A Unified Approach to the Study of Tail Probabilities of Compound Distributions*

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Abstract

We consider the tail probabilities of a general class of compound distributions. First, the relations between reliability distribution classes and heavy-tailed distributions are discussed. These relations reveal that many previous results on estimating the tail probabilities are not applicable to heavy-tailed distributions.

Then, a generalized Wald's identity and identities for compound geometric distributions are presented in terms of renewal processes. Using these identities, lower and upper bounds for the tail probabilities are derived for the class of compound distributions, both under the conditions of NBU and NWU tails, which include exponential tails, as well as for heavy tails.

Many previous results are shown to be special cases of these results, which are also improved. In addition, simplified bounds are derived by the technique of stochastic ordering. It also allows for the correction of errors in the proof of some previous results.

Key words: Compound distribution, renewal process, Wald's identity, compound geometric distribution, reliability distributions, NWU distributions,

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1 Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of nonnegative independent identically distributed (i.i.d.) random variables with common distribution function F and $F(0) = 0$. Now let N be a counting random variable, independent of $\{X_i, i \geq 1\}$ with $\Pr\{N = n\} = p_n$ for $n \geq 0$.

The distribution function of the random sum $S = X_1 + \cdots + X_N$, (with $S = 0$ if $N = 0$), is called a compound distribution and is given by

$$\Pr\{S \leq x\} = \sum_{n=0}^{\infty} p_n \, F^{(n)}(x) \, , \tag{1.1}$$

where $F^{(n)}$ is the distribution function of the n-fold convolution of F with itself, $F^{(0)}(x) = 1$ if $x \geq 0$, and 0 otherwise. The tail probability of the compound distribution is defined by

$$\psi(x) = \Pr\{S > x\} = \sum_{n=1}^{\infty} p_n \, \overline{F}^{(n)}(x) \, , \tag{1.2}$$

where $\overline{F}^{(n)} = 1 - F^{(n)}$ is the survival function of $F^{(n)}$.

The tail of the compound distribution arises in many applied probability models such as insurance risk, queueing and reliability theory. However, explicit and closed expressions for the tail are not available, except for a few special cases. Hence, there has been being great interest in estimating the tail probability.

Recently, in a series of works Lin (1996), Willmot (1994, 1997^a) and Willmot and Lin (1994, 1997), have considered more general compound distributions, for which there exists a constant $0 < \phi < 1$ such that the probability distribution $\{p_n, n \geq 0\}$ of N satisfies

$$a_{n+1} \leq \phi \, a_n, \quad n = 0, 1, 2, \cdots , \tag{1.3}$$

or

$$a_{n+1} \geq \phi\, a_n, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$a_n = \Pr\{N > n\} = \sum_{k=n+1}^{\infty} p_k \, .$$

Such a class of compound distributions includes many interesting models [see, for example, Panjer and Willmot (1992) and Willmot and Lin (1994)]. The main results and methods for estimating the tail probabilities of this class of compound distributions are summarized below.

Suppose that B is a life distribution function [a life distribution function is a distribution function of a nonnegative random variable, *i.e.* with support $[0, \infty)$] and satisfies the following equation

$$\int_0^{\infty} \left[\overline{B}(y)\right]^{-1} dF(y) = \frac{1}{\phi} \, . \tag{1.5}$$

By the induction method, Willmot (1994) shows that if (1.3) holds and (1.5) is satisfied by an NWU distribution function B [a life distribution function B is said to be New Worse than Used (NWU) if for any $x, y \geq 0$, $\overline{B}(x+y) \geq \overline{B}(x)\,\overline{B}(y)$ and be New Better than Used (NBU) if the reversed inequality holds], then

$$\psi(x) \, \leq \, \frac{1-p_0}{\phi}\, \alpha_2(x)\, \overline{B}(x), \quad x \geq 0 \, , \tag{1.6}$$

where α_2 is given below.

By the renewal recursive method, Willmot (1997^a) proved that if (1.4) holds and (1.5) is satisfied by an NBU distribution function B, then

$$\psi(x) \, \geq \, \frac{1-p_0}{\phi}\, \alpha_1(x)\, \overline{B}(x), \quad x \geq 0 \, , \tag{1.7}$$

where

$$[\alpha_1(x)]^{-1} = \sup_{0 \leq h \leq x,\ \overline{F}(h) > 0} \alpha(h) \ , \quad [\alpha_2(x)]^{-1} = \inf_{0 \leq h \leq x,\ \overline{F}(h) > 0} \alpha(h) \ ,$$

and

$$\alpha(h) = \frac{\int_h^{\infty} [\overline{B}(y)]^{-1} \, dF(y)}{[\overline{B}(h)]^{-1}\, \overline{F}(h)} \, .$$

Using a generalized Wald's identity, Lin (1996) shows that if (1.3) holds and (1.5) is satisfied by an NWU distribution function B, then

$$\psi(x) \le \frac{1-p_0}{\phi}\,\Delta_2(x), \quad x \ge 0 , \tag{1.8}$$

while if (1.4) holds and (1.5) is satisfied by an NBU distribution function B, then

$$\psi(x) \ge \frac{1-p_0}{\phi}\,\Delta_1(x), \quad x \ge 0 , \tag{1.9}$$

where

$$[\Delta_1(x)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)>0} \Delta(h) , \quad [\Delta_2(x)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)>0} \Delta(h) ,$$

and

$$\Delta(h) = \frac{\int_h^\infty [\,\overline{B}(x-h+y)]^{-1}\,dF(y)}{\overline{F}(h)} .$$

Willmot (1997^a) shows that if B has a Decreasing Failure Rate (DFR, a subclass of NWU), then

$$\alpha_2(x)\,\overline{B}(x) \le \Delta_2(x), \quad x \ge 0 ,$$

and if B has an Increasing Failure Rate (IFR, a subclass of NBU), then

$$\alpha_1(x)\,\overline{B}(x) \ge \Delta_1(x), \quad x \ge 0 .$$

That is to say that the upper bound in (1.6) is tighter than that in (1.8) if B is DFR and the lower bound in (1.7) is tighter than that in (1.9) if B is IFR.

As shown by Cai and Wu (1997), conditions (1.3) and (1.4) imply respectively that

$$\psi(x) \le \frac{1-p_0}{\phi}\,\psi^*(x), \quad x \ge 0 , \tag{1.10}$$

and

$$\psi(x) \ge \frac{1-p_0}{\phi}\,\psi^*(x), \quad x \ge 0 , \tag{1.11}$$

where

$$\psi^*(x) = \sum_{n=1}^{\infty} (1-\phi)\, \phi^n\, \overline{F}^{(n)}(x) \;, \tag{1.12}$$

is the tail of a compound geometric distribution.

