Some Optimal Dividend Problems in a Markov-modulated Risk Model

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Abstract

In this paper, we derive some results on the dividend payments prior to ruin in a Markov-modulated risk model in which the claim inter-arrivals, claim sizes and premiums are influenced by an external Markovian environment process. A system of integro-differential equations with boundary conditions satisfied by the n-th moment of present value of the total dividend payments prior to ruin, given the initial environment state, is derived and solved. We show that both the probabilities that the surplus process attains a given level from the initial surplus without first falling below zero and the Laplace transforms of the time that the surplus process first hits a barrier without ruin occuring can be expressed in term of the solution of the above mentioned system of integro-differential equations. In the two-state model, explicit solutions to the integro-differential equations are obtained when both claim size distributions are exponentially distributed. Finally, a numerical example and the comparison with the results obtained from the associated averaged compound Poisson risk model are also given.

Keywords: Markov-modulated processes; Dividend barrier; Present value of dividend payments; Time of ruin; Integro-differential equation; Laplace transform.

1 Introduction

Consider a risk model in continuous time. Denote by $\{I(t);\, t \geq 0\}$ the external environment process, which influences the frequencies of the claims, the distributions of the claims, and the premiums. As pointed out by Asmussen (1989), in health insurance, sojourns of $\{I(t);\, t \geq 0\}$ could be certain types of epidemics or, in automobile insurance, these could be weather types (for example, icy, foggy,...). Suppose that $\{I(t);\, t \geq 0\}$ is a homogeneous, irreducible and recurrent Markov process with finite state space $J = \{1, 2, \ldots, m\}$. Denote the intensity matrix of $\{I(t);\, t \geq 0\}$ by

$$\Lambda = (\alpha_{ij})_{i,j=1}^{m}\,, \qquad \alpha_{ii} := -\alpha_i = -\sum_{j \neq i} \alpha_{ij}\,, \qquad i \in J\,.$$

The transition probability matrix of the embedded Markov chain is then given by

$$Q = (p_{ij})_{i,j=1}^{m}\,, \qquad p_{ij} = \begin{cases} 0, & i = j, \\ \frac{\alpha_{ij}}{\alpha_i}, & i \neq j, \end{cases} \qquad i,\ j \in J\,. \tag{1}$$

Further assume that at time t claims occur according to a Poisson process with constant intensity rate $\lambda_i \in \mathbb{R}^+$, when $I(t) = i$ and the corresponding claim amounts have distributions $F_i(x)$, with density function $f_i(x)$ and finite mean μ_i ($i \in J$). Moreover, we assume that premiums are received continuously at a positive constant rate c_i during any time interval when the environment process remains in state i. Denote by W_n and X_n, respectively, the arrival time and the amount of the n-th claim, and by $T_n = W_n - W_{n-1}$ the inter-arrival time of the $(n-1)$-th claim and the n-th claim, with $W_0 = X_0 = T_0 = 0$.

Let J_n be the state of the process $\{I(t);\, t \geq 0\}$ at the arrival of the n-th claim, i.e.,

$$J_n = I(W_n)\,, \qquad n \in \mathbb{N}\,,$$

and $\pi = (\pi_1, \ldots, \pi_m)$ be its unique stationary probability distribution. Reinhard (1984) shows that

$$\pi_i = \frac{\frac{\lambda_i \eta_i}{\alpha_i}}{\sum_{k=1}^{m} \frac{\lambda_k \eta_k}{\alpha_k}}\,, \qquad i \in J\,, \tag{2}$$

where $\eta = (\eta_1, \ldots, \eta_m)$ is the unique stationary probability distribution of the embedded Markov chain of process I, such that $\eta Q = 0$, in which transition probability matrix Q is given by (1).

Suppose that the sequences of random variables $\{X_n;\, n \geq 0\}$ and $\{T_n;\, n \geq 0\}$ are conditionally independent given $\{I(t);\, t \geq 0\}$.

