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Roland Hodler and Hadi Yektas

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Department of Economics
The University of Melbourne
Parkville VIC 3010
www.economics.unimelb.edu.au

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Roland Hodler† and Hadi Yektaş‡

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Abstract

We study a model of war in which the outcome is uncertain not because of luck on the battlefield (as in standard models), but because the involved countries lack information about their opponent. In our model their production and military technologies are common knowledge, but their resources are private information. Each country decides how to allocate its resources to production and warfare. The country with the stronger military wins and receives aggregate production. In equilibrium the country with a comparative advantage in warfare allocates all resources to warfare for low resource levels and follows a non-decreasing concave strategy thereafter. The opponent allocates a constant fraction of its resources to warfare for low resource levels and follows an increasing non-linear strategy thereafter. From an ex ante perspective the country with a comparative advantage in warfare is likely to win the war unless its military technology is much weaker than the opponent's.

Keywords: Conflict; war; all-pay auction; private information

JEL classification: D44; D74; H56

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†Study Center Gerzensee; and Department of Economics, University of Melbourne, VIC 3010, Australia. Email: rhodler@unimelb.edu.au

‡Department of Economics, University of Melbourne, VIC 3010, Australia. Email: hyektas@unimelb.edu.au

1 Introduction

The outcome of many conflicts and wars is uncertain from the perspective of the involved parties or countries, as well as from the outsiders' perspective. Standard models of conflicts and wars account for this uncertainty by assuming that luck plays a crucial role on the battlefield. In this paper we study an alternative model in which countries are uncertain about the outcome not because of luck on the battlefield, but because of a lack of information about their opponent and, consequently, its endogenous military power.

In this model there are two countries (or regions) characterized by their production and military technologies, which are common knowledge, and their resources. Each country only knows the level of its own resources, and can choose how to allocate them to production and warfare. The resource allocation and the technologies determine domestic production and military power. The country with the greater miliary power wins and can consume all goods that have been produced in the two countries, while the loosing country gets nothing. Our model therefore resembles an all-pay auction in which the winner's prize is endogenous and decreasing in both bids.

The assumption of incomplete information about the opponent's resources can have various interpretations. First, the two countries can be imperfectly informed about each other's labor-force or stock of human and physical capital. We imagine that throughout history this was often the case when two tribes, possibly from remote forests or mountainous areas, were fighting against each other. But even nowadays, most countries lack precise estimates of their opponent's resources and, therefore, its productive and military potential.[1] Second, even countries that know the size of their opponent's labor-force during peacetime might not know how many people who are normally out of the labor-force are willing and able to help out on the home front (i.e. in production) or the battlefield during wartime.[2] Finally, a broader understanding of resources does not only include the available labor-force and the stocks of

[1]One reason is that official figures on, e.g., labor supply and production are often biased, and that the size of these biases are typically unknown. Shleifer and Treisman (2005) argue that official figures tend to overestimate true resources and production in communist countries in which managers routinely inflate production figures. In contrast, official figures may underestimate true resources and production in capitalist countries in which individuals and businesses may want to evade taxation.

[2]As an example, many were surprised by the dramatic increase in women's labor force participation in the United States during World War II.

human and physical capital, but also how dedicated and motivated people are to use their labor and their human capital for the best of their country during wartime. Even a country that can accurately guess the size of its opponent's labor-force and capital stocks during wartime may lack accurate information about the dedication and the morale of the opponent country's people on the home front and the battlefield.[3]

We characterize monotone continuous equilibrium strategies for all possible values of the parameters representing the countries' production and military technologies. Interestingly, these strategies depend on absolute as well as comparative advantages in warfare.[4] They are straightforward if the country with a comparative advantage in warfare has a large absolute disadvantage in warfare. For any resource level this country then allocates all resources to warfare, while its opponent only allocates some fraction of its resources to warfare. Because of its better military technology, the opponent nevertheless has the stronger military at any resource level. From an ex ante perspective, the opponent is therefore likely to win the war.

Equilibrium strategies are more involved if the country with a comparative advantage in warfare has also an absolute advantage or only a modest absolute disadvantage in warfare. This country then allocates all resources to warfare up to some threshold and follows a non-decreasing and concave strategy for higher resource levels. Its opponent allocates a constant fraction of its resources to warfare up to some threshold and follows an increasing non-linear strategy for higher resource level. Hence, at low resource levels it is again the country with a comparative advantage in warfare that allocates more resources to warfare. However, at high resource levels absolute advantages matter: the country with an absolute advantage in warfare allocates less resources to warfare in order to avoid diverting many more resources away from production when already winning the war with high probability. The country with a comparative advantage in warfare nevertheless has the stronger military at any resource level. From an ex ante perspective this country is therefore likely to win the war.

The theoretical literature on conflicts and wars contains two main strands. The first looks

[3]As an example, many were surprised by the (initial) reluctance of Iraqis to fight when the United States and its allies invaded Iraq to overthrow the regime of Saddam Hussein. Many were also surprised by the fierce resistance of some Iraqi factions in later years.

[4]The country with the better miliary technology is said to have an absolute advantage in warfare, and the country with the higher ratio of miliary to production technology a comparative advantage in warfare.

at reasons why conflicts emerge, and the second studies how conflicts are fought.[5] Our model contributes to the second strand. It is closely related to the standard models of conflicts and wars that go back to Haavelmo (1954) and have been popularized by Garfinkel (1990), Grossman (1991), Hirshleifer (1991, 2001), and Skaperdas (1992). Garfinkel and Skaperdas (2007) present a synthesis of these models, which typically have four key features: First, there is a war taking place for exogenous reasons. Second, each country can choose how to allocate its resources to production and warfare. Third, the mapping from the resources that the different countries allocate to warfare to the outcome of the war is probabilistic. Fourth, the winning country can consume all production. While keeping the first two and the last of these features, we assume that the country with the stronger military wins for sure. Moreover, we add the assumption that countries are imperfectly informed about their opponent's resources. Our model thus offers a complementary view according to which countries are uncertain about the outcome of the war not because of luck on the battlefield, but because of a lack of information about their opponent. This view allows us to study how aggressively countries behave at different resource levels, i.e., depending on the level of their resources relative to the level that their opponent expected.

Despite these differences in the setup, our results share some properties with the standard models of conflict and war: Countries with few resources tend to allocate all resources to warfare, and countries with a comparative advantage in warfare tend to allocate more resources to warfare than their opponent. However, there are some interesting differences: First, in the standard models each player's equilibrium strategy is simply a choice of a particular resource allocation that depends on his and the opponent's known resource level, while in our model it is a function of his own resource level.[6] Hence we can present the results for different resource levels in a unified way, as the same player may allocate all resources to warfare at low resource levels, but not at higher levels. Second, when the resource constraint is not binding, the share of resources allocated to warfare is linear in the resource level in standard models (e.g., Garfinkel and Skaperdas, 2007), but non-linear in ours. This non-linearity in our model

[5] Garfinkel and Skaperdas (2007) and Blattman and Miguel (2010) review the literature. See also Jackson and Morelli (forthcoming) for a survey of the first strand of this literature.

[6] This follows because the strategy space is a subset of $\Re_+$ in the standard models, but a subset of functions from $\Re_+$ to $\Re_+$ in our model.

follows from the players' comparative advantages. Lastly, and from an economic point most interestingly, we find that equilibrium strategies depend not only on comparative but also on absolute advantages in warfare.

Most other models of conflicts in which countries have some private information contribute to the first strand of the theoretical literature on conflicts and wars by studying the emergence of conflicts. Thereby they typically take military power as given (e.g., Fearon, 1995). Building on these models, Meirowitz and Satori (2008) present a model with a similar flavor as ours in that war can occur between two countries that have invested in military power but cannot observe each other's investment. In their model private information follows from countries playing mixed strategies when deciding how much to invest in military power.[7] The complexity and generality of their model comes however at the cost that equilibrium strategies cannot be derived.

As we model war as an asymmetric auction with incomplete information, our paper also relates to the literature on auction theory. Our model thereby shares some features with both all-pay and winner-pay auctions. Like in all-pay auctions, all bids need to be paid, i.e., no resources allocated to warfare can be used to produce consumption goods.[8] And like in winner-pay auctions, the loser ends up with a payoff of zero independently of his bid. The main difference to both all-pay and winner-pay auctions is that in our model the winner's payoff decreases in his own bid as well as the loser's bid.