Thus, bounds for the tail of the class of compound distributions can be deduced from the bounds for the tail $\psi^*(x)$ of the compound geometric distribution. With this idea, we derive here new lower and upper bounds for $\psi(x)$, which are uniformly sharper than the bounds in (1.6), (1.7), (1.8) and (1.9).

In addition, based on the bounds in (1.6), (1.7), (1.8) and (1.9), many simplified bounds have been derived in the references mentioned above by imposing additional assumptions on B and F. However, as pointed out by Schmidli (see MR 97k: 60265), some proofs of these simplified bounds [for example, those of Willmot and Lin (1997)] are wrong in general cases, due to an improper use of integrations by parts. But, "by making the same mistake twice, the results turn out to be correct (Schmidli, MR 97k: 60265)".

In Section 4 we use the technique of stochastic ordering to derive simplified bounds. The method is simple and unifying. The error is corrected and these simplified bounds are tighter than those in previous results.

On the other hand, the compound geometric distributions is of independent interest. Many useful results on the compound geometric distribution have been obtained in different disciplines. One celebrated such result for the tail of the geometric compound distribution is Lundberg's inequality, which states that if there exists a constant κ satisfying the following condition, called Cramér-Lundberg's condition,

$$\int_0^\infty e^{\kappa y}\, dF(y) = \frac{1}{\phi} \;, \tag{1.13}$$

then

$$\psi^*(x) \;\le\; e^{-\kappa x}, \quad x \ge 0 \;. \tag{1.14}$$

Heavy-tailed distributions, such as the Pareto or lognormal, do not satisfy Cramér -Lundberg's condition (1.13) [a distribution is said to be heavy-tailed if its moment generating function does not exist in a neighborhood of the origin]. Hence, a generalized Cramér-Lundberg's condition has been proposed by Willmot (1994), in which the exponential distribution function in (1.13) is replaced by an NWU or NBU distribution function B, *i.e.* condition (1.5). Indeed, with the generalized Cramér-Lundberg's condition, Lundberg's inequality has been improved and generalized. Various inequalities can be obtained by choosing B as a special NWU or NBU distribution function, see, for example, the references mentioned above.

Therefore, as we discuss in Section 2, many results based on the generalized Cramér-Lundberg's condition are still not applicable to heavy-tailed distributions. This motivates us to consider more general conditions that can be satisfied by more general distributions, especially by heavy-tailed distributions. For this purpose, two truncated versions of Cramér-Lundberg's condition are proposed.

The first version is obtained by replacing the exponential function $e^{\kappa y}$ in (1.13) by a truncated function $\min(e^{\rho_t y},\ e^{\rho_t t})$ that satisfies the following equation

$$\int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\, \overline{F}(t) = \frac{1}{\phi}\ , \tag{1.15}$$

for a given $t > 0$ and ρ_t.

The second type is obtained by replacing the exponential function $e^{\kappa y}$ in (1.13) by a truncated function $e^{\kappa_t y}\, I_{(\infty,t)}(y)$ that satisfies the following equation

$$\int_0^t e^{\kappa_t y}\, dF(y) = \frac{1}{\phi}\ , \tag{1.16}$$

for a given $t > 0$ and κ_t.

Conditions (1.15) and (1.16) can be satisfied by any life distribution F with positive (possibly infinite) mean, by choosing a sufficiently large value of t. Hence these are also generalizations of Cramér-Lundberg's condition (1.13) in the sense that if (1.13) holds, then $\rho_t \to \kappa$ and $\kappa_t \to \kappa$ as $t \to \infty$, and (1.15) and (1.16) result in (1.13) by taking $t \to \infty$. In addition, conditions (1.9) of Broeckx *et al.* (1986), (2.1) of Dickson (1994) and (15) of Taylor (1976), in which F is an integrated tail distribution, are special cases of (1.15) and (1.16). Under the two truncated conditions, lower and upper bounds for $\psi^*(x)$ are derived. The upper bounds improve upon (2.4) of Broeckx *et al.* (1986), (2.2) of Dickson (1994) and (16) of Taylor (1976), which are obtained as special cases. The corresponding lower bounds are also given. The proofs are obtained under the unifying approach of the generalized Wald's identity and the identities of the compound geometric distribution in terms of renewal processes. These are unified and simpler than the method of error functions of Broeckx *et al.* (1986), the inductive method of Dickson (1994) and the use of integro-differential inequalities of Taylor (1976). Their application to the tail of the class of compound distributions is given. These bounds are especially interesting for heavy-tailed distributions.

The study is organized as follows. In Section 2, some relations between NBU, NBUE and heavy-tailed distributions are considered [a life distribution function B is said to be New Better than Used in Expectation (NBUE) if for any $y \geq 0$, $\int_y^\infty \overline{B}(x)dx \leq \mu_B \overline{B}(y)$, where $\mu_B = \int_0^\infty \overline{B}(x)dx$, and New Worse than Used in Expectation (NWUE) if the reversed inequality holds]. These relations show that many results based on condition (1.5) and reliability distribution classes are not applicable to heavy-tailed distributions.

In Section 3, some useful identities of compound geometric distributions and the generalized Wald's identity are presented in terms of renewal processes. The truncated conditions (1.15) and (1.16) are expressed as the expectations of the transformations of the distribution F.

In Section 4, lower and upper bounds for $\psi^*(x)$ are derived under conditions (1.5), (1.15) and (1.16), in a unifying way, by the identities given in section 3. As applications, improved lower and upper bounds for $\psi(x)$ are derived.

Section 5 gives some simplied bounds, based on the results in section 4, that are derived in a unifying way, by stochastic ordering. This refines the bounds of Willmot and Lin (1997) and avoids inappropriate uses of integration by parts.

2 Some relations between reliability distribution classes and heavy tailed distributions

Property 2.1 If F is a NBUE distribution function [see Section 1] with finite mean μ_F, then it can not be a heavy-tailed distribution function.