Now define $N(t) = \sup\{n \in \mathbb{N} \,\big|\, \sum_{i=1}^{n} T_i \leq t\}$ as the number of claims that have occured before time t. The counting process $\{N(t);\, t \geq 0\}$ is called a Markov-modulated Poisson process, which is a special case of Cox processes. It also can be seen as a Poisson process with parameters driven by an external environment process. The corresponding surplus process $\{U(t);\, t \geq 0\}$ is given by

$$U(t) = u + C(t) - \sum_{n=1}^{N(t)} X_n \,, \qquad t \geq 0 \,, \tag{3}$$

where $C(t)$ denotes the aggregate premium received during interval $(0, t]$ and $u(\geq 0)$ is the initial reserve. Let Z_n be the time at which the n-th transition of the environment process occurs and I_n be the state of the environment after its n-th transition. Reinhard (1984) shows that

$$C(t) = \sum_{k=1}^{N_e(t)} c_{I_{k-1}}(Z_k - Z_{k-1}) + c_{I_{N_e(t)}}(t - T_{N_e(t)}) \,, \qquad t \geq 0 \,,$$

where $N_e(t) = \sup\{n \in \mathbb{N} : Z_n \leq t\}$. We also assume that the positive loading condition satisfies [see Reinhard (1984)], i.e.,

$$d = \sum_{i=1}^{m} \pi_i \Big(\frac{c_i}{\lambda_i} - \mu_i \Big) > 0 \,, \tag{4}$$

where π_i is given by (2).

The definition of this Markov-modulated risk model using an environmental Markov chain $\{J(t);\, t \geq 0\}$ is first given in Asmussen (1989), where the process J models the random environment in which an insurance business is assumed to be an irreducible continuous time Markov chain, with finite state space.

Models of this type have also been investigated by some authors. Reinhard (1984) and Lu and Li (2005) consider the probability of ruin in a class of Markov-modulated risk models. Wu (1999) develops generalized bounds and Schmidli (1997) studies the estimation of the Lundberg coefficient for the probability of ruin of this model. Bäuerle (1996) also considers the expected ruin time of the same model. The severity of ruin, in the particular case where one has two possible states for the environmental process, is studied by Snoussi (2002) and Lu (2005).

In this paper, we consider the above defined surplus process modified by the payment of dividends. When the surplus exceeds a constant barrier $b \geq u$, dividends are paid continuously so the surplus stays at the level b until a new claim occurs. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the barrier strategy above and define $T_b = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin. Let $\delta > 0$ be the force of interest for valuation and define

$$D_{u,b} = \int_0^{T_b} e^{-\delta\, t} dD(t), \qquad 0 \leq u \leq b,$$

to be the present value of all dividends until time of ruin T_b, where $D(t)$ is the aggregate dividends paid by time t. Define the mean of $D_{u,b}$, given that the initial environment state is i, by

$$V_i(u;\, b) = E[D_{u,\,b} | I(0) = i], \qquad 0 \leq u \leq b,\; i \in J.$$

Then the expected present value of the total dividend payments until ruin in the stationary case is

$$V(u; b) = \sum_{i=1}^{m} \eta_i\, V_i(u; b)\,, \qquad 0 \leq u \leq b.$$

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies have been studied in a number of papers and books. These references include Bühlmann (1970), Segerdahl (1970), Gerber (1972, 1979, 1981), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Lin et al. (2003), Dickson and Waters (2004), Li and Garrido (2004), Albrecher et al. (2005), and Li and Dickson (2005). The main focus is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. For the risk process modeled by a Brownian motion, detailed analysis can be found in Gerber and Shiu (2004).

2 Integro-differential equations with boundary conditions

We now derive a system of integro-differential equations for $V_i(u; b)$, $i \in J$. Considering a small time interval $[0, h]$, with $h > 0$, there are four possible events regarding to the occurrence of the claim and the change of the environment:

(i) no claim and no change of environment occur in $[0,\, h]$,

(ii) a claim occurs in $[0,\, h]$ (it can either cause the ruin or not),

(iii) the environment changes in $[0,\, h]$, and

(iv) two or more events occur in $[0,\, h]$.