The remainder of the paper is organized as follows: Section 2 introduces the model. Section 3 presents some preliminary results. Section 4 derives and discusses the equilibrium. Section 5 concludes. The appendix contains all proofs.

[7] Jackson and Morelli (2009) study a model similar to Meirowitz and Satori (2008), but assume that investments in miliary power are observable.

[8] The literature on all-pay auctions with incomplete information goes back to Amann and Leininger (1996) and Krishna and Morgan (1997). Feess et al. (2008) study an all-pay auction with incomplete information in which one player may have a handicap in the same way as one country may have a lower military technology in our model.

2 The Model

There are two countries that are at war for some exogenous reason. Each country $i = 1, 2$ acts as a single player, and the two countries must simultaneously decide how to allocate their resources r_i to production and warfare. We assume that r_1 and r_2 are independently drawn from a uniform distribution on the unit interval, and that their realizations are private information while their distribution is common knowledge.[9]

The military power of country i is $\lambda_i b_i$, where $\lambda_i > 0$ is its military technology, and b_i the resources it allocates to warfare. The production of consumption goods of country i is $\beta_i(r_i - b_i)$, where $\beta_i > 0$ is its production technology, and $(r_i - b_i)$ the resources it allocates to production. The resource constraint requires $b_i \in [0, r_i]$. The technology parameters β_i and λ_i are common knowledge, but may differ across countries.

The outcome of the war is deterministic in that the country with the higher military power $\lambda_i b_i$ wins for sure. The winning country can consume all goods that have been produced in the two countries. Therefore, given choices b_i and b_j, and resources r_i and r_j, the payoff of country i is

$$\tilde{u}_i(b_i, b_j; r_i, r_j) = \begin{cases} 0 & \text{for} \quad \lambda_i b_i < \lambda_j b_j \\ \beta_i(r_i - b_i) & \text{for} \quad \lambda_i b_i = \lambda_j b_j \\ \beta_1(r_1 - b_1) + \beta_2(r_2 - b_2) & \text{for} \quad \lambda_i b_i > \lambda_j b_j. \end{cases}$$

In this game, the strategy space is such that country i's strategies are of the form $b_i = f_i(r_i)$: $[0, 1] \to [0, r_i]$. We look for a Bayesian Nash equilibrium in monotone continuous strategies that are differentiable almost everywhere.

We define $\lambda \equiv \frac{\lambda_1}{\lambda_2}$ and $\beta \equiv \frac{\beta_1}{\beta_2}$, and we assume without loss of generality that $\beta \leq \lambda$, which implies $\frac{\beta_1}{\lambda_1} \leq \frac{\beta_2}{\lambda_2}$. That is, we call the country with a comparative advantage in warfare country 1, and the country with a comparative advantage in the production of consumption

[9] We think of resources as a composite measure that include a country's labor-force and its stock of human and physical capital as well as the motivation and dedication of the people to work and fight hard during wartime. As argued in the introduction, the private information of resource levels can represent a situation in which each country is uncertain about the labor-force and the stocks of human and physical capital available to its opponent; or one in which each country is uncertain about how motivated and dedicated the opponent's people are to use their labor-force and their capital stocks during wartime.

goods country 2. Subsequently we refer to countries as players, thereby calling player 1 "she" and player 2 "he". Moreover, we call their choices of b_i their bids or *real bids*, while referring to $\lambda_i b_i$ as their *effective bids*. Effective bids play a key role in this game because the player with the higher effective bid wins the war.

3 Preliminary Results

In this section we first present an important lemma. We then study a simplified version of the game introduced in the previous section to understand some of the main forces at work.

Lemma 1 *In any monotone equilibrium it holds that* $f_1(0) = f_2(0) = 0$*, that* $f_1(.)$ *and* $f_2(.)$ *are non-decreasing, and that* $\lambda f_1(1) = f_2(1)$*.*

Lemma 1 already puts some structure on the players' bidding strategies. It directly follows from the resource constraint that players with zero resources cannot allocate any resources to warfare. As a consequence, monotone strategies must be non-decreasing. Moreover, no player ever bids more than necessary to win with probability one because the winner's payoff decreases in the resources he or she has allocated to warfare. Effective bids must thus coincide at the top, i.e. if $r_1 = r_2 = 1$.

We next solve our game assuming that the three properties specified in Lemma 1 hold, but abstracting from the resource constraint for $r_i > 0$ and $i = 1, 2$. This simplified version of the game has a closed-form solution that is easy to interpret and helpful to understand how the players' incentives shape their behavior. To avoid confusion, we call the equilibrium of this simplified version of our game a quasi-equilibrium.

We start by focusing on the bidding strategy chosen by player 1 assuming that player 2 chooses the non-decreasing strategy $f_2(r_2)$. Player 1 wins if and only if she bids $y > \frac{f_2(r_2)}{\lambda}$, i.e., if and only if $r_2 < f_2^{-1}(\lambda y)$. Hence her expected payoff is

$$u_1(y; r_1) \equiv \int_0^1 \tilde{u}_1(y, f_2(r_2); r_1, r_2) dr_2 = \int_0^{f_2^{-1}(\lambda y)} [\beta_1(r_1 - y) + \beta_2(r_2 - f_2(r_2))] dr_2. \qquad (1)$$

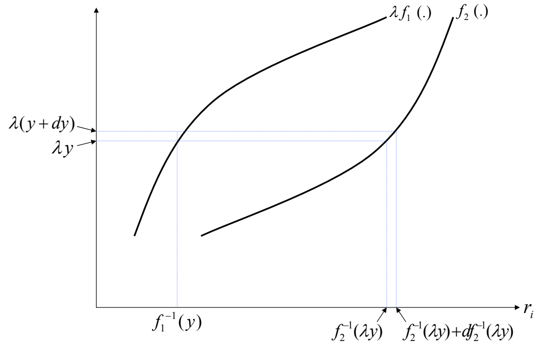

Figure 1: Player 1's trade-off

She faces a trade-off as a marginal increase in y increases the probability of winning, but reduces the prize, i.e. aggregate production of consumption goods. It follows from the first-order condition that the optimal bid $y = f_1(r_1)$ must satisfy

$$-\beta f_2^{-1}(\lambda y) + [\beta(f_1^{-1}(y) - y) + f_2^{-1}(\lambda y) - \lambda y]\frac{df_2^{-1}(\lambda y)}{dy} = 0, \tag{2}$$

or, equivalently,

$$[\beta_1(f_1^{-1}(y) - y) + \beta_2(f_2^{-1}(\lambda y) - \lambda y)]df_2^{-1}(\lambda y) = \beta_1 dy f_2^{-1}(\lambda y). \tag{3}$$

Condition (3) and Figure 1 illustrate the trade-off that player 1 faces. Consider a type of player 1 that bids y and thinks about bidding $y + dy$ (such that her effective bid would increase from λy to $\lambda(y + dy)$). The benefit from increasing the bid by dy occurs if this increase turns her into a winner. This event occurs with probability $df_2^{-1}(\lambda y)$ and generates an expected marginal benefit as represented on the left-hand side of (3). The marginal cost of increasing the bid is borne if player 2 is already a winner when bidding y. This event occurs with probability $f_2^{-1}(\lambda y)$. The opportunity cost of increasing the bid is the forgone production $\beta_1 dy$. Hence the right-hand side of (3) represents the expected marginal cost of increasing the bid.

Similarly, if player 1 chooses the non-decreasing strategy $f_1(r_1)$, then player 2's optimal bid $y = f_2(r_2)$ must satisfy

$$-f_1^{-1}(y) + [\beta(f_1^{-1}(y) - y) + f_2^{-1}(\lambda y) - \lambda y]\frac{df_1^{-1}(y)}{\lambda dy} = 0. \tag{4}$$

It follows from the system of the two differential equations (2) and (4):

Lemma 2 *Disregarding any constraints, the players' strategies are mutual best responses if they are of the form*

$$\begin{aligned} f_1(r_1) &= \frac{\beta}{\beta + 2\lambda} r_1 + \frac{1}{2\beta + \lambda} K_0 r_1^{\frac{\beta}{\lambda}} + K_1 r_1^{-(1+\frac{\beta}{\lambda})} && (5) \\ f_2(r_2) &= \frac{\lambda}{2\beta + \lambda} r_2 + \frac{\lambda\beta}{\beta + 2\lambda} K_0^{-\frac{\lambda}{\beta}} r_2^{\frac{\lambda}{\beta}} + K_2 r_2^{-\left(1+\frac{\lambda}{\beta}\right)}, && (6) \end{aligned}$$

where K_0, K_1 and K_2 are constants.

