cdot)$ in such a way that $gv\phi \in L^2(R_+)$ and

$$\int_0^\infty \int_0^\infty (k^{fg\phi}(x,y))^2 dxdy < \infty. \tag{57}$$

It is sufficient to select ϕ in such a way that

$$\int_0^\infty \phi^2(x)dx < \infty \tag{58}$$

and

$$\int_0^\infty (k^{fg\phi}(x,x))^2 \phi^2(x)dx < \infty. \tag{59}$$

This is always possible. Indeed, let

$$\xi(x) = (k^{fg\phi}(x,x))^2 \tag{60}$$

and

$$\zeta(x) = \xi(0) + \int_0^x \max(0, \xi'(t))dt. \tag{61}$$

Then it is sufficient to choose

$$\phi = \sqrt{\zeta'(x)\exp(-\zeta(x))}. \tag{62}$$

Q. E. D.

The major example of densities for which conditions of Theorem 3 will be violated are the densities which have a singularity along line $z = x$, i. e.

$$f(x, z) = O(|x - z|^{\alpha}) \tag{63}$$

for some $\alpha < 1$. Note, however, that conditions of Theorem 3 are sufficient but not necessary for the existence of the solution. Let us demonstrate this point by the following example.

Example 2 *Let*

$$f(x, z) = \begin{cases} \frac{1}{4\sqrt{\eta z}\sqrt{|x-z|}}, \text{ for } x \in [0, 2z] \\ 0, \text{ otherwise} \end{cases} \tag{64}$$

and assume that

$$v(z) = z^b. \tag{65}$$

Though the conditions of Theorem 3 are not satisfied the solution of equation (29), which in this case takes the form

$$\frac{1}{4\sqrt{\eta}} \int_0^{2z} w(x) \frac{1}{\sqrt{z}\sqrt{|x-z|}} dx = z^b, \tag{66}$$

can still be found. Indeed, let us look for the solution in a form

$$w(x) = cx^b. \tag{67}$$

Substituting it into equation (66) and making a change of variables

$$x = tz \tag{68}$$

one obtains:

$$c = \frac{4}{I}, \tag{69}$$

where

$$I = \int_0^1 t^{-1/2}(1+t)^b dt + \int_0^1 t^{-1/2}(1-t)^b dt. \tag{70}$$

Evaluating the integrals one obtains:

$$I = \Gamma(b+1)(\frac{\sqrt{\pi}}{\Gamma(b+3/2)} + 2\sum_{n=0}^{\infty} \frac{1}{n!(2n+1)\Gamma(b+1-n)}). \qquad (71)$$

Here $\Gamma(\cdot)$ *is Euler's gamma function. If* $\operatorname{Re} z > 0$ *then* $\Gamma(z)$ *is defined by:*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \exp(-t) dt. \qquad (72)$$

Otherwise, it should be understood as the analytical continuation of function (72).

Note that the sum on the right hand side converges. Moreover, if $b \in N$ *it has only finitely many non-zero terms, since* $1/\Gamma(w) = 0$, *for non-positive integer values of* w.

Having solved examples with multiplicative technological uncertainty and uncertainty with a singular density, let us finally solve an example with additive technological uncertainty.

Example 3 *Let us assume that the production technology is additive, i. e.*

$$x = z + \varepsilon. \qquad (73)$$

Let us also assume that ε is distributed normally with mean zero and variance $1/2$. *Then equation (29) becomes*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} w(x) \exp(-(x-z)^2) dx = v(z). \tag{74}$$

Proof of the existence is similar to Example 1 and is omitted. Let us find the solution in the case when $v(z) = z^n$, where $n \in N$.Let us look for a solution in a form

$$w(x) = cH_n(x), \tag{75}$$

where Hermite polynomials $H_n(\cdot)$ are defined by:

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n(\exp(-x^2))}{dx^n} \tag{76}$$