Conditioning on the event occurs in the interval $[0,\, h]$, we have

$$\begin{aligned} V_i(u;b) \;=\; & (1-\alpha_i\, h - \lambda_i\, h)\, e^{-\delta\, h}\, V_i(u+c_i\, h;\, b) \\ & +\lambda_i\, h\, e^{-\delta\, h} \int_0^{u+c_i h} V_i(u+c_i\, h - x;\, b)\, dF_i(x) \\ & +\alpha_i\, h\, e^{-\delta\, h} \sum_{k=1}^{m} p_{ik}\, V_k(u+c_i\, h;\, b) + o(h)\,, \qquad 0 \le u < b\,,\ i \in J\,, \end{aligned}$$

where $o(h)/h \to 0$, when $h \to 0$. Since $e^{-\delta\, h} = 1 - \delta\, h + o(h)$, we then get

$$\begin{aligned} V_i(u;\, b) = {} & [1-(\alpha_i+\lambda_i+\delta)\, h]\, V_i(u+c_i\, h;\, b) + \lambda_i\, h \int_0^{u+c_i h} V_i(u+c_i\, h - x;\, b)\, dF_i(x) \\ & + \alpha_i\, h \sum_{k=1}^{m} p_{ik}\, V_k(u+c_i\, h;\, b) + o(h)\,, \qquad 0 \le u < b\,,\ i \in J\,. \end{aligned} \tag{5}$$

Equation (5) can be rewritten for $0 \le u < b$ as

$$\begin{aligned} \frac{V_i(u+c_i\, h;\, b) - V_i(u;\, b)}{h} = {} & (\alpha_i+\lambda_i+\delta)\, V_i(u+c_i\, h;\, b) \\ & - \lambda_i \int_0^{u+c_i h} V_i(u+c_i\, h - x;\, b)\, dF_i(x) \\ & - \alpha_i \sum_{k=1}^{m} p_{ik}\, V_k(u+c_i\, h;\, b) + \frac{o(h)}{h}\,, \qquad i \in J\,. \end{aligned} \tag{6}$$

Letting $h \to 0$ in (6), we get a system of integro-differential equations satisfied by $V_i(u;\, b)$:

$$\begin{aligned} c_i\, V_i'(u;\, b) = {} & (\alpha_i+\lambda_i+\delta)\, V_i(u;\, b) - \lambda_i \int_0^{u} V_i(u-x;\, b)\, dF_i(x) \\ & - \alpha_i \sum_{k=1}^{m} p_{ik}\, V_k(u;\, b)\,, \qquad 0 \le u < b\,,\ i \in J\,. \end{aligned} \tag{7}$$

For the case $u = b$, similarly, we condition on the event occurs in $[0, h]$ to obtain

$$\begin{aligned} V_i(b;b) &= (1-\alpha_i\,h-\lambda_i\,h)\left[e^{-\delta\,h}V_i(b;\,b)+c_i\int_0^h e^{-\delta s}ds\right] \\ &\quad +\lambda_i\,h\,e^{-\delta\,h}\left[\int_0^b V_i(b-x;\,b)\,dF_i(x)+\int_0^h c_i e^{\delta s}ds\right] \\ &\quad +\alpha_i\,h\,e^{-\delta\,h}\left[\sum_{k=1}^m p_{ik}\,V_k(b;\,b)+\int_0^h c_i e^{\delta s}ds\right]+o(h)\,, \qquad i\in J\,. \end{aligned}$$

Using the same arguments as the above gives for $i \in J$,

$$(\alpha_i+\lambda_i+\delta)V_i(b;\,b)-\lambda_i\int_0^b V_i(b-x;\,b)\,dF_i(x)-\alpha_i\sum_{k=1}^m p_{ik}\,V_k(b;\,b)=c_i\,. \qquad (8)$$

Setting $u = b$ in equation (7) and using equation (8) show that $V_i(u;\,b)$ satisfies the following boundary conditions:

$$V_i'(u;b)\Big|_{u=b}=1, \qquad i\in J. \qquad (9)$$

We remark that if $m = 1$, equations (7) and (9) can be found in Bühlmann (1970) and Gerber (1979).