Lemmas 1 and 2 imply that the quasi-equilibrium strategies must be of form (5) and (6), respectively, and satisfy the boundary conditions $f_1(0) = f_2(0) = 0$ and $\lambda f_1(1) = f_2(1)$. It follows:

Corollary 1 *The players' quasi-equilibrium strategies are*

$$\begin{aligned} f_1(r_1) &= \frac{\beta}{\beta + 2\lambda} r_1 + \frac{1}{2\beta + \lambda} r_1^{\frac{\beta}{\lambda}} && (7) \\ f_2(r_2) &= \frac{\lambda}{2\beta + \lambda} r_2 + \frac{\lambda\beta}{\beta + 2\lambda} r_2^{\frac{\lambda}{\beta}}. && (8) \end{aligned}$$

These quasi-equilibrium strategies are increasing. Moreover, they are linear and reduce to $f_1(r_1) = \frac{1+\beta}{3\beta} r_1$ and $f_2(r_2) = \frac{\beta(1+\beta)}{3\beta} r_2$ if $\beta = \lambda$. Hence, if none of the players has a comparative advantage in warfare, the one with lower β_i and λ_i bids so much more than his or her opponent that their effective bids exactly coincide for any resource level.

The more interesting situation arises if $\beta < \lambda$. Then player 1's quasi-equilibrium strategy is strictly concave, and player 2's quasi-equilibrium strategy strictly convex. Since $\lambda f_1(0) = f_2(0)$ and $\lambda f_1(1) = f_2(1)$, it follows that $\lambda f_1(r) > f_2(r)$ for all $r \in (0, 1)$. That is, in the absence of resource constraints, player 1 who has a comparative advantage in warfare chooses the higher effective bid, i.e. the stronger military, at any resource level $r \in (0,1)$. Player 1 thus wins the war when having weakly more resources than player 2, and even when having slightly less resources. From an ex ante perspective, i.e. in expectation before nature draws r_1 and r_2, player 1 is therefore more likely to win the war than her opponent.

Turning from effective to real bids, it directly follows from $\lambda f_1(r) > f_2(r)$ for all $r \in (0,1)$ that $f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\lambda \leq 1$. Hence player 1 chooses a higher real bid and allocates more resources to warfare than her opponent for any resource level when she has a comparative advantage, but an absolute disadvantage in warfare. This is necessary for her to build the stronger military. However, if $\lambda > 1$, there exists a unique threshold $\hat{r} \in (0,1)$ such that $f_1(r) > f_2(r)$ for $r \in (0,\hat{r})$ and $f_1(r) < f_2(r)$ for $r \in (\hat{r},1)$.[10] That is, when player 1 has a comparative and an absolute advantage in warfare, she allocates more resources to warfare than her opponent for relatively low resource levels, but less resources for relatively high resource levels. The former is driven by her incentive to specialize in warfare, and the latter by her incentive not to allocate many more resources to warfare when already winning the war with high probability.

The quasi-equilibrium strategies (7) and (8) characterize equilibrium behavior if and only if they satisfy the resource constraint $f_i(r_i) \leq r_i$ for $r_i > 0$ and $i = 1,2$. This is the case if and only if $\beta = \lambda \in \left[\frac{1}{2}, 2\right]$. If $\beta = \lambda \notin \left[\frac{1}{2}, 2\right]$, the quasi-equilibrium strategy of the player with lower β_i and λ_i violates the resource constraint for all resource levels. And if $\beta < \lambda$, player 1's quasi-equilibrium strategy and any other strategy of form (5) violate the resource constraint for r_1 sufficiently close to zero.[11]

4 Equilibrium

In this section we first characterize the players' equilibrium strategies for all possible values of β and λ. We then compare their real and effective bids. The general pattern will be similar

[10] Existence and uniqueness of this threshold can be established using the following observations. First, the quasi-equilibrium strategy $f_1(r_1)$ is continuously increasing and concave, while $f_2(r_2)$ is continuously increasing and convex. Second, $f_1(0) = f_2(0)$ and $\lim_{r\to 0^+} f_1'(r) > \lim_{r\to 0^+} f_2'(r)$ since $\beta < \lambda$. Third, $f_1(1) < f_2(1)$ since $\lambda f_1(1) = f_2(1)$ and $\lambda > 1$.

[11] Note that $\lim_{r_1\to 0^+} f_1'(r_1) = \infty$ if $f_1(r_1)$ is characterized by (5) and $\beta < \lambda$. Since $f_1(0) = 0$, it follows that $f_1(r_1) > r_1$ for $r_1 \to 0^+$.

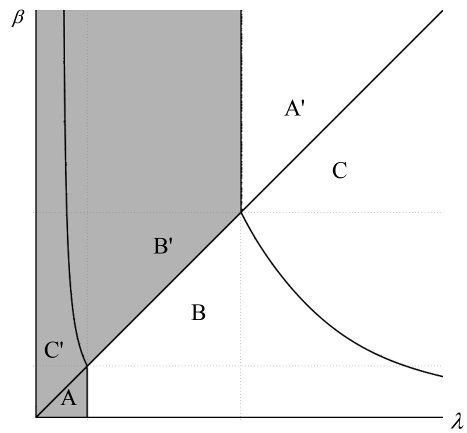

Figure 2: Regions in the parameter space

as in the quasi-equilibrium. The behavioral differences that will occur are due to the resource constraints that the players are facing, and not due to changes in their incentives. The insights that we have gained in the previous section will therefore be helpful to understand equilibrium behavior.

We know from the previous section that strategies satisfying (5) and (6) are mutual best responses, and that they are non-linear unless $\beta = \lambda$. Also we know that any strategy of type (5) violates player 1's resource constraint for r_1 sufficiently close to zero if $\beta < \lambda$. We thus conjecture that player 1's equilibrium strategy includes bidding all resources r_1 up to some threshold $c_l > 0$, and possibly to follow a non-linear strategy of type (5) for $r_1 > c_l$. The following result will therefore be useful:

Lemma 3 *Suppose player* 1 *follows a non-decreasing strategy with* $f_1(r_1) = r_1$ *for* $r_1 \leq c_l$. *Then player 2's best response that is lower than* λc_l *is* $f_2(r_2) = \frac{r_2}{2}$.

Suppose player 2 *follows a non-decreasing strategy with* $f_2(r_2) = \frac{r_2}{2}$ *for* $r_2 \leq 2\lambda c_l$. *Then player 1's best response is* $f_1(r_1) = r_1$ *for* $r_1 \leq c_l$.

We next derive the equilibrium strategies separately for different regions of the parameter space. These regions are shown in Figure 2. We focus on regions A, B and C, which are consistent with our assumption $\beta \leq \lambda$. We do not explicitly derive equilibrium strategies for regions A', B' and C' in which $\beta > \lambda$. However it is straightforward to show that these equilibrium strategies are symmetrical to those in regions A, B and C, respectively.

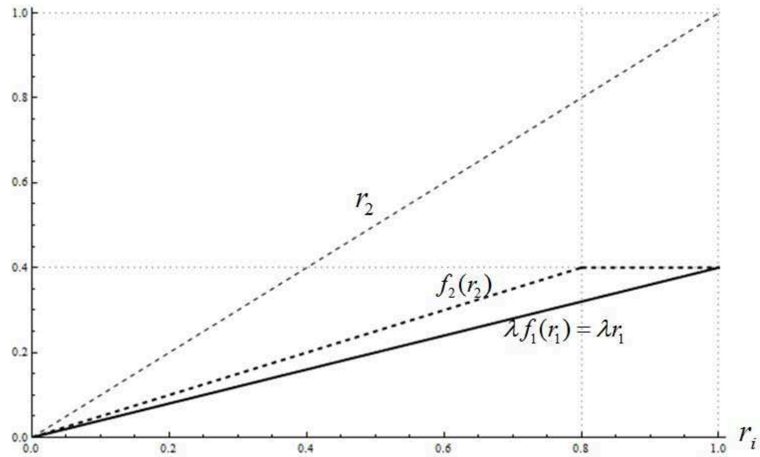

Figure 3: Effective equilibrium bids in region A (with $\beta = 0.3$ and $\lambda = 0.4$)

Region A is defined by $\beta \leq \lambda \leq \frac{1}{2}$. Hence player 1 has a comparative advantage but a large absolute disadvantage in warfare. She has therefore little incentive to allocate resources to production because she can produce relatively little anyway, and because she needs to bid much more aggressively than her opponent if she ever wants to win the war. We can thus explain equilibrium behavior using Lemmas 1 and 3 only.