Proof. Since NBUE $\rightarrow$ HNBUE [a life distribution function B is said to be Harmonic New Better than Used in Expectation (HNBUE) if for any $y \geq 0$, $\int_y^\infty \overline{B}(x)dx \leq \mu_B e^{-y/\mu_B}$], thus by Theorem 3 of Klefsjö (1982), we know that for $x \geq \mu_F$,

$$\overline{F}(x) \;\leq\; \exp\left\{\frac{\mu_F - x}{\mu_F}\right\}\,. \tag{2.1}$$

This implies that F has an exponential tail, *i.e.* there exists some $\alpha > 0$ such that $\int_0^\infty e^{\alpha y}\, dF(y) < \infty$. Hence, F is not heavy-tailed. □

Since the following relations hold between reliability distribution classes [for their definitions and relations, see Barlow and Proschan (1981) and Cao and Wang (1991)]:

$$\begin{array}{ccccccccc}
 & & \text{IFRA} & \longrightarrow & \text{NBU} & & & & \\
 & \nearrow & & & & \searrow & & & \\
\text{IFR} & & & & & & \text{NBUC} & \longrightarrow & \text{NBUE} \\
 & \searrow & & & & \nearrow & & & \\
 & & & \text{DMRL} & & & & &
\end{array}$$

This, together with Property 2.1, implies that all bounds based on the condition that F is a member of IFR, IFRA, DMRL, NBU, NBUC and NBUE distribution classes, are not applicable to heavy-tailed distributions. For example (2.8), (2.13) and (2.14) of Lin (1996) and (2.7) and (3.8) of Willmot and Lin (1997).

Property 2.2 If B is a NBU distribution function, then condition (1.5) can not be satisfied by heavy-tailed distributions F.

Proof. Since condition (1.5) implies that $\left[\overline{B}(x)\right]^{-1}\overline{F}(x) \to 0$ as $x \to \infty$, there must exist some $M > 0$ such that $\overline{F}(x) \leq \overline{B}(x)$ for $x \geq M$. Thus, the relation NBU $\to$ HNBUE and (2.1) imply for $x \geq \max(M,\ \mu_B)$ that

$$\overline{F}(x) \;\leq\; \overline{B}(x) \;\leq\; \exp\left\{\frac{\mu_B - x}{\mu_B}\right\}\,. \tag{2.2}$$

This implies that F has an exponential tail and can not be heavy-tailed. □

Remark 2.1 Property 2.2 implies that all the lower bounds of Lin (1996), Willmot (1994, 1997) and Willmot and Lin (1997) do not apply to heavy-tailed distributions, since all are based on condition (1.5) and B being a NBU distribution function.

3 Some useful identities in terms of renewal processes

For $n \geq 0$, define $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ and $N(x) = \sup\{n \geq 0 : S_n \leq x\}$, then $\{N(x), x \geq 0\}$ is a renewal process associated with the underlying random sequence $\{X_i, i \geq 1\}$. For $x \geq 0$

$$\Pr\{N(x) \geq n\} = \Pr\{S_n \leq x\} = F^{(n)}(x) \ ,$$

and

$$0 \ \leq \ S_{N(x)} \ \leq \ x \ \leq \ S_{N(x)+1} \ .$$

The tail probability $\psi^*(x)$ of the compound geometric distribution has a connection to the renewal process $\{N(x), x \geq 0\}$, namely, for $x \geq 0$,

$$\psi^*(x) = \sum_{n=1}^{\infty} (1-\phi)\, \phi^n\, \overline{F}^{(n)}(x) = E\left[\phi^{N(x)+1}\right] . \tag{3.1}$$

This is a simple and useful representation of the tail of the compound geometric distribution in terms of the renewal process $\{N(x), x \geq 0\}$. Its proof follows simply from summation by parts [for example, see Kalashnikov (1994, 1996)].

Since we can always choose a $t > 0$ such that it is a continuous point of F and $F(t) > 0$, without the loss of generality we assume in what follows that these two conditions hold for this given $t > 0$.

Define F_t to be the conditional distribution function of X, given $X \leq t$. That is

$$F_t(x) = \begin{cases} F(x)/F(t), & 0 \leq x < t \\ 1, & x \geq t \end{cases} \tag{3.2}$$

There exists a lifetime random variable, say X_t, whose distribution is F_t and survival function

$$\overline{F}_t(x) = 1 - F_t(x) = \begin{cases} \frac{\overline{F}(x) - \overline{F}(t)}{F(t)}, & 0 \leq x < t \\ 0, & x \geq t \end{cases}$$

Thus, condition (1.16) is expressed equivalently as

$$\frac{1}{\phi F(t)} = \int_0^t e^{\kappa_t\, y}\, \frac{dF(y)}{F(t)} = \int_0^t e^{\kappa_t\, y}\, dF_t(y) = \int_0^\infty e^{\kappa_t\, y}\, dF_t(y) = E\left[e^{\kappa_t\, X_t}\right] \tag{3.3}$$

Define $X_t^* = \min\{X,\ t\}$, then the distribution function G_t of X_t^* is

$$G_t(x) = \begin{cases} F(x), & 0 \le x < t \\ 1, & x \ge t \end{cases} \tag{3.4}$$

and the survival function

$$\overline{G}_t(x) = 1 - G_t(x) = \begin{cases} \overline{F}(x), & 0 \le x < t \\ 0, & x \ge t \end{cases}$$

Thus, condition (1.15) is expressed equivalently as

$$\frac{1}{\phi} = \int_0^t e^{\rho_t\, y}\, dF(y) + e^{\rho_t\, t}\, \overline{F}(t) = \int_0^\infty e^{\rho_t\, y}\, dG_t(y) = E\left[e^{\rho_t\, X_t^*}\right]. \tag{3.5}$$

Suppose that $\{X_i^t, i \ge 1\}$ (respectively, $\{X_i^{*t}, i \ge 1\}$) is a sequence of i.i.d. nonnegative random variables with common distribution function F_t (G_t), $N_t(x) = \sup\{n \ge 0 :\ X_1^t + \cdots + X_n^t \le x\}$ (resp. $N_t^*(x) = \sup\{n \ge 0 :\ X_1^{*t} + \cdots + X_n^{*t} \le x\}$) is the renewal process associated with the underlying random sequence $\{X_i^t, i \ge 1\}$ (resp. $\{X_i^{*t}, i \ge 1\}$), thus we have representations of $\psi^*(x)$ in terms of the renewal process $\{N_t(x), x \ge 0\}$ and $\{N_t^*(x), x \ge 0\}$, that will be used in next section.