Now letting $v_i(u), 0 \le u < \infty$, $i \in J$, be the solutions of the integro-differential equations, we get

$$\begin{aligned} c_i\,v_i'(u) &= (\alpha_i+\lambda_i+\delta)\,v_i(u)-\lambda_i\int_0^u v_i(u-x)\,dF_i(x) \\ &\quad -\alpha_i\sum_{k=1}^m p_{ik}\,v_k(u)\,, \qquad i\in J. \end{aligned} \qquad (10)$$

The solutions of (10) are uniquely determined by the initial conditions $v_i(0), i \in J$. Further for $j \in J$, let $v_{1,j}(u), v_{2,j}(u), \ldots, v_{m,j}(u)$ be the particular solutions of (10) with the following initial conditions

$$v_{i,j}(0)=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Then the general solutions of (10) are of the form

$$v_i(u)=\sum_{j=1}^m a_j\,v_{i,j}(u)\,, \qquad i\in J\,,$$

where $a_1, a_2, \ldots, a_m$ are any real numbers. It follows that the solutions to the integro-differential equations (7) with the boundary conditions (9) can be expressed as

$$V_i(u;\, b) = \sum_{j=1}^{m} a_j(b)\, v_{i,j}(u)\,, \qquad 0 \le u \le b,\ i \in J\,,$$

where $a_1(b), a_2(b), \ldots, a_m(b)$ are the solutions of the following system of linear equations

$$\sum_{j=1}^{m} a_j(b)\, v'_{i,j}(b) = 1\,, \qquad i \in J\,.$$

The particular solutions $v_{i,j}(u)$ will be analyzed in Section 6.

3 Moment generating function of $D_{u,b}$ and higher moment

In this section, we study the moment generating function of $D_{u,b}$, through which we can analyze the higher moment of the present value of all dividend payments prior to ruin. Define the moment generating function of $D_{u,b}$, given that the initial environment state is i, by

$$M_i(u, y;\, b) = E[e^{y\,D_{u,b}} | U(0) = u,\, I(0) = i], \qquad 0 \le u \le b,\ i \in J,$$

where y is such that $M_i(u, y;\, b)$ exists.

Similar arguments as in Section 2, we condition on the events which could occur in the small interval $[0, h]$,

$$\begin{aligned}
M_i(u, y;\, b) &= E\left[e^{y\,D_{u,b}} \middle| U(0) = u,\, I(0) = i\right] \\
&= (1 - \alpha_i h - \lambda_i h) M_i(u + c_i h, e^{-\delta h} y;\, b) \\
&= \lambda_i h \left[\int_0^{u+c_i h} M_i(u + c_i h - x, e^{-\delta h} y;\, b) dF_i(x) + \bar{F}_i(u + c_i h)\right] \\
&= \alpha_i h \sum_{k=1}^{m} p_{ik} M_k(u + c_i h, e^{-\delta h} y;\, b) + o(h)\,, \qquad 0 \le u < b,\ i \in J\,,
\end{aligned}$$

where $\bar{F}_i = 1 - F_i$ is the tail of the distribution function F_i.

Taylor's expansion gives

$$\begin{aligned} M_i(u+c_ih, e^{-\delta h}y;\, b) &= M_i(u,y;\, b) + c_ih\frac{\partial M_i(u,y;\, b)}{\partial u} \\ &\quad -\delta yh\frac{\partial M_i(u,y;\, b)}{\partial y} + o(h). \end{aligned} \tag{11}$$

Substituting (11) into the expression of $M_i(u,y;\, b)$, dividing both sides by h and letting $h \to 0$, we have

$$\begin{aligned} & c_i\frac{\partial M_i(u,y;\, b)}{\partial u} - \delta y\frac{\partial M_i(u,y;\, b)}{\partial y} - (\lambda_i+\alpha_i)M_i(u,y;\, b) \\ & +\lambda_i\left[\int_0^u M_i(u-x,y;\, b)dF_i(x) + \bar{F}_i(u)\right] \\ & +\alpha_i\sum_{k=1}^m p_{ik}M_k(u,y;\, b) = 0\,, \qquad 0 \le u < b,\ i \in J\,. \end{aligned} \tag{12}$$

For the case $u = b$,

$$\begin{aligned} M_i(b,y;\, b) &= (1-\alpha_ih-\lambda_ih)e^{y\,c_i\,h}M_i(b,e^{-\delta h}y;\, b) \\ &= \lambda_ih\,e^{y\,c_i\,h}\left[\int_0^b M_i(b-x,e^{-\delta h}y;\, b)dF_i(x) + \bar{F}_i(b)\right] \\ &= \alpha_ih\,e^{y\,c_i\,h}\sum_{k=1}^m p_{ik}M_k(b,e^{-\delta h}y;\, b) + o(h)\,, \qquad i \in J\,. \end{aligned}$$