Proposition 1 *Assume $\lambda \leq \frac{1}{2}$. Then player 1's equilibrium strategy is $f_1(r_1) = r_1$, and player 2's equilibrium strategy is $f_2(r_2) = \min\left\{\frac{r_2}{2}, \lambda\right\}$.*

Figure 3 illustrates the equilibrium strategies described in Proposition 1. Note that the proposition states our results in real bids, while the figure shows effective bids. Player 1 bids all resources for any resource level r_1. Player 2's best response is to bid half his resources, but never more than necessary to win with probability one. Figure 3 further shows that $\lambda f_1(r) \leq f_2(r)$ for all $r \in [0,1]$ despite $f_1(r) \geq 2f_2(r)$ for all $r \in [0,1]$. We will come back to comparisons of the players' real and effective bids after characterizing the equilibrium strategies for $\lambda > \frac{1}{2}$.

The strategies described in Proposition 1 cannot explain equilibrium behavior when $\lambda > \frac{1}{2}$, as player 1 would have an incentive to deviate and to allocate some resources to production for $r_1 > \frac{1}{2\lambda}$. Nevertheless she still has an incentive to bid all resources for low r_1. To derive the equilibrium strategies we therefore use Lemmas 1 and 3 as well as Lemma 2. In particular, we conjecture that player 1's equilibrium strategy is $f_1(r_1) = r_1$ for $r_1 \in [0, c_l]$, where $c_l \in (0,1)$, and of type (5) for $r_1 \in (c_l, 1]$, and that player 2's equilibrium strategy is $f_2(r_2) = \frac{r_2}{2}$ for $r_2 \in [0, 2\lambda c_l]$ and of type (6) for $r_2 \in (2\lambda c_l, 1]$. Given these conjectured equilibrium strategies, the system of equations (5) and (6) must satisfy the boundary condition

$$\lambda f_1(c_l) = f_2(2\lambda c_l) = \lambda c_l. \tag{9}$$

It follows:

Lemma 4 *Suppose player 1 follows a non-decreasing strategy with*

$$f_1(r_1) = c_l h\left(\frac{r_1}{c_l}\right) \tag{10}$$

for $r_1 > c_l$, *where* $c_l \in \left(0, \max\{1, \frac{1}{2\lambda}\}\right)$ *and* $h(x) \equiv \frac{\beta}{\beta+2\lambda}x + \frac{2\lambda}{2\beta+\lambda}x^{\frac{\beta}{\lambda}} + \frac{2\lambda(\beta-\lambda)}{(\beta+2\lambda)(2\beta+\lambda)}x^{-\left(1+\frac{\beta}{\lambda}\right)}$. *Then player 2's best response that is higher than* λc_l *is*

$$f_2(r_2) = \lambda c_l h\left(\left(\frac{r_2}{2\lambda c_l}\right)^{\frac{\lambda}{\beta}}\right). \tag{11}$$

Suppose player 2 follows a non-decreasing strategy with $f_2(r_2)$ *given by (11) for* $r_2 \geq 2\lambda c_l$. *Then player 1's best response* $f_1(r_1)$ *that is higher than* c_l *is given by (10). It holds that* $f_1'(.) > 0$, $f_1'(c_l) = 1$, $f_1''(.) < 0$ *and* $f_2'(.) > 0$.

It is straightforward to see that the conjectured equilibrium strategies do not violate the players' resource constraints for $r_1 \leq c_l$ and $r_2 \leq 2\lambda c_l$. Also player 1's conjectured equilibrium strategy does not violate her resource constraint for any $r_1 > c_l$, as her strategy described by (10) satisfies $f_1'(c_l) = 1$ and is concave for $r_1 > c_l$. However it is a priori unclear whether or not player 2's conjectured equilibrium strategy violates the resource constraint for some $r_2 > 2\lambda c_l$. We know from Lemma 1 that the strategies described by (10) and (11) must satisfy the boundary condition $\lambda f_1(1) = f_2(1)$ if $f_2(r_2)$ does not violate the resource constraint for any $r_2 > 2\lambda c_l$. This boundary condition and (10) and (11) imply $c_l = (2\lambda)^{\frac{\lambda}{\beta-\lambda}}$. An equilibrium of the type conjectured therefore exists if and only if the strategy described by (11) satisfies $f_2(r_2) \leq r_2$ for all $r_2 > 2\lambda c_l$ when $c_l = (2\lambda)^{\frac{\lambda}{\beta-\lambda}}$. The following proposition establishes that

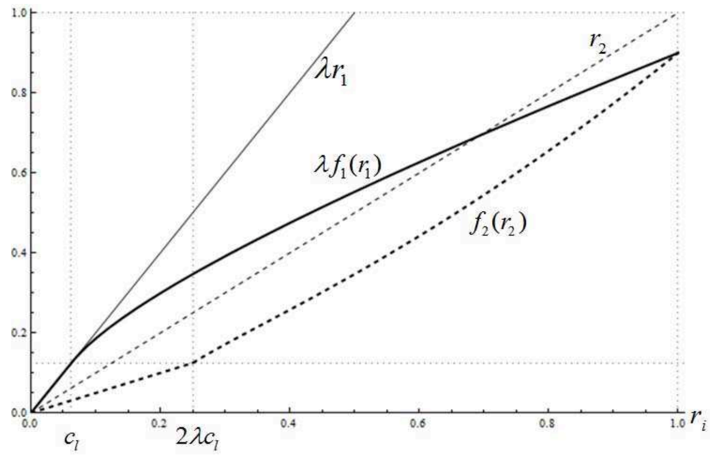

Figure 4: Effective equilibrium bids in region B (with $\beta = 1$ and $\lambda = 2$)

this is the case if and only if

$$\lambda \leq \Lambda(\beta, \lambda) \equiv \left[(2\lambda)^{\frac{\lambda}{\beta-\lambda}} h\left((2\lambda)^{\frac{\lambda}{\lambda-\beta}} \right) \right]^{-1}, \tag{12}$$

and that $\Lambda(\beta, \lambda) > 2$ if $\max\left\{\beta, \frac{1}{2}\right\} < \lambda \leq \Lambda(\beta, \lambda)$. This proposition thus applies to region B, which is characterized by $\beta \leq \lambda$ and $\frac{1}{2} < \lambda \leq \Lambda(\beta, \lambda)$

Proposition 2 *Assume $\frac{1}{2} < \lambda \leq \Lambda(\beta, \lambda)$, which implies $\Lambda(\beta, \lambda) > 2$. Then player 1's equilibrium strategy is $f_1(r_1) = r_1$ for $r_1 \in [0, c_l]$ and as described by (10) for $r_1 \in (c_l, 1]$, with $c_l = (2\lambda)^{\frac{\lambda}{\beta-\lambda}} < 1$. Player 2's equilibrium strategy is $f_2(r_2) = \frac{r_2}{2}$ for $r_2 \in [0, 2\lambda c_l]$ and as described by (11) for $r_2 \in (2\lambda c_l, 1]$, with $2\lambda c_l < 1$.*

Figure 4 illustrates the equilibrium strategies described in Proposition 2. It shows that player 1's resource constraint is binding for $r_1 \leq c_l$, while player 2 responds by bidding half his resources for $r_2 \leq 2\lambda c_l$. For higher resource levels, both players' strategies are non-linear and their effective bids coincide at the top.