Property 3.1 For $0 \le x \le t$

$$\psi^*(x) = \frac{(1-\phi)E\left\{[\phi F(t)]^{N_t(x)+1}\right\} + \phi\, \overline{F}(t)}{1 - \phi + \phi\, \overline{F}(t)}\ , \tag{3.6}$$

and

$$\psi^*(x) = E\left[\phi^{N_t^*(x)+1}\right]\ . \tag{3.7}$$

Proof. Since, for $0 \le x \le t$, $F_t(x) = F(x)/F(t)$, this implies that for $0 \le x \le t$

$$\begin{aligned} F_t^{(2)}(x) &= \int_0^x F_t(x-y)\, dF_t(y) = \int_0^x F(x-y)\, dF(y)\ /\ [F(t)]^2\ , \\ &= F^{(2)}(x)\ /\ [F(t)]^2\ . \end{aligned}$$

By induction, we get that for $0 \le x \le t$ and $n \ge 1$

$$F_t^{(n)}(x) = F^{(n)}(x) \,/\, [F(t)]^n \ ,$$

hence, for $0 \le x \le t$

$$\overline{F}_t^{(n)}(x) = 1 - F_t^{(n)}(x) = 1 - \frac{F^{(n)}(x)}{[F(t)]^n} \ ,$$

Replacing ϕ, $F^{(n)}(x)$ and $N(x)$ in (3.1) by $\phi\, F(t)$, $F_t^{(n)}(x)$ and $N_t(x)$, respectively, we get for $0 \le x \le t$

$$\begin{aligned}
E\left\{[\phi F(t)]^{N_t(x)+1}\right\} &= \sum_{n=1}^{\infty} [1-\phi F(t)]\,[\phi F(t)]^n\, \overline{F}_t^{(n)}(x) \ ,\\
&= \sum_{n=1}^{\infty} [1-\phi F(t)]\,[\phi F(t)]^n \left\{1 - \frac{F^{(n)}(x)}{[F(t)]^n}\right\} \ ,\\
&= \phi F(t) - \sum_{n=1}^{\infty} [1-\phi F(t)]\,\phi^n\, F^{(n)}(x) \ ,\\
&= \phi F(t) - \sum_{n=1}^{\infty} [1-\phi F(t)]\,\phi^n \left[1 - \overline{F}^{(n)}(x)\right] \ ,\\
&= \phi F(t) - \frac{\phi\,[1-\phi F(t)]}{1-\phi} + [1-\phi F(t)] \sum_{n=1}^{\infty} \phi^n\, \overline{F}^{(n)}(x) \ ,\\
&= \frac{-\phi\, \overline{F}(t)}{1-\phi} + \frac{[1-\phi F(t)]}{1-\phi}\, \psi^*(x) \ .
\end{aligned}$$

this implies (3.6).

Similarly, since for $0 \le x \le t$, $G_t(x) = F(x)$, this implies that for $0 \le x \le t$, $G_t^{(n)}(x) = F^{(n)}(x)$ and $\overline{G}_t^{(n)}(x) = \overline{F}^{(n)}(x)$. Thus, by (3.1), we get for $0 \le x \le t$,

$$\begin{aligned}
E\left[\phi^{N_t^*(x)+1}\right] &= \sum_{n=1}^{\infty} (1-\phi)\,\phi^n\, \overline{G}_t^{(n)}(x) \ ,\\
&= \sum_{n=1}^{\infty} (1-\phi)\,\phi^n\, \overline{F}^{(n)}(x) = \psi^*(x) \ ,
\end{aligned}$$

i.e. (3.7) holds. □

Suppose that X has the same distribution function F as $\{X_i, i \ge 1\}$. Define $T(x) = \inf\{n : S_n > x\}$, then $T(x) = N(x) + 1$, and (1.14) of Lin (1996), a generalized Wald's identity, can be also expressed in terms of $N(x)$ as the following property.

Property 3.2 If a nonnegative function g on $[0, \infty)$ satisfies

$$E[g(X)] = \int_0^\infty g(x)\, dF(x) = \frac{1}{\phi} \ , \tag{3.8}$$

then, for any $x \geq 0$

$$E\left[\phi^{N(x)+1} \prod_{i=1}^{N(x)+1} g(X_i)\right] = 1 \ . \tag{3.9}$$

4 Lower and upper bounds derived from the above identities

Theorem 4.1 If there exists an NBU distribution function B such that

$$\int_0^\infty [\,\overline{B}(y)]^{-1}\, dF(y) = \frac{1}{\phi} \ , \tag{4.1}$$

then for any $x \geq 0$

$$E\left[\phi^{N(x)+1}\right] \ \geq \ K_1(x) \ . \tag{4.2}$$

If there exists an NWU distribution function B such that (4.1) holds, then for any $x \geq 0$,

$$E\left[\phi^{N(x)+1}\right] \ \leq \ K_2(x) \ , \tag{4.3}$$

where

$$[K_1(x)]^{-1} = \sup_{0 \leq h \leq x,\ \overline{F}(h) > 0} K(h), \qquad [K_2(x)]^{-1} = \inf_{0 \leq h \leq x,\ \overline{F}(h) > 0} K(h) \ ,$$

and

$$K(h) = \frac{[\,\overline{B}(x-h)]^{-1} \int_h^\infty [\,\overline{B}(y)]^{-1}\, dF(y)}{\overline{F}(h)} \ . \tag{4.4}$$

Proof. By (4.1), Property 3.2 and the definition of NBU, we get that

$$\begin{aligned}
1 &= E\left\{\phi^{N(x)+1}\prod_{i=1}^{N(x)+1}[\overline{B}(X_i)]^{-1}\right\} , \\
&= E\left\{\phi^{N(x)+1}\,[\overline{B}(X_{N(x)+1})]^{-1}\prod_{i=1}^{N(x)}[\overline{B}(X_i)]^{-1}\right\} , \\
&\le E\left\{\phi^{N(x)+1}\,[\overline{B}(X_{N(x)+1})]^{-1}\,[\overline{B}(X_1+\cdots+X_{N(x)})]^{-1}\right\} , \qquad (4.5)\\
&= E\left\{\phi^{N(x)+1}\,[\overline{B}(X_{N(x)+1})]^{-1}\,[\overline{B}(S_{N(x)})]^{-1}\right\} , \\
&= E\left\{\phi^{N(x)+1}\,E\left\{[\overline{B}(X_{N(x)+1})]^{-1}\mid N(x),\,S_{N(x)}\right\}\,[\overline{B}(S_{N(x)})]^{-1}\right\} \qquad (4.6)
\end{aligned}$$

By the renewal property, we know that

$$\begin{aligned}
E\left\{[\overline{B}(X_{N(x)+1})]^{-1}\mid N(x),\ S_{N(x)}\right\} &= E\left\{[\overline{B}(X_1)]^{-1}\mid X_1 > x - S_{N(x)}\right\} , \\
&= \frac{\int_{x-S_{N(x)}}^{\infty}[\overline{B}(y)]^{-1}\,dF(y)}{\overline{F}(x-S_{N(x)})} . \qquad (4.7)
\end{aligned}$$

Now, since $0 \le x - S_{N(x)} \le x$, we get that

$$\begin{aligned}
[\overline{B}(S_{N(x)})]^{-1} &\times E\left\{[\overline{B}(X_{N(x)+1})]^{-1}\mid N(x),\ S_{N(x)}\right\} \\
&= \frac{\{\overline{B}[x-(x-S_{N(x)})]\}^{-1}\int_{x-S_{N(x)}}^{\infty}[\overline{B}(y)]^{-1}\,dF(y)}{\overline{F}(x-S_{N(x)})} , \\
&= K(x-S_{N(x)}) \ \le\ [K_1(x)]^{-1} . \qquad (4.8)
\end{aligned}$$

Substituting in (4.6), implies that

$$1 \ \le\ E\left[\phi^{N(x)+1}\right][K_1(x)]^{-1} ,$$

i.e. (4.2) holds.