(see, Pugachev and Sinitsyn, 1999). Substituting (75) into (74) one obtains:

$$cI_n(z) = z^n, \tag{77}$$

where $I_n(z)$ *is defined by:*

$$I_n(z) = \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{\infty} H_n(x) \exp(-(x-z)^2) dx. \qquad (78)$$

Using definition (76) and integrating (78) by parts one can prove that $I_n(z)$ *solves:*

$$I_n(z) = 2zI_{n-1}(z),\ I_0(z) = 1. \qquad (79)$$

Therefore,

$$I_n(z) = 2^n z^n \qquad (80)$$

and $c = \frac{1}{2^n}$*.Finally, if function* $v(\cdot)$ *is analytical at zero, i. e. in some neighborhood of* $z = 0$ *it can be represented as a sum of a convergent series:*

$$v(z) = \sum_{n=0}^{\infty} a_n z^n. \qquad (81)$$

Then according to formula (80) and the superposition principle for linear equations:

$$w(x) = \sum_{n=0}^{\infty} \frac{a_n}{2^n} H_n(x). \qquad (82)$$

5 When does hidden action entails a welfare loss?

In the previous section we established that independence of technology of the private information of the agents is the basic economic assumption that makes hidden action problem irrelevant from the welfare point of view. In this section we are going to provide an economically natural example, when this condition does not hold and find the optimal contract. The example is particularly interesting, because in the situation it describes neither hidden action nor hidden information on their own lead to welfare losses, while their combination does.

Assume different research institutions compete for government grants for research projects. Each institution has a research project which is characterized by the potential success probability $q \in [0,1]$. The actual success probability is $q\theta$ where $\theta \in [0,1]$ is the effort level. The cost of a project is normalized to be zero. The cost of effort is given by an increasing, convex and twice continuously differentiable function $h(\theta)$. If successful, the project results into production of a public good. The value of it to the society is one. Both the success probability and the cost of the project are private information to the research institution. There is a continuum of the research institutions.

The government knows the population density $f(q)$ of different types of projects, which is assumed to have a compact support and be strictly positive at any point in the support. It aims to maximize the expected payoffs to the projects net of the funding costs subject to the incentive and participation constraints (we assume that the social cost of a \$1 transfer exceeds \$1 because of the deadweight loss of taxation, for simplicity we normalize the cost of raising this revenue to be \$1). Denote the value of the outside option by U_0. Then the government solves:

$$\max \int_0^1 (q\theta(1-y) - t(y))f(q,x)dq \tag{83}$$

$$(y,\theta) \in \arg\max(q\theta y - h(\theta) - t(y)) \tag{84}$$

$$\max_{(y,\theta)}(q\theta y - h(\theta) - t(y)) \geq U_0. \tag{85}$$

For this purpose, the government offers a menu of pairs (t, y) where t is an up-front payment and y is a success prize. Incentive compatibility requires that $y_1 = y_2$ implies $t_1 = t_2$. Therefore, without loss of generality we can assume that the government offers a schedule $t(y)$.

Note that the government observes the signal $z \in \{0, 1\}$, where $z = 1$ if and only if the project is successful. Since the probability that $z = 1$ is $q\theta$, and depends on q for a given value of θ, the production technology depends on the private information of the consumers.

Timing is the following. At time zero the government announces the schedule $t(y)$. An institution decides whether to participate and if so, which contract to choose. That is it picks up y^* and receives $t(y^*)$. Then it chooses research effort θ. If the research is successful an institution gets the prize y^*. The institution is assumed to be risk neutral.

We will solve the problem by backward induction. After y is chosen, the institution selects θ to solve

$$\max(q\theta y - h(\theta)). \tag{86}$$

The first order condition is

$$h'(\theta) = qy. \tag{87}$$

Define the indirect utility of an institution by

$$v(qy) = qy\theta(qy) - h(\theta(qy)) \tag{88}$$

where $\theta(qy)$ is given by (87). It is straightforward to show that

$$v'(z) = \theta(z) \geq 0,\ \ v''(z) = \frac{1}{h''(\theta(z))} > 0. \tag{89}$$

Since

$$\frac{\partial^2 v}{\partial q \partial y} = qyv''(z) \geq 0 \tag{90}$$

the single crossing property is satisfied and $y(q)$ is implementable if and only if it is increasing. For a proof, see Mussa and Rosen (1978).