It remains to explain equilibrium behavior in region C, which is characterized by $\beta \leq \lambda$ and $\lambda > \Lambda(\beta, \lambda)$. We know from the definition of $\Lambda(\beta, \lambda)$ that player 2's resource constraint must be binding at the top in this region, which of course affects player 1's strategy for high resource levels. We conjecture that in this case the strategy profile satisfies the boundary

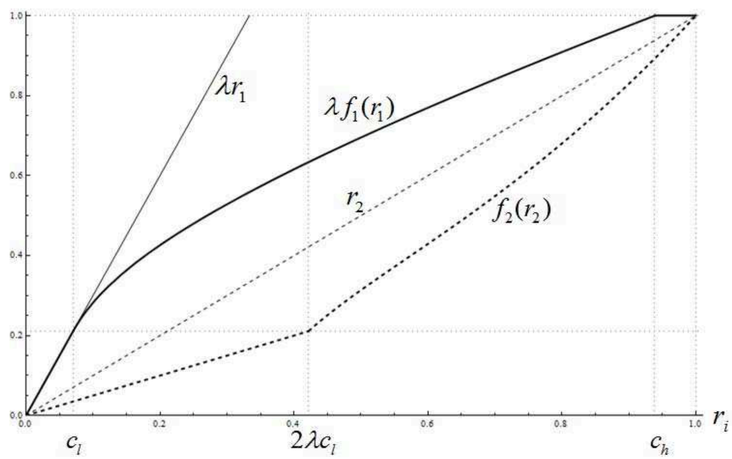

Figure 5: Effective equilibrium bids in region C (with $\beta = 1$ and $\lambda = 3$)

condition

$$\lambda f_1(c_h) = f_2(1) = 1, \tag{13}$$

where $c_h < 1$. It then follows from Lemma 1 that player 1 bids $f_1(r_1) = \frac{1}{\lambda}$ for all $r_1 \geq c_h$. Therefore:

Proposition 3 *Assume $\lambda > \Lambda(\beta, \lambda)$. Then player 1's equilibrium strategy is $f_1(r_1) = r_1$ for $r_1 \in [0, c_l]$, as described by (10) for $r_1 \in (c_l, c_h]$ and $f_1(r_1) = \frac{1}{\lambda}$ for $r_1 \in (c_h, 1]$, with c_l being unique and implicitly defined by $c_l = \left(\lambda h\left((2\lambda c_l)^{-\frac{\lambda}{\beta}}\right)\right)^{-1}$ and with $c_h = (2\lambda)^{-\frac{\lambda}{\beta}} c_l^{\frac{\beta-\lambda}{\beta}}$ satisfying $c_l < c_h < 1$. Player 2's equilibrium strategy is $f_2(r_2) = \frac{r_2}{2}$ for $r_2 \in [0, 2\lambda c_l]$ and as described by (11) for $r_2 \in (2\lambda c_l, 1]$, with $2\lambda c_l < 1$.*

Figure 5 illustrates the equilibrium strategies described in Proposition 3. Unlike in Figure 4, player 2's resource constraint is now binding at the top, and player 1 responds by never submitting any bid higher than necessary to win the war with probability one.

Having derived the players' equilibrium strategies for all possible values of β and λ, we next compare their real and effective bids. We start by looking at the case in which $\beta = \lambda$ such that no player has a comparative advantage in warfare:

Proposition 4 *Assume $\beta = \lambda$. In equilibrium it then holds for all $r \in (0, 1)$ that $f_1(r) > f_2(r)$ if $\lambda < 1$, $f_1(r) = f_2(r)$ if $\lambda = 1$, and $f_1(r) < f_2(r)$ if $\lambda > 1$; and that $\lambda f_1(r) < f_2(r)$ if $\lambda < \frac{1}{2}$, $\lambda f_1(r) = f_2(r)$ if $\lambda \in [\frac{1}{2}, 2]$, and $\lambda f_1(r) > f_2(r)$ if $\lambda > 2$.*

Proposition 4 states that the weaker player with lower β_i and λ_i chooses higher real bids $f_i(r)$ for any resource level $r \in (0,1)$, just as in the quasi-equilibrium. As long as $\lambda \in [\frac{1}{2}, 2]$, allocating more resources to warfare allows this player to compensate for his (or her) poorer military technology λ_i and to end up with the same effective bid $\lambda_i f_i(r)$ for any $r \in (0,1)$. However if his technologies β_i and λ_i are less than half as good as the opponent's technologies, i.e. if $\lambda < \frac{1}{2}$ or $\lambda > 2$, then this weaker player ends up with the lower effective bid for any $r \in (0,1)$. This result, which did not obtain in the quasi-equilibrium, is not due to the weaker player not wanting to bid more to compensate for his poor military technology, but due to his resource constraint. As Proposition 1 implies, this player bids all of his resources, but this is not enough to reach the same effective bid as the stronger opponent who generally bids half her resources (but never more than necessary to win with probability one). From an ex ante perspective the two players are thus equally likely to win the war unless their technologies are sufficiently dissimilar.

We next compare real and effective bids for the more interesting case in which $\beta < \lambda$, such that player 1 has a comparative advantage in warfare.

Proposition 5 *Assume $\beta < \lambda$. In equilibrium it then holds that $f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\lambda \leq 1$. Otherwise, $f_1(r) > f_2(r)$ for r below or sufficiently close to $2\lambda c_l$, and $f_1(r) < f_2(r)$ for r sufficiently close to 1. Further it holds for all $r \in (0,1)$ that $\lambda f_1(r) < f_2(r)$ if $\lambda < \frac{1}{2}$, $\lambda f_1(r) = f_2(r)$ if $\lambda = \frac{1}{2}$, and $\lambda f_1(r) > f_2(r)$ if $\lambda > \frac{1}{2}$.*

It follows from Proposition 5 that results relating to the players' real bids are again similar in equilibrium as in the quasi-equilibrium discussed in section 3. When player 1, who has a comparative advantage in warfare, has an absolute disadvantage in warfare, then she allocates at any resource level a higher share of her resource to warfare than her opponent. But when having a comparative as well as an absolute advantage in warfare, she allocates more resources to warfare than her opponent at low resource levels, but less at high resource levels.

Proposition 5 also states (and Figures 4 and 5 illustrate) that player 1 chooses the higher effective bid than her opponent for any resource level if her military technology is at least half as good as her opponent's. As argued earlier, player 1 has this incentive to build a stronger military because of her comparative advantage in warfare. But for any resource level, player

1 ends up with the weaker military if her military technology is not even half as good as her opponent's. The reason for this result, which did not obtain in the quasi-equilibrium, is not that player 1 does not want to choose a higher effective bid, but again that her resource constraint rules this out. She can only bid all her resources r_1, which she does for any r_1 if $\lambda < \frac{1}{2}$. She then ends up with the lower effective bid than player 2, because his best response is to generally bid half of his resources, and because his military technology is more than twice as good. From an ex ante perspective, the player with a comparative advantage in warfare consequently wins the war with higher probability than her opponent if and only if her military technology is at least half as good as her opponent's military technology. These results are illustrated in Figure 2, where white regions indicate that player 1 is more likely to win, and grey regions that player 2 is more likely to win.

Hirshleifer (1991) discusses the Paradox of Power, i.e. why weak players often win against stronger opponents. Our model helps to understand in what circumstances the Paradox of Power emerges. It suggests that the player with the poorer military technology is more likely to win the war if and only if she has a comparative advantages in warfare *and* her military technology is at least half as good as her opponent's.

5 Conclusions

We have presented a model of war and conflict that offers a different perspective than standard models of conflicts. In our model the outcome of the war is uncertain from the countries' perspective because they lack information about their opponents' resources, not because luck plays any role on the battlefield. We have characterized monotone continuous equilibrium strategies and have shown how they depend on absolute and comparative advantages in warfare. We have seen that if the country with a comparative advantage in warfare has a large absolute disadvantage in warfare, then it allocates all resources to warfare, but is still unlikely to win the war against its much stronger opponent that only allocates some fraction of its resources to warfare. But if the country with a comparative advantage in warfare has also an absolute advantage or only a modest absolute disadvantage in warfare, then it chooses the stronger military at any resource level and is therefore likely to win the war. It is the country with a comparative advantage in warfare that allocates more resources to warfare at low resource levels, while absolute advantages matter at high resource levels because no country wants to divert many more resources away from production when already winning the war with high probability.

It is noteworthy that the equilibrium strategies are similar in an alternative version of our game in which the winner receives the resources that the loser allocated to production (rather than the produced goods) and can then use these resources to produce consumption goods with its own production technology. In this alternative version, production technologies play no crucial role and the equilibrium strategies coincide with the equilibrium strategies in our model when $\beta = 1$, implying that no country would ever allocate all resources to warfare irrespective of its resource level. This simpler game could also represent contests in firms or political parties. In a firm, two groups may invest resources to convince the CEO that it is their product that should be developed and/or marketed, and the winning group can then use all remaining resources to develop, produce and market this product. In a political party, two politicians may collect campaign contributions in the primaries to become their party's candidate, and the winner can then exhaust the contribution potential of all donors supporting this party in the main electoral race.