Similarly, we can get (4.3) in the case of NWU by reversing inequalities (4.5) and (4.8) and replacing $K_1(x)$ by $K_2(x)$. □

Combining Theorem 4.1, (3.1), (1.10) and (1.11), we get directly the following corollary.

Corollary 4.1 If (1.4) holds and (4.1) is satisfied by an NBU distribution function B, then

$$\psi(x) \ \ge\ \frac{1-p_0}{\phi}\,K_1(x) , \qquad (4.9)$$

and if (1.3) holds and (4.1) is satisfied by an NWU distribution function B, then

$$\psi(x) \;\le\; \frac{1-p_0}{\phi}\, K_2(x)\;. \tag{4.10}$$

Remark 4.1 The lower and upper bounds in Corollary 4.1 are uniformly tighter than the lower bounds in (1.7) and (1.9), and the upper bounds in (1.6) and (1.8) respectively, which are the main results of Lin (1996) and Willmot (1994, 1997[a]). This follows easily from the fact that if B is NBU, then for any $x \ge 0$

$$K_1(x) \ge \alpha_1(x)\,\overline{B}(x) \quad \text{and} \quad K_1(x) \ge \Delta_1(x)\;, \tag{4.11}$$

and if B is NWU, then for any $x \ge 0$

$$K_2(x) \le \alpha_2(x)\,\overline{B}(x) \quad \text{and} \quad K_2(x) \le \Delta_2(x)\;. \tag{4.12}$$

Theorem 4.2 Given $t > 0$, if there exists a constant κ_t such that

$$\int_0^t e^{\kappa_t\, y}\, dF(y) = \frac{1}{\phi}\;, \tag{4.13}$$

then for any $0 \le x \le t$

$$\frac{(1-\phi)c_1(x,t)e^{-\kappa_t\, x} + \phi\overline{F}(t)}{(1-\phi)\;+\;\phi\,\overline{F}(t)} \;\le\; \psi^*(x) \;\le\; \frac{(1-\phi)c_2(x,t)e^{-\kappa_t\, x} + \phi\overline{F}(t)}{(1-\phi)\;+\;\phi\,\overline{F}(t)} \tag{4.14}$$

In particular, for any $t > 0$

$$\frac{(1-\phi)c_1(t)e^{-\kappa_t\, t}\;+\;\phi\,\overline{F}(t)}{(1-\phi)\;+\;\phi\,\overline{F}(t)} \;\le\; \psi^*(t) \;\le\; \frac{(1-\phi)c_2(t)e^{-\kappa_t\, t}\;+\;\phi\,\overline{F}(t)}{(1-\phi)\;+\;\phi\,\overline{F}(t)}\;, \tag{4.15}$$

where

$$[c_1(x,t)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)\ne\overline{F}(t)} c(h,t), \qquad [c_2(x,t)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)\ne\overline{F}(t)} c(h,t),$$

$$c(h,t) = \frac{\int_h^t e^{\kappa_t\, y}\, dF(y)}{e^{\kappa_t\, h}\,[\,\overline{F}(h) - \overline{F}(t)\,]},$$

and $c_1(t) = c_1(t,t),\;\; c_2(t) = c_2(t,t)$.

Proof. By (3.3), we know that (4.13) is equivalent to

$$E[e^{\kappa_t X_t}] = \int_0^\infty e^{\kappa_t y}\, dF_t(y) = \frac{1}{\phi F(t)} \;.$$

Thus, taking $\overline{B}(x) = e^{-\kappa_t x}$, by (4.2) in Theorem 4.1, we get that for $0 \le x \le t$

$$\begin{aligned} E\left\{[\phi F(t)]^{N_t(x)+1}\right\} &\ge \left\{ \sup_{0\le h\le x,\ \overline{F}_t(h)>0} \frac{e^{\kappa_t x} \int_h^\infty e^{\kappa_t y}\, dF_t(y)}{e^{\kappa_t h}\,\overline{F}_t(h)} \right\}^{-1}, \quad (4.16)\\ &= e^{-\kappa_t x} \left\{ \sup_{0\le h\le x,\ \overline{F}(h)\ne\overline{F}(t)} \frac{\int_h^t e^{\kappa_t y}\, dF(y)}{e^{\kappa_t h}\,[\overline{F}(h) - \overline{F}(t)]} \right\}^{-1},\\ &= c_1(x,\, t)\, e^{-\kappa_t x} \;. \quad (4.17)\end{aligned}$$

Thus, by (3.6) and (4.17), we get the lower bound in (4.14).

The proof of the upper bound in (4.14) is similar. Setting $x = t$ in (4.14) gives (4.15). □

Theorem 4.3 Given $t > 0$, if there exists a constant ρ_t such that

$$\int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\,\overline{F}(t) = \frac{1}{\phi}\;, \qquad (4.18)$$

then for any $0 \le x \le t$

$$\beta_1(x,t)\, e^{-\rho_t x} \;\le\; \psi^*(x) \;\le\; \beta_2(x,t)\, e^{-\rho_t x}\;. \qquad (4.19)$$

In particular, for any $t > 0$

$$\beta_1(t)\, e^{-\rho_t t} \;\le\; \psi^*(t) \;\le\; \beta_2(t)\, e^{-\rho_t t}\;, \qquad (4.20)$$

where

$$[\beta_1(x,t)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)>0} \beta(h,t), \qquad [\beta_2(x,t)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)>0} \beta(h,t),$$

$$\beta(h,t) = \frac{\int_h^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\,\overline{F}(t)}{e^{\rho_t h}\,\overline{F}(h)},$$

and $\beta_1(t) = \beta_1(t,t),\;\; \beta_2(t) = \beta_2(t,t)$.