Define the surplus of an institution by

$$u(q) = \max_y (v(qy) - t(y)) \tag{91}$$

Applying the envelope theorem to (91) and using the definition of $v(\cdot)$, one obtains

$$u'(q) = \frac{\theta h'(\theta)}{q}. \tag{92}$$

Now the government's objective (83) can be rewritten as:

$$\max \int_0^1 [q\theta - h(\theta) - u(q)]f(q)dq \tag{93}$$

Integrating by parts the term

$$\int_0^1 u(q)f(q)dq \tag{94}$$

one obtains the following problem

$$\max_{\theta\in[0,1]} \int_0^1 (q\theta - C_{SB}(q,\theta))f(q)dq, \tag{95}$$

where the second best cost $C_{SB}(q,\cdot)$ is defined by

$$C_{SB}(q,\theta) = h(\theta) + \frac{1-F(q)}{qf(q)}\theta h'(\theta).$$

The cost function can easily be understood intuitively. The first term represents the physical cost of effort. In the absence of adverse selection it gives the implementation cost. The second term captures an increase in the information rents earned by the types on interval $[q,1]$ due to the increase in effort.

Assume that $2h''(\theta) + \theta h'''(\theta) \geq 0$. Then $C_{SB}(q, \cdot)$ is convex for any distribution $F(\cdot)$. In this case the unique solution to (95) will be interior and increasing in q, which implies that all types will participate. If the resulting $y(q)$ is increasing, there is full separation of types, otherwise one should apply ironing procedure developed by Mussa and Rosen (1978). For $q = 1$ the second best cost coincides with $h(\theta)$ which implies no distortions at the top, in accordance with the general result in the screening literature.

For $h(\theta) = \theta^2/2$ and a uniform distribution of the success probability is distributed on $[0, 1]$ one obtains:

$$C_{SB}(q, \theta) = \frac{2-q}{2q}\theta^2.$$

Hence,

$$\begin{aligned} \theta &= \frac{q^2}{2-q} \\ y &= \frac{q}{2-q}. \end{aligned}$$

Since $y(\cdot)$ is increasing it is implementable.

It is easy to check that the first best outcome is implementable under either adverse selection or moral hazard alone. Indeed, if q is observable then an institution receives an up-front payment $U_0 - v(q)$ and a prize one if and only if it succeeds. If θ is observable then the government will offer an up-front payment $t = h(\theta)$ and no success prize.

The main feature of the above example is that the effort improves the type. Such situations are quite general, especially in the cases when a party can make an unobservable relation-specific investment. Hence, we conclude that the interaction moral hazard and adverse selection can generate a welfare loss in an economically interesting environment even if the both parties are risk-neutral.

6 Conclusions

In this paper we consider a general model with coexisting hidden action and hidden information. Starting from a model without any specific assumptions on the production technology we characterize the first order properties of the solution and prove that, with minor technical qualifications, independence of the production technology from the consumer type is necessary and sufficient for the welfare irrelevance of hidden action. The most important case when this criterion might break is when the distribution of output has interior singularities. We also solve some examples explicitly.

Our result clarifies and confirms the main conclusion drawn in the existing literature on mixed models, that if the parties are risk neutral and the production technology is not correlated with private information, then hidden action is irrelevant. However it makes it clear that even under risk neutrality this conclusion does not extend to the correlated case, which in practice occurs quite frequently. We illustrate this with an example motivated by research contracting, where the project type affects both the success probability and the marginal effect of effort. The example is particularly striking because, in the situation it describes, neither hidden action nor hidden information on their own lead to welfare losses, while their combination does.

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