Appendix: Proofs

Proof of Lemma 1: It directly follows from the requirement $f_i(r_i) \in [0, r_i]$ that $f_1(0) = f_2(0) = 0$. Together with the required monotonicity of $f_i(r_i)$, this implies that $f_1(.)$ and $f_2(.)$ must be non-decreasing. We prove $\lambda f_1(1) = f_2(1)$ by contradiction. Suppose $\lambda_i f_i(1) > \lambda_j f_j(1)$. For $r_i = 1$, player i is then better off by deviating and playing $b_i = \frac{f_j(1)\lambda_j}{\lambda_i} < f_i(1)$, as this increases the winner's payoff while i still wins with probability one. Hence it must hold in any monotone equilibrium that $\lambda_i f_i(1) = \lambda_j f_j(1)$. ■

Proof of Lemma 2: The system of the differential equations (2) and (4), which is defined for $y \in A \subseteq \left[0, \min\left\{\frac{1}{\lambda}, 1\right\}\right]$, characterizes mutual best responses. The terms in the square brackets on the left-hand sides of (2) and (4) are the same, which implies $\left(\ln\left[f_2^{-1}(\lambda y)\right]\right)' = \frac{\beta}{\lambda}\left(\ln\left[f_1^{-1}(y)\right]\right)'$ and, consequently,

$$f_2^{-1}(\lambda y) = K_0 f_1^{-1}(y)^{\frac{\beta}{\lambda}}, \tag{14}$$

where K_0 is a constant. Substituting this expression into (2) and (4), we obtain two independent differential equations:

$$-\beta f_2^{-1}(\lambda y) + \left[\beta\left(K_0^{-\frac{\lambda}{\beta}} f_2^{-1}(\lambda y)^{\frac{\lambda}{\beta}} - y\right) + f_2^{-1}(\lambda y) - \lambda y\right]\frac{df_2^{-1}(\lambda y)}{dy} = 0 \tag{15}$$

$$-f_1^{-1}(y) + \left[\beta(f_1^{-1}(y) - y) + K_0 f_1^{-1}(y)^{\frac{\beta}{\lambda}} - \lambda y\right]\frac{df_1^{-1}(y)}{\lambda dy} = 0 \tag{16}$$

We rename the variable $y = \frac{z}{\lambda}$ in (15) to obtain

$$-\beta f_2^{-1}(z) + \left[\beta\left(K_0^{-\frac{\lambda}{\beta}} f_2^{-1}(z)^{\frac{\lambda}{\beta}} - \frac{z}{\lambda}\right) + f_2^{-1}(z) - z\right]\frac{\lambda df_2^{-1}(z)}{dz} = 0, \tag{17}$$

where $z \in \lambda A$. After rewriting (16) and (17) using $y = f_1(r_1)$ and $z = f_2(r_2)$, and rearranging terms, we get

$$\lambda\frac{df_1(r_1)}{dr_1} = \beta + K_0 r_1^{\frac{\beta}{\lambda}-1} - (\lambda+\beta)\frac{f_1(r_1)}{r_1} \tag{18}$$

$$\beta\frac{df_2(r_2)}{dr_2} = \lambda + \lambda\beta K_0^{-\frac{\lambda}{\beta}} r_2^{\frac{\lambda}{\beta}-1} - (\lambda+\beta)\frac{f_2(r_2)}{r_2}. \tag{19}$$

Note that $r_1 \in f_1^{-1}(A)$ in (18) and $r_2 \in f_2^{-1}(\lambda A)$ in (19). Equations (5) and (6) are the solution to (18) and (19). ■

Proof of Corollary 1: Equations (5) and (6) satisfy $f_1(0) = f_2(0) = 0$ only if $K_1 = K_2 = 0$, and then $\lambda f_1(1) = f_2(1)$ only if $K_0 = 1$. Inserting $K_0 = 1$ and $K_1 = K_2 = 0$ into (5) and (6) gives (7) and (8). ■

Proof of Lemma 3: Given player 1's strategy characterized in the first statement, player 2's best response lower than λc_l follows from inserting $f_1(r_1) = r_1$ into condition (4), which then reduces to $f_2^{-1}(\lambda y) = 2\lambda y$, implying $f_2(r_2) = \frac{r_2}{2}$.

Given player 2's strategy characterized in the second statement, it follows that $\frac{\partial u_1(y;r_1)}{\partial y} = 2\lambda[\beta_1(r_1-2y)+\beta_2\lambda y]$ for $r_1 \le c_l$, which is positive since $y \le r_1$ and $\beta \le \lambda$. Hence it is optimal for player 1 to bid all resources whenever $r_1 \le c_l$. ■

Proof of Proposition 1: It follows from Lemma 3 that $f_1(r_1) = r_1$ is player 1's best response. It follows from Lemma 3 that $f_2(r_2) = \frac{r_2}{2}$ is player 2's best response for $r_2 \in [0, 2\lambda]$, and from Lemma 1 and $f_1(1) = 1$ that player 2 should bid $f_2(r_2) \le \lambda$ for all r_2. Hence player 2's best response is $f_2(r_2) = \min\left\{\frac{r_2}{2}, \lambda\right\}$. ■

Proof of Lemma 4: Evaluate (14) at $y = c_l$ to get $K_0 = 2\lambda c_l^{1-\frac{\beta}{\lambda}}$. Then substitute K_0 into (5) and (6), evaluate (5) at $r_1 = c_l$ and (6) at $r_2 = 2\lambda c_l$, and use boundary condition (9) to get

$$K_1 = \left(\frac{\lambda}{\beta+2\lambda} - \frac{\lambda}{2\beta+\lambda}\right) 2c_l^{2+\frac{\beta}{\lambda}} \tag{20}$$

$$K_2 = \left(\frac{\lambda}{\beta+2\lambda} - \frac{\lambda}{2\beta+\lambda}\right) (2\lambda c_l)^{2+\frac{\lambda}{\beta}}. \tag{21}$$

Then plug K_0, K_1 and K_2 into (5) and (6) to obtain (10) and (11).

It follows from the definition of $h(x)$ that $h'(x) > 0$, $h''(x) < 0$ and $h(1) = h'(1) = 1$; and from (10) and (11) that $f_1'(r_1) = h'\left(\frac{r_1}{c_l}\right)$, $f_2'(r_2) = \frac{1}{2}h'\left(\left(\frac{r_2}{2\lambda c_l}\right)^{\frac{\lambda}{\beta}}\right)\frac{\lambda}{\beta}\left(\frac{r_2}{2\lambda c_l}\right)^{\frac{\lambda-\beta}{\beta}}$ and $f_1''(r_1) = \frac{1}{c_l}h''\left(\frac{r_1}{c_l}\right)$. Consequently, $f_1'(r_1) > 0$, $f_1'(c_l) = 1$, $f_1''(r_1) < 0$ and $f_2'(r_2) > 0$. ■

Proof of Proposition 2: To avoid confusion, we denote the strategies described by (10) and (11) by $\tilde{f}_1(r_1)$ and $\tilde{f}_2(r_2)$, respectively.

First, we prove that $c_l < 1$ and $2\lambda c_l < 1$. Note that $2\lambda c_l = \left(\frac{1}{2\lambda}\right)^{\frac{\beta}{\lambda-\beta}}$, where $\frac{1}{2\lambda} < 1$ and $\frac{\beta}{\lambda-\beta} > 0$ since $\max\{\frac{1}{2}, \beta\} < \lambda$. Hence $2\lambda c_l < 1$. It follows from $2\lambda c_l < 1$ and $\frac{1}{2} < \lambda$ that $c_l < 1$.