Proof. By (3.5), we know that (4.18) is equivalent to

$$\int_0^\infty e^{\rho_t\, y}\, dG_t(y) = E[e^{\rho_t\, X_t^*}] = \frac{1}{\phi}\,.$$

Thus, taking $\overline{B}(x) = e^{-\rho_t\, x}$, by (4.2) in Theorem 4.1, we get that for $0 \le x \le t$

$$E\left[\phi^{N_t^*(x)+1}\right] \;\ge\; \left\{ \sup_{0\le h\le x,\ \overline{G}_t(h)>0} \frac{e^{\rho_t\, x}\int_h^\infty e^{\rho_t\, y}\, dG_t(y)}{e^{\rho_t\, h}\,\overline{G}_t(h)} \right\}^{-1}, \tag{4.21}$$

$$= \; e^{-\rho_t\, x} \left\{ \sup_{0\le h\le x,\ \overline{F}(h)>0} \frac{\int_h^t e^{\rho_t\, y}\, dF(y) + e^{\rho_t\, t}\,\overline{F}(t)}{e^{\rho_t\, h}\,\overline{F}(h)} \right\}^{-1}, \tag{4.22}$$

$$= \; \beta_1(x,t)\, e^{-\rho_t\, x}\,. \tag{4.23}$$

Thus, the lower bound in (4.19) follows from (3.7).

The proof of the upper bound in (4.19) is similar. Setting $x = t$ in (4.19) gives (4.20). □

Combining Theorems 4.2, 4.3 with expressions (1.10) and (1.11) directly gives the following two corollaries.

Corollary 4.2 Under the conditions of Theorem 4.2, if (1.4) holds then for any $0 \le x \le t$

$$\psi(x) \;\ge\; \frac{(1-p_0)\left[(1-\phi)\, c_1(x,t)\, e^{-\kappa_t\, x} \;+\; \phi\,\overline{F}(t)\right]}{\phi\left[(1-\phi) \;+\; \phi\,\overline{F}(t)\right]}\,, \tag{4.24}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\ge\; \frac{(1-p_0)\left[(1-\phi)\, c_1(t)\, e^{-\kappa_t\, t} \;+\; \phi\,\overline{F}(t)\right]}{\phi\left[(1-\phi) \;+\; \phi\,\overline{F}(t)\right]}\,. \tag{4.25}$$

Alternatively, if (1.3) holds, then for any $0 \le x \le t$

$$\psi(x) \;\le\; \frac{(1-p_0)\left[(1-\phi)\, c_2(x,t)\, e^{-\kappa_t\, x} \;+\; \phi\,\overline{F}(t)\right]}{\phi\left[(1-\phi) \;+\; \phi\,\overline{F}(t)\right]}\,, \tag{4.26}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\le\; \frac{(1-p_0)\left[(1-\phi)\, c_2(t)\, e^{-\kappa_t\, t} \;+\; \phi\,\overline{F}(t)\right]}{\phi\left[(1-\phi) \;+\; \phi\,\overline{F}(t)\right]}\,. \tag{4.27}$$

Corollary 4.3 Under the conditions of Theorem 4.3, if (1.4) holds then for any $0 \le x \le t$

$$\psi(x) \;\ge\; \frac{1-p_0}{\phi}\,\beta_1(x,t)\,e^{-\rho_t x}\,, \tag{4.28}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\ge\; \frac{1-p_0}{\phi}\,\beta_1(t)\,e^{-\rho_t t}\,. \tag{4.29}$$

Alternatively, if (1.3) holds, then for any $0 \le x \le t$

$$\psi(x) \;\le\; \frac{1-p_0}{\phi}\,\beta_2(x,t)\,e^{-\rho_t x}\,, \tag{4.30}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\le\; \frac{1-p_0}{\phi}\,\beta_2(t)\,e^{-\rho_t t}\,. \tag{4.31}$$

Remark 4.2

1. It is easily shown that $0 \le c_1(x,t) \le c_2(x,t) \le 1$, $0 \le \beta_1(x,t) \le \beta_2(x,t) \le 1$ and that if Cramér-Lundberg's condition (1.13) holds, then $\kappa_t \searrow \kappa$ while $\rho_t \searrow \kappa$ as $t \to \infty$. Thus, Lundberg's inequality (1.14) can be obtained and improved by letting $t \to \infty$ in the upper bounds in (4.14) and (4.19).

2. The upper bound in (4.19) is uniformly sharper on $[0, t]$ than that in Lundberg's inequality (1.14), since $0 \le \beta_2(x,t) \le 1$, and (4.18) with (1.13) mean that $\int_0^\infty e^{\kappa y}\,dF(y) = \int_0^\infty e^{\rho_t \min\{y,t\}}\,dF(y) = 1/\phi$. This implies that $\rho_t \ge \kappa$.

3. The upper bounds of (1.9) of Broeckx *et al.* (1986), (2.1) of Dickson (1994) and (16) of Taylor are all special cases of the upper bounds in (4.14) and (4.19), and improved upon by the new upper bounds. In addition, corresponding lower bounds are given in Theorems 4.2 and 4.3. Cai and Garrido (1997) gives a special case of Theorem 4.2, for the ruin probability in the compound Poisson risk model, derived by the renewal recursive method.

4. Putting (4.18) and (4.13) together

$$\int_0^t e^{\kappa_t y}\, dF(y) = \int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\overline{F}(t) = 1/\phi \,,$$

implies that $\kappa_t \geq \rho_t$. This suggests that the upper bounds in (4.14) and (4.15) may be tighter than those in (4.19) and (4.20). A numerical example of Dickson (1994) shows that in that special case, the upper bounds in (4.14) and (4.15) are superior to those in (4.19) and (4.20) for large values of x, but inferior for small values of x. From the same example, we can see the lower bounds in (4.14) and (4.15) are also superior to those in (4.19) and (4.20) for large values of x, but inferior for small values of x.

5. The bounds in Theorems 4.2 and 4.3, Corollaries 4.2 and 4.3 can apply to any life distribution F with positive (possibly infinite) mean. Especially, as shown by numerical examples in Cai and Garrido (1997), the bounds in Theorem 4.2 and Corollary 4.2 are very effective for heavy-tailed distributions. Even when Cramér-Lundberg's condition holds, in some cases, they are also superior to Lundberg's inequality.

5 Simplified bounds derived from stochastic orderings

In this section, we derive simplified bounds for $\psi(x)$ by studying the function K in (4.4) and the use of stochastic ordering. This approach helps unify the theory, is simpler than integrations by parts, used by Lin (1996) and Willmot and Lin (1997) as it does not require continuity conditions, and yields new, sharper bounds for $\psi(x)$.

