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ABSTRACT. This note considers finite state Markov chains which overlap supports. While the overlapping supports condition is known to be necessary and sufficient for stability of these chains, the result is typically presented in a more general context. As such, one objective of the note is to provide an exposition, along with simple proofs corresponding to the finite case. Second, the note provides an additional equivalent condition which should be useful in applications.

1. INTRODUCTION

It is a standard result in the literature that every Markov matrix on a finite state space which is both irreducible and aperiodic is asymptotically stable—a unique stationary distribution exists, and iterating on any initial distribution with the matrix generates a trajectory which converges to the stationary distribution. (This situation is also called ergodicity.) Note however that the conditions are not necessary. For example, it is easy to construct matrices which are asymptotically stable but not irreducible.

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In what follows we review the notion of Markov chains which overlap supports. The property of overlapping supports has been investigated by authors such as Lasota (1994), who showed that a certain class of positive linear operators mapping L_1 into itself were asymptotically stable if and only if they overlapped supports and had the additional property of Lagrange stability. For finite state Markov chains, every Markov matrix can be identified with one of these operators, and is automatically Lagrange stable. Hence the overlapping supports property is necessary and sufficient.[1]

These results are not well known in the literature on finite state Markov chains. A brief exposition is provided, along with simple proofs specialized to the finite case. Second, the note adds an additional condition (Condition (3) of Theorem 2.1 below) which is equivalent to overlapping supports in the finite state case, and is relatively easy to check in applications.

2. Results

Let S be the finite set of size N, and let $\mathscr{P}(S)$ be the set of distributions on S. That is, $\mathscr{P}(S)$ is all $\varphi \in \mathbb{R}^S$ such that $\varphi(s) \geq 0$, for all $s \in S$, and $\sum_{s \in S} \varphi(s) = 1$. By a Markov matrix is meant an $N \times N$ matrix $\mathbf{M}$, where each row is an element of $\mathscr{P}(S)$. For general $x \in \mathbb{R}^S$ we impose the ℓ_1 norm $\|x\| := \sum_{s \in S} |x(s)|$. If $\varphi, \psi \in \mathscr{P}(S)$, then $\|\varphi - \psi\|$ is propotional to the total variation distance between these distributions.

[1] Recently, Zaharopol (2000) showed that every such operator which overlaps supports and has an invariant distribution is asymptotically stable. In the finite state case, every Markov chain has an invariant distribution.

A $\psi^* \in \mathbb{R}^S$ is called a *stationary distribution* for Markov matrix $\mathbf{M}$ if $\psi^* \in \mathscr{P}(S)$ and $\psi^* = \psi^* \mathbf{M}$, where, as is traditional, we are treating distributions as row vectors. The matrix is called *ergodic* or *asymptotically stable* if there is one and only one such ψ^* in $\mathscr{P}(S)$, and, moreover, $\|\psi \mathbf{M}^t - \psi^*\| \to 0$ as $t \to \infty$ for every $\psi \in \mathscr{P}(S)$.

We say that two distributions φ and ψ *overlap* if $\varphi \wedge \psi \neq 0$; alternatively, if $\exists s \in S$ with $\varphi(s) > 0$ and $\psi(s) > 0$. Also, we say that $T \colon \mathscr{P}(S) \to \mathscr{P}(S)$ is a strict contraction if

$$\forall \varphi, \psi \in \mathscr{P}(S) \text{ with } \varphi \neq \psi, \quad \|T\varphi - T\psi\| < \|\varphi - \psi\|.$$

The definitions lead us to several equivalent conditions for asymptotic stability:

Theorem 2.1. *Let* $\mathbf{M}$ *be a Markov matrix on* S*. The following statements are all equivalent:*

(1) $\mathbf{M}$ *is asymptotically stable;*
(2) $\forall \varphi, \psi \in \mathscr{P}(S)$*,* $\exists t \in \mathbb{N}$ *such that* $\varphi \mathbf{M}^t$ *and* $\psi \mathbf{M}^t$ *overlap;*
(3) $\exists t \in \mathbb{N}$ *such that any two rows of* $\mathbf{M}^t$ *overlap; and*
(4) $\exists t \in \mathbb{N}$ *such that* $\mathbf{M}^t$ *is a strict contraction on* $\mathscr{P}(S)$*.*

Remark. The second property is usually identified with the notion that $\mathbf{M}$ overlaps supports. The third property is relatively easy to check in applications. For example, if a column of $\mathbf{M}^t$ is strictly positive then (3) clearly holds for this t. This is a well-known stability condition.[2] However, the fact that any two rows of $\mathbf{M}^t$ overlap does not imply that $\mathbf{M}^t$ has a strictly positive column. As a result, (3) can hold for smaller

[2]For a recent exposition see Stroock (2005).

t, which means that asymptotic stability can potentially be verified for a smaller power of $\mathbf{M}$.

Before beginning the proofs we make some preliminary observations. One is that distributions φ and ψ overlap if and only if $\|\psi - \varphi\| < 2$. This is because $\|\varphi - \psi\| = \sum_{s \in S} |\varphi(s) - \psi(s)|$, and because nonnegative real numbers a and b satisfy $|a - b| \leq a + b$, with strict inequality if and only if both are strictly positive.