Second, we prove that $f_i(r_i) \le r_i$ for all $r_i \in [0,1]$ and $i \in \{1,2\}$. It is straightforward that $f_1(r_1) = r_1 \le r_1$ for all $r_1 \in [0, c_l]$, and it holds that $f_1(r_1) \le r_1$ for all $r_1 \in (c_l, 1]$ since $\tilde{f}_1'(c_l) = 1$ and $\tilde{f}_1''(r_1) < 0$ for $r_1 > c_l$. It is also straightforward that $f_2(r_2) = \frac{r_2}{2} \le r_2$ for all $r_2 \in [0, 2\lambda c_l]$. Hence we only need to identify the region in the parameter space in which $\tilde{f}_2(r_2) \le r_2$ for all $r_2 \in (2\lambda c_l, 1]$ when $c_l = (2\lambda)^{\frac{\lambda}{\beta-\lambda}}$, or, equivalently, $h\left(\omega^{\frac{\lambda}{\beta}}\right) \le 2\omega$ for all $\omega \equiv \frac{r_2}{2\lambda c_l} \in \left(1, \frac{1}{2\lambda c_l}\right] = \left(1, (2\lambda)^{\frac{\beta}{\lambda-\beta}}\right]$. Using the definition of $h(x)$, $h\left(\omega^{\frac{\lambda}{\beta}}\right) \leq 2\omega$ can be rewritten as

$$\frac{\beta}{\beta+2\lambda}\omega^{\frac{\beta+2\lambda}{\beta}} - \frac{4\beta}{2\beta+\lambda}\omega^{\frac{2\beta+\lambda}{\beta}} \leq \frac{2\lambda(\lambda-\beta)}{(\beta+2\lambda)(2\beta+\lambda)}. \tag{22}$$

The first derivative of the left-hand side of (22) is zero only when $\omega = 4^{\frac{\beta}{\lambda-\beta}}$, and the second derivative of the left-hand side evaluated at $\omega = 4^{\frac{\beta}{\lambda-\beta}}$ is strictly positive. Hence the left-hand side of (22) must be U-shaped with respect to ω. Thus, since the resource constraint is not violated at $r_2 = 2\lambda c_l$, we only need to verify that it is not violated at the top, i.e., at $r_2 = 1$. We therefore substitute $r_2 = 1$ and $c_l = (2\lambda)^{\frac{\lambda}{\beta-\lambda}}$ into $\tilde{f}_2(r_2) \leq r_2$ and rearrange to get $\lambda \leq \Lambda(\beta, \lambda)$.

Third, we prove that each player's equilibrium strategy is their global best response against their opponent's equilibrium strategy. We start with player 1. It directly follows from Lemma 3 that $f_1(r_1) = r_1$ is player 1's best response for $r_1 \leq c_l$. (Note that a deviation to some $y > c_l$ is not feasible in this case.) Now suppose $r_1 > c_l$. We know from section 2 and Lemma 4 that $f_1(r_1) = \tilde{f}_1(r_1)$ is player 1's best response above c_l. Hence we only need to show that player 1 has no incentive to bid some $y \leq c_l$. When bidding some $y \leq c_l$, the payoff of player 1 would be $u_1(y; r_1) = \int_0^{2\lambda y} \left[\beta_1(r_1 - y) + \beta_2 \frac{r_2}{2}\right] dr_2$. The first derivative is

$$\frac{\partial u_1(y;r_1)}{\partial y} = [\beta_1(r_1 - y) + \beta_2 \lambda y]2\lambda - \beta_1 2\lambda y = \frac{2\lambda}{\beta_2}(\beta(r_1 - y) + (\lambda - \beta)y), \tag{23}$$

and it must be positive since $\beta < \lambda$ and $y \leq c_l < r_1$. Hence player 1 has an incentive to increase his bid whenever $y \in [0, c_l]$ and $r_1 > c_l$.

We now turn to player 2. Given player 1's equilibrium strategy, the payoff of player 2 when bidding y is

$$u_2(y; r_2) = \begin{cases} \int_0^{\frac{y}{\lambda}} \beta_2(r_2 - y)dr_1 & \text{for} \quad y \leq \min\{\lambda c_l, r_2\} \\ \beta_1 \int_{c_l}^{\tilde{f}_1^{-1}(\frac{y}{\lambda})}(r_1 - \tilde{f}_1(r_1))dr_1 + \beta_2 \int_0^{\tilde{f}_1^{-1}(\frac{y}{\lambda})}(r_2 - y)dr_1 & \text{for} \quad \lambda c_l \leq y \leq r_2. \end{cases} \tag{24}$$

Suppose $r_2 \leq 2\lambda c_l$. We know from Lemma 3 that player 2's optimal bid less than $\min\{\lambda c_l, r_2\}$ is $y = \frac{r_2}{2}$. Hence we only need to show that player 1 has no incentive to bid some $y \in [\lambda c_l, r_2]$.

For $y \in [\lambda c_l, r_2]$, it follows from (24) that

$$\frac{\partial u_2(y;r_2)}{\partial y} = \left[\beta_1 \left(\tilde{f}_1^{-1}\left(\frac{y}{\lambda}\right) - \frac{y}{\lambda}\right) + \beta_2(r_2 - y)\right] \frac{d\tilde{f}_1^{-1}\left(\frac{y}{\lambda}\right)}{dy} - \beta_2 \tilde{f}_1^{-1}\left(\frac{y}{\lambda}\right). \tag{25}$$

By construction of $\tilde{f}_2(r_2)$, this derivative is zero when $r_2 = \tilde{f}_2^{-1}(y)$. Since $r_2 \leq 2\lambda c_l \leq \tilde{f}_2^{-1}(y)$ and $\frac{d\tilde{f}_1^{-1}\left(\frac{y}{\lambda}\right)}{dy} > 0$, $\frac{\partial u_2(y;r_2)}{\partial y}$ must be negative. Hence player 2 has an incentive to reduce his bid whenever $y \in [\lambda c_l, r_2]$ and $r_2 \leq 2\lambda c_l$. Now suppose $r_2 > 2\lambda c_l$. We know from section 2 and Lemma 4 that $f_2(r_2) = \tilde{f}_2(r_2)$ is player 2's best response above λc_l. Hence we only need to show that player 1 has no incentive to bid some $y \leq \lambda c_l$. For $y \leq \lambda c_l$, it follows from (24) that

$$\frac{\partial u_2(y;r_2)}{\partial y} = \frac{\beta_2}{\lambda}(r_2 - 2y), \tag{26}$$

which must be positive since $r_2 \geq 2\lambda c_l$ and $y \leq \lambda c_l$. Hence player 2 has an incentive to increase his bid whenever $y \leq \lambda c_l$ and $r_2 > 2\lambda c_l$.

Finally, we prove that $\max\left\{\beta, \frac{1}{2}\right\} < \lambda \leq \Lambda(\beta,\lambda)$ implies $\Lambda(\beta,\lambda) > 2$. It can be shown that $\frac{h(x)}{x}$ is decreasing whenever $x > 1$. Hence $\Lambda(\beta,\lambda) = \left[\frac{h(x)}{x}\right]^{-1}$ increases as $x = (2\lambda)^{\frac{\lambda}{\lambda-\beta}} > 1$ increases. Since $\frac{\partial x}{\partial \lambda} > 0$ and $\frac{\partial x}{\partial \beta} > 0$ whenever $\lambda \geq \beta$, the chain rule implies that $\frac{\partial \Lambda(\beta,\lambda)}{\partial \lambda} = \frac{\partial \Lambda(\beta,\lambda)}{\partial x}\frac{\partial x}{\partial \lambda} > 0$ and $\frac{\partial \Lambda(\beta,\lambda)}{\partial \beta} = \frac{\partial \Lambda(\beta,\lambda)}{\partial x}\frac{\partial x}{\partial \beta} > 0$ whenever $\lambda \geq \beta$. Thus, in the set defined by $\lambda \leq \Lambda(\beta,\lambda)$, $\Lambda(\beta,\lambda)$ is smallest at the boundary characterized by $\lambda = \Lambda(\beta,\lambda)$. Now, we look for the point at which the level curve $\lambda = \Lambda(\beta,\lambda)$ intersects with $\lambda = \beta$. It can be shown that $\lim_{\lambda \to \beta^+} \Lambda(\beta,\lambda) = \left(\frac{1}{3} + \frac{1}{3\beta}\right)^{-1} = \frac{3\lambda}{\lambda+1}$ since $\lambda > \frac{1}{2}$ and $\lambda \geq \beta$. At the intersection it must hold that $\lim_{\lambda \to \beta^+} \Lambda(\beta,\lambda) = \frac{3\lambda}{\lambda+1}$ and $\Lambda(\beta,\lambda) = \lambda$, which requires $\lambda = 2$. Therefore $\max\left\{\beta, \frac{1}{2}\right\} < \lambda \leq \Lambda(\beta,\lambda)$ implies $\Lambda(\beta,\lambda) > 2$.

Proof of Proposition 3: We again denote the strategies described by (10) and (11) by $\tilde{f}_1(r_1)$ and $\tilde{f}_2(r_2)$, respectively.