In this section, we denote by F_h the residual life distribution function of F, *i.e.*

$$F_h(x) = \Pr\{X \leq x + h | X > h\} = \frac{F(x+h) - F(h)}{\overline{F}(h)} \,,$$

and δ_d the degenerate distribution function of the probability measure concentrated at d, *i.e.*

$$\delta_d(x) = \begin{cases} 0, & x < d \\ 1, & x \geq d \end{cases}$$

Thus, given $h \geq 0$, note that $F(0) = 0$ and we know that for any $y \in (-\infty, \infty)$,

$$\begin{aligned} F(y+h) &= \overline{F}(h) \frac{F(y+h) - F(h)}{\overline{F}(h)} + F(h)\, \delta_{-h}(y)\ , \\ &= \overline{F}(h)\, F_h(y) + F(h)\, \delta_{-h}(y)\ . \end{aligned} \tag{5.1}$$

Hence, by (4.4), (5.1) and $\int_0^\infty [\,\overline{B}(y+h)]^{-1}\, F(h)\, d\delta_{-h}(y) = 0$ for $h \geq 0$, we get that for any $0 \leq h \leq x$,

$$\begin{aligned} K(h) &= \frac{[\,\overline{B}(x-h)]^{-1} \int_h^\infty [\,\overline{B}(y)]^{-1}\, dF(y)}{\overline{F}(h)}\ , \\ &= \frac{[\,\overline{B}(x-h)]^{-1} \int_0^\infty [\,\overline{B}(y+h)]^{-1}\, dF(y+h)}{\overline{F}(h)}\ , \qquad (5.2) \\ &= [\,\overline{B}(x-h)]^{-1} \int_0^\infty [\,\overline{B}(y+h)]^{-1}\, dF_h(y)\ . \qquad (5.3) \end{aligned}$$

Theorem 5.1 Suppose that condition (1.3) holds and (1.5) is satisfied by an NWU distribution function B.

1. If Q_x is an increasing function satisfying, for any $y \geq 0$,

$$Q_x(y) \leq \inf_{0 \leq h \leq x} \left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1}\ , \tag{5.4}$$

   and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\overline{H}_x(y) \ \leq\ \inf_{0 \leq h \leq x} \overline{F}_h(y)\ , \tag{5.5}$$

   then for any $x \geq 0$

$$\psi(x) \ \leq\ \frac{1 - p_0}{\phi} \left[\int_0^\infty Q_x(y)\, dH_x(y)\right]^{-1} . \tag{5.6}$$

2. If Q_x is an increasing convex function satisfying (5.4) and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\int_y^\infty \overline{H}_x(u)\, du \ \leq\ \inf_{0 \leq h \leq x} \int_y^\infty \overline{F}_h(u)\, du\ , \tag{5.7}$$

   then, for any $x \geq 0$, (5.6) holds.

Proof. 1. (5.5) implies that for any $y \geq 0$ and $0 \leq h \leq x$, $\overline{H}_x(y) \leq \overline{F}_h(y)$, *i.e.* $H_x <_{st} F_h$. Since Q_x is increasing, by the equivalent condition of "$<_{st}$" [see, for example, Theorem B of Szekli (1995), page 6] and (5.3), we get for any $0 \leq h \leq x$,

$$K(h) \geq \int_0^\infty Q_x(y)\, dF_h(y) \geq \int_0^\infty Q_x(y)\, dH_x(y)\,. \tag{5.8}$$

This implies that in (4.10), $[K_2(x)]^{-1} \geq \int_0^\infty Q_x(y)\, dH_x(y)$. Thus (5.6) follows from (4.10).

2. (5.7) implies that for any $y \geq 0$ and $0 \leq h \leq x$, $\int_y^\infty \overline{H}_x(u)\, du \leq \int_y^\infty \overline{F}_h(u)\, du$, *i.e.* $H_x <_{icx} F_h$. Thus, since Q_x is an increasing convex function, by the equivalent condition of "$<_{icx}$" [see, for example, Theorem 3.A.1. of Shaked and Shanthikumar (1994)] and (5.3), we get for any $0 \leq h \leq x$,

$$K(h) \geq \int_0^\infty Q_x(y)\, dF_h(y) \geq \int_0^\infty Q_x(y)\, dH_x(y)\,. \tag{5.9}$$

This implies now that $[K_2(x)]^{-1} \geq \int_0^\infty Q_x(y)\, dH_x(y)$, and hence (5.6) still follows from (4.10) when (5.7) holds. □

Similarly, it is easy to prove the two following results.

Theorem 5.2 Suppose that condition (1.4) holds and (1.5) is satisfied by an NBU distribution function B.

1. If Q_x is an increasing function satisfying, for any $y \geq 0$,

$$Q_x(y) \geq \sup_{0 \leq h \leq x} \left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1}\,, \tag{5.10}$$

   and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\overline{H}_x(y) \geq \sup_{0 \leq h \leq x} \overline{F}_h(y)\,, \tag{5.11}$$

   then, for any $x \geq 0$,

$$\psi(x) \geq \frac{1-p_0}{\phi} \left[\int_0^\infty Q_x(y)\, dH_x(y)\right]^{-1}\,. \tag{5.12}$$

2. If Q_x is an increasing convex function satisfying (5.10) and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\int_y^\infty \overline{H}_x(u)\,du \;\geq\; \sup_{0\leq h\leq x} \int_y^\infty \overline{F}_h(u)\,du\;, \tag{5.13}$$

then, for any $x \geq 0$, (5.12) holds.

Corollary 5.1 Under the conditions of Theorem 5.1-1.,

1. If F has an Increasing Failure Rate (IFR), then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(y-x)\,dF(y)\right]^{-1}\overline{F}(x)\;. \tag{5.14}$$

2. If F is NWU, then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dF(y)\right]^{-1}\;. \tag{5.15}$$

3. If F is Used Better than Aged (UBA) with $L_F > 0$, then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi\, L_F}\left[\int_0^\infty e^{-L_F\, y}\, Q_x(y)\,dy\right]^{-1}\;, \tag{5.16}$$

where, $L_F = \lim_{x\to\infty} r_F(x)$ and $r_F(x)$ is the failure rate function of F.

Proof. 1. Since F is IFR if and only if $\overline{F}_h$ is decreasing in $h \geq 0$, then $\inf_{0\leq h\leq x} \overline{F}_h(y) = \overline{F}_x(y)$. Thus, setting $\overline{H}_x(y) = \overline{F}_x(y)$ in Theorem 5.1-1. gives, for any $x \geq 0$,

$$\begin{aligned}
\psi(x) &\leq \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dF_x(y)\right]^{-1}\;,\\
&= \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dF(y+x)\right]^{-1}\overline{F}(x)\;,\\
&= \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(u-x)\,dF(u)\right]^{-1}\overline{F}(x)\;,
\end{aligned}$$

where the first equality follows from (5.1), similarly to the argument for (5.3).