Second, it is easy to show and well-known that if $\mathbf{M}$ is any Markov matrix and $\varphi, \psi \in \mathscr{P}(S)$, then $\|\varphi\mathbf{M} - \psi\mathbf{M}\| \leq \|\varphi - \psi\|$ always holds. From this we conclude that if $\varphi\mathbf{M}^t$ and $\psi\mathbf{M}^t$ overlap, then so do $\varphi\mathbf{M}^{t+k}$ and $\psi\mathbf{M}^{t+k}$ for every $k \in \mathbb{N}$, because

$$\|\varphi\mathbf{M}^{t+k} - \psi\mathbf{M}^{t+k}\| \leq \|\varphi\mathbf{M}^t - \psi\mathbf{M}^t\| < 2.$$

Proof of Theorem 2.1. (1) $\implies$ (2). Let $\varphi, \psi \in \mathscr{P}(S)$. By (1) there is a $t \in \mathbb{N}$ such that $\|\varphi\mathbf{M}^t - \psi\mathbf{M}^t\| < 2$, which implies (2).

(2) $\implies$ (3) Let $e_1, \ldots, e_N$ be the cannonical basis vectors for $\mathbb{R}^S$.[3] Let $t(n,m) \in \mathbb{N}$ be such that $e_n\mathbf{M}^{t(n,m)}$ and $e_m\mathbf{M}^{t(n,m)}$ overlap, and let $t := \sup_{n,m} t(n,m)$. Now consider the n-th and m-th row of $\mathbf{M}^t$. These are precisely $e_n\mathbf{M}^t$ and $e_m\mathbf{M}^t$, which overlap.

(3) $\implies$ (4) Pick any $\varphi, \psi \in \mathscr{P}(S)$, where $\varphi \neq \psi$. Let t be as in (3), and let $p(s, s')$ be a typical element of $\mathbf{M}^t$. We have

$$\|\varphi\mathbf{M}^t - \psi\mathbf{M}^t\| = \sum_{s' \in S} \left| \sum_{s \in S} (\varphi(s) - \psi(s)) p(s, s') \right|.$$

Let $a(s) := \varphi(s) - \psi(s)$. Observe that for some $s_1 \in S$ we have $a(s_1) > 0$, and for some $s_2 \in S$ we have $a(s_2) < 0$. Using the fact

[3] That is, $e_n(s) = 1$ if $s = n$ and zero otherwise.

that rows s_1 and s_2 of $\mathbf{M}^t$ overlap, choose a further $s'' \in S$ such that $p(s_1, s'')$ and $p(s_2, s'')$ are both strictly positive. For this s'' we have

$$|\sum_{s \in S} a(s)p(s, s'')| < \sum_{s \in S} |a(s)p(s, s'')|,$$

owing to the fact that at least some of the terms such as $a(s_1)p(s_1, s'')$ are strictly positive, while other such as $a(s_2)p(s_2, s'')$ are strictly negative. It now follows that

$$\sum_{s' \in S} \left| \sum_{s \in S} a(s)p(s, s') \right| < \sum_{s' \in S} \sum_{s \in S} |a(s)p(s, s')| = \sum_{s \in S} |a(s)| \sum_{s' \in S} p(s, s').$$

Since rows of $\mathbf{M}^t$ sum to one, the last term is just $\sum_{s \in S} |a(s)|$, which is $\|\varphi - \psi\|$.

(4) $\implies$ (1). Every strict contraction mapping a compact metric space into itself is known to be asymptotically stable.[4] Therefore $\mathbf{M}^t$ has a unique fixed point ψ^* in $\mathscr{P}(S)$, and $\psi \mathbf{M}^{t \cdot k} \to \psi^*$ as $k \to \infty$. It remains to show that $\mathbf{M}$ is asymptotically stable. To see that this is the case, pick any $\varepsilon > 0$, and choose $k \in \mathbb{N}$ so that $\|(\psi^* \mathbf{M})\mathbf{M}^{t \cdot k} - \psi^*\| < \varepsilon$. Then

$$\|\psi^* \mathbf{M} - \psi^*\| = \|(\psi^* \mathbf{M}^{t \cdot k})\mathbf{M} - \psi^*\| = \|(\psi^* \mathbf{M})\mathbf{M}^{t \cdot k} - \psi^*\| < \varepsilon.$$

Since ε was arbitrary, it follows that $\|\psi^* \mathbf{M} - \psi^*\| = 0$, and ψ^* is a fixed point of $\mathbf{M}$.

Stability: Fix $\psi \in \mathscr{P}(S)$, and choose $k \in \mathbb{N}$ so that $\|\psi \mathbf{M}^{t \cdot k} - \psi^*\| < \varepsilon$. Then $n \geq t \cdot k$ implies

$$\|\psi \mathbf{M}^n - \psi^*\| = \|(\psi \mathbf{M}^{t \cdot k})\mathbf{M}^{n - t \cdot k} - \psi^* \mathbf{M}^{n - t \cdot k}\| \leq \|\psi \mathbf{M}^{t \cdot k} - \psi^*\| < \varepsilon.$$

□

[4]For a proof, see for example Stachurski (2003).

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