First, we derive the thresholds c_l and c_h, and prove the uniqueness of c_l, $2\lambda c_l < 1$ and $c_l < c_h < 1$. Boundary condition (13) and Lemma 4 imply

$$\lambda c_l h\left(\frac{c_h}{c_l}\right) = \lambda c_l h\left(\left(\frac{1}{2\lambda c_l}\right)^{\frac{\lambda}{\beta}}\right) = 1. \tag{27}$$

Since $h(\cdot)$ is strictly increasing, the first equality implies $c_h = (2\lambda)^{-\frac{\lambda}{\beta}} c_l^{\frac{\beta-\lambda}{\beta}}$. The second equality gives the implicit definition of c_l. To prove existence and uniqueness of c_l, we rewrite this second equality as $x = \phi(x)$, where $x = 2\lambda c_l$ and $\phi(x) = 2\left(h\left(x^{-\frac{\lambda}{\beta}}\right)\right)^{-1}$. Note that $\phi(\cdot)$ is not well-defined when $x = 0$, and that $\phi : (0,1] \to (0,2]$ is a continuous and increasing function with $\lim_{x\to 0^+} \phi(\cdot) = 0$ and $\phi(1) = 2$. Suppose condition (12) is violated and let $\varepsilon = (2\lambda)^{-\frac{\beta}{\lambda-\beta}} < 1$. Then it can be shown that $\phi(\varepsilon) < \varepsilon$. Hence $\phi(\cdot)$ has a fixed point $x^* \in (0,1)$ satisfying $x^* = \phi(x^*)$ whenever condition (12) is violated. Moreover, this fixed point is unique since $\phi'(x^*) > 1$ whenever $x^* = \phi(x^*)$. Hence there exists a unique c_l, and it must hold that $2\lambda c_l < 1$. It follows from $2\lambda c_l < 1$ that $1 < (2\lambda c_l)^{-\frac{\lambda}{\beta}}$ and, consequently, $c_l < c_h$; and it follows from $\lambda > \max\left\{\frac{1}{2}, \beta\right\}$ that $c_l < (2\lambda c_l)^{\frac{\lambda}{\beta}}$ and, consequently, $c_2 < 1$.

Second, we prove that $f_i(r_i) \leq r_i$ for all $r_i \in [0,1]$ and $i \in \{1,2\}$. It is straightforward that $f_1(r_1) = r_1 \leq r_1$ for all $r_1 \in [0, c_l]$, and it holds that $f_1(r_1) \leq r_1$ for all $r_1 \in (c_l, 1]$ since $\tilde{f}_1'(c_l) = 1$, $\tilde{f}_1''(r_1) < 0$ for $r_1 \in (c_l, c_h]$ and $\tilde{f}_1(r_1) = \tilde{f}_1(c_h)$ for $r_1 \in (c_h, 1]$. It is also straightforward that $f_2(r_2) = \frac{r_2}{2} \leq r_2$ for all $r_2 \in [0, 2\lambda c_l]$; and c_l is chosen such that player 2's resource constraint is binding when $r_2 = 1$. It can be shown along the same lines as in the proof of that player 2 also bids strictly less then his resources for $r_2 \in (2\lambda c_l, 1)$.

Third, we prove that each player's equilibrium strategy is their global best response against their opponent's equilibrium strategy. The corresponding part of the proof of Proposition 2 applies here as well, as the arguments do not assume a value for c_l. Hence we only need to show that player 1 has no incentive to deviate for $r_1 \in (c_h, 1]$. Therefore, suppose $r_1 \in (c_h, 1]$ and consider a bid $y \in \left[c_l, \frac{1}{\lambda}\right]$. Differentiating player 1's payoff with respect to y then yields

$$\frac{\partial u_1(y; r_1)}{\partial y} = \left[\beta_1(r_1 - y) + \beta_2\left(\tilde{f}_2^{-1}(\lambda y) - \lambda y\right)\right] \frac{d\tilde{f}_2^{-1}(\lambda y)}{dy} - \beta_1 \tilde{f}_2^{-1}(\lambda y). \tag{28}$$

By construction of $\tilde{f}_1(r_1)$, this derivative is zero when $r_1 = \tilde{f}_1^{-1}(y) \leq c_h$. Thus, since $\frac{d\tilde{f}_2^{-1}(\lambda y)}{dy} > 0$, $\frac{\partial u_1(y, r_1)}{\partial y}$ must be positive when $r_1 \geq c_h$, implying that in this case player 1 can profitably increase his bid y. ■

Proof of Proposition 4: Results for $\lambda \in [\frac{1}{2}, 2]$ directly follow from Corollary 1 and our discussion thereafter. Results for $\lambda < \frac{1}{2}$ directly follow from Proposition 1. Results for $\lambda > 2$

also follow from Proposition 1 after renaming player 1 as player 2, and vice versa. ■

Proof of Proposition 5: We first prove the last statement comparing effective bids. It directly follows from the equilibrium strategies described in Proposition 1 that $\lambda f_1(r) < f_2(r)$ for all $r \in (0,1)$ if $\lambda < \frac{1}{2}$, and that $\lambda f_1(r) = f_2(r)$ for all $r \in (0,1)$ if $\lambda = \frac{1}{2}$. To prove that $\lambda f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\lambda > \frac{1}{2}$, we first consider the case in which $\frac{1}{2} < \lambda \leq \Lambda(\beta, \lambda)$. Proposition 2 characterizes the equilibrium strategies for this case. Consider a particular $\widetilde{y} \in A$ such that $\widetilde{y} = \lambda f_1(r_1) = f_2(r_2)$. We need to show that $r_2 > r_1$. For $\widetilde{y} \leq \lambda c_l$, it follows from $f_1(r_1) = r_1$ for $r_1 \in [0, c_l]$, $f_2(r_2) = \frac{r_2}{2}$ for $r_2 \in [0, 2\lambda c_l]$, and $\lambda > \frac{1}{2}$ that $r_2 \geq r_1$ must hold. For $\widetilde{y} > \lambda c_l$, it follows from (10) and (11) and $h'(x) > 0$ that $\lambda f_1(r_1) = f_2(r_2)$ requires $r_1 = c_l \left(\frac{r_2}{2\lambda c_l}\right)^{\frac{\lambda}{\beta}} = r_2^{\frac{\lambda}{\beta}}$, where the second equality follows from $c_l = (2\lambda)^{\frac{\lambda}{\beta - \lambda}}$. Since $\beta < \lambda$ and $r_i \in (0,1)$ for $i = 1, 2$, $r_1 = r_2^{\frac{\lambda}{\beta}}$ implies $r_2 > r_1$. Hence $\lambda f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\frac{1}{2} < \lambda \leq \Lambda(\beta, \lambda)$. It remains to consider the case in which $\lambda > \Lambda(\beta, \lambda)$. Proposition 3 characterizes the equilibrium strategies for this case. Using the same strategy as above, we can prove that $\lambda f_1(r) > f_2(r)$ for all $r \in (0, c_h)$. Moreover, it directly follows from $f_1(r_1) = \frac{1}{\lambda}$ for $r_1 \geq c_h$ and $f_2(r_2) \leq r_2$ that $\lambda f_1(r) > f_2(r)$ must also hold for all $r \in [c_h, 1)$.

We next prove the two statements comparing real bids. For $\lambda \leq \frac{1}{2}$, it directly follows from the equilibrium strategies described in Proposition 1 that $f_1(r) > f_2(r)$ for all $r \in (0,1]$. We have shown above that $\lambda f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\lambda > \frac{1}{2}$. Hence it must hold that $f_1(r) > f_2(r)$ for all $r \in (0,1)$ if $\lambda \in \left(\frac{1}{2}, 1\right]$. For $\lambda > 1$, Propositions 2 and 3 imply $f_1(r) > f_2(r)$ for $r \in (0, 2\lambda c_l]$. Further it follows from Lemma 1 that $f_1(1) < f_2(1)$ if $\lambda > 1$. Hence the continuity of $f_1(r_1)$ and $f_2(r_2)$ and the intermediate value theorem imply that there must exists an odd number of thresholds $\hat{r}$ in the interval $(2\lambda c_l, 1)$ that satisfy $f_1(\hat{r}) = f_2(\hat{r})$. It holds that $f_1(r) > f_2(r)$ for all r below the lowest threshold and $f_1(r) < f_2(r)$ for all r above the highest threshold. ■

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