2. Since F is NWU if and only if for any $h \geq 0$ and $y \geq 0$,

$$\overline{F}(y) \;\leq\; \overline{F}_h(y) \;,$$

this implies that $\overline{F}(y) \;\leq\; \inf_{0\leq h\leq x} \overline{F}_h(y)$. Thus, setting $\overline{H}_x(y) = \overline{F}(y)$ in Theorem 5.1-1. gives (5.15).

3. Since F is UBA [see, for example, Alzaid (1994)] if and only if for any $h \geq 0$ and $y \geq 0$,

$$\overline{F}_h(y) \;\geq\; e^{-L_F\, y} \;,$$

which implies that $e^{-L_F\, y} \;\leq\; \inf_{0\leq h\leq x} \overline{F}_h(y)$. Thus, setting $\overline{H}_x(y) = e^{-L_F\, y}$ in Theorem 5.1-1. gives (5.16). □

Corollary 5.2 Under the conditions of Theorem 5.1-2.,

1. If F has a Decreasing Mean Residual Life (DMRL), then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi} \left[\int_x^\infty Q_x(y-x)\, dF(y)\right]^{-1} \overline{F}(x) \;. \qquad (5.17)$$

2. If F is New Worse than Used in Convex ordering (NWUC), then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi} \left[\int_0^\infty Q_x(y)\, dF(y)\right]^{-1} \;. \qquad (5.18)$$

Proof. 1. By the equivalence of these conditions [see, for example, Theorem 3.A.18 of Shaked and Shanthikumar (1994)], we know that F is DMRL if and only if, for any $0 \leq h_1 \leq h_2$

$$F_{h_2} \;<_{icx}\; F_{h_1} \;,$$

which implies that, for any $0 \leq h \leq x$ and $y \geq 0$,

$$\int_y^\infty \overline{F}_x(u)\, du \;\leq\; \int_y^\infty \overline{F}_h(u)\, du \;,$$

and hence, $\int_y^\infty \overline{F}_x(u)\, du \;\leq\; \inf_{0\leq h\leq x} \int_y^\infty \overline{F}_h(u)\, du$. Thus, setting $\overline{H}_x(u) = \overline{F}_x(u)$ in Theorem 5.1-2. gives (5.17).

2. By definition [see, for example, Cao and Wang (1991)], F is NWUC if and only if, for any $h \geq 0$ and $y \geq 0$,

$$\int_y^\infty \overline{F}(u)\,du \leq \int_y^\infty \overline{F}_h(u)\,du\ ,$$

which implies that $\int_y^\infty \overline{F}(u)\,du \leq \inf_{0\leq h\leq x} \int_y^\infty \overline{F}_h(u)\,du$. So here, setting $\overline{H}_x(u) = \overline{F}(u)$ in Theorem 5.1-2. gives (5.18). $\square$

Similarly, using the dual classes DFR, NBU, UWA, IMRL and NBUC to those in Corollaries 5.1 and 5.2 and by use of Theorem 5.2, we get the following two corollaries.

Corollary 5.3 Under the conditions of Theorem 5.2-1.,

1. If F has a Decreasing Failure Rate (DFR), then for $x \geq 0$,

$$\psi(x) \ \geq \ \frac{1-p_0}{\phi} \left[\int_x^\infty Q_x(y-x)\,dF(y)\right]^{-1} \overline{F}(x)\ . \tag{5.19}$$

2. If F is NBU, then for $x \geq 0$,

$$\psi(x) \ \geq \ \frac{1-p_0}{\phi} \left[\int_0^\infty Q_x(y)\,dF(y)\right]^{-1} . \tag{5.20}$$

3. If F is Used Worse than Aged (UWA) with $L_F > 0$, then for $x \geq 0$,

$$\psi(x) \ \geq \ \frac{1-p_0}{\phi\, L_F} \left[\int_0^\infty e^{-L_F\, y}\, Q_x(y)\,dy\right]^{-1} , \tag{5.21}$$

where $L_F = \lim_{x\to\infty} r_F(x)$ and $r_F(x)$ is the failure rate function of F.

Corollary 5.4 Under the conditions of Theorem 5.2-2.,

1. If F has an Increasing Mean Residual Life (IMRL), then for $x \geq 0$,

$$\psi(x) \ \geq \ \frac{1-p_0}{\phi} \left[\int_x^\infty Q_x(y-x)\,dF(y)\right]^{-1} \overline{F}(x)\ . \tag{5.22}$$

2. If F is New Better than Used in Convex ordering (NBUC) [that is for any $h \geq 0$ and $y \geq 0$, $\int_y^\infty \overline{F}(u)du \geq \int_y^\infty \overline{F}_h(u)du$,], then for $x \geq 0$,

$$\psi(x) \ \geq \ \frac{1-p_0}{\phi} \left[\int_0^\infty Q_x(y)\,dF(y)\right]^{-1} . \tag{5.23}$$

Remark 5.1 1. It is clear that if B is NWU, then

$$[\overline{B}(x+y)]^{-1} \leq \inf_{0\leq h\leq x} \left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1} ,$$

while, if B is NBU, then

$$[\overline{B}(x+y)]^{-1} \geq \sup_{0\leq h\leq x} \left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1} .$$

Thus, Theorems 1 and 2 of Willmot and Lin (1997) are obtained as special cases of Theorems 5.1-1. and 5.2-1., respectively, by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$. With stochastic orderings, the proofs do not require the condition that B and F, in $\int_x^\infty [\overline{B}(y)]^{-1}\,dF(y)$, have no common discontinuities [the proof of Corollaries 2.2 and 3.5 of Lin (1996) would also require it].

2. Theorems 3.1 and 5.1 of Willmot (1997^b) are also special cases of Theorems 5.1-2. and 5.2, respectively, by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$. Note that the condition that B be twice differentiable is not necessary here which yields a simpler proof.

   Similarly, Corollaries 2.2, 2.4 and 3.5 of Lin (1996), Corollaries 1 and 3 of Willmot and Lin (1997) and Corollaries 3.1, 3.2 , 5.1 and 5.2 of Willmot (1997^b) can all be obtained as special cases of Corallaries 5.1, 5.2, 5.3 and 5.4 by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$.

3. The bounds in Theorems 5.1 and 5.2 are sharp, in the sense that if F and B are exponential distributions and N is a geometric random variable with $\Pr\{N=n\} = (1-\phi)\phi^n, n = 0, 1, 2, \cdots$, then the inequalities become equalities.

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