Using Taylor's expansion, we have, for $i \in J$,

$$\begin{aligned} & \delta y\,\frac{\partial M_i(b,y;\, b)}{\partial y} + (\lambda_i+\alpha_i+c_iy)M_i(b,y\ \ b) \\ & = \lambda_i\left[\int_0^b M_i(b-x,y;\, b)dF_i(x) + \bar{F}_i(b)\right] + \alpha_i\sum_{k=1}^m p_{ik}M_k(b,y;\, b) = 0\,. \end{aligned}$$

Comparing these equations with the corresponding equations in (12) for $u = b$, we have the following boundary conditions

$$\left.\frac{\partial M_i(u,y;\, b)}{\partial u}\right|_{u=b} = yM_i(b,y;\, b), \qquad i \in J\,. \tag{13}$$

For $0 \le u \le b$ and $i = 1, 2, \ldots, m$, define

$$V_{i,n}(u;\, b) = E\left[D_{u,b}^n \middle| I(0) = i\right], \qquad n \in \mathbb{N},$$

to be the n-th moment of $D_{u,b}$, with $V_{i,0}(u;\,b) = 1$ and $V_{i,1}(u;\,b) = V_i(u;\,b)$. Substituting $M_i(u,y;\,b) = 1+\sum_{n=1}^{\infty}(y^n/n!)V_{i,n}(u;\,b)$ into (12) and comparing the coefficient of y^n yields the following integro-differential equations

$$\begin{aligned} c_i V'_{i,n}(u;\,b) &= (\alpha_i+\lambda_i+n\delta)V_{i,n}(u;\,b) - \lambda_i \int_0^u V_{i,n}(u-x;\,b)dF_i(x) \\ &\quad -\alpha_i \sum_{k=1}^{m} p_{ik} V_{k,n}(u;\,b), \qquad 0 \le u < b,\ i \in J\,. \end{aligned} \tag{14}$$

It follows from (13) that

$$V'_{i,n}(u;\,b)\Big|_{u=b} = nV_{i,n-1}(b;\,b), \qquad i \in J,\ n \in \mathbb{N}\,, \tag{15}$$

with $V_{i,0}(b;\,b) = 1$.

We remark that the way of solving the integro-differential equations (14) with boundary conditions (15) is the same as that of solving equations (7) with boundary conditions (9), and the only difference is to replace δ by $n\delta$.

4 The time to reach the dividend barrier

In this section, we consider how long it takes for the surplus process to reach the dividend barrier b from the initial surplus u without ruin occuring. We define τ_b to be the first time that the surplus reaches b without ruin having previously occured, and for $\delta > 0$, define

$$\mathcal{L}_i(u;\,b) = E\left[e^{-\delta\,\tau_b}\middle|U(0)=u, I(0)=i\right], \quad 0 \le u \le b,\, i \in J.$$

Here $\mathcal{L}_i(u;\,b)$ can be viewed as the expected present value of one dollar payable at time τ_b, given that the initial environment state is i, or, alternatively, it can be viewed as the Laplace transform of τ_b with respect to the parameter δ.

Theorem 1 $\mathcal{L}_i(u;\,b)$ satisfies the following integro-differential equations

$$\begin{aligned} c_i\,\mathcal{L}'_i(u;\,b) &= (\alpha_i+\lambda_i+\delta)\,\mathcal{L}_i(u;\,b) - \lambda_i \int_0^u \mathcal{L}_i(u-x;\,b)\,dF_i(x) \\ &\quad -\alpha_i \sum_{k=1}^{m} p_{ik}\,\mathcal{L}_k(u;\,b)\,, \qquad 0 \le u < b\,,\ i \in J\,. \end{aligned} \tag{16}$$

with the following boundary conditions:

$$\mathcal{L}_i(b;\,b) = 1, \qquad i \in J\,. \tag{17}$$

Proof: Using the same arguments as in deriving (7), we can prove that the integro-differential equations (16) holds. The boundary conditions (17) is from the fact that $\tau_b = 0$ and $E[e^{-\delta\tau_b}|U(0) = u, I(0) = i] = 1$ when $u = b$. □

The solutions to equations (16) with boundary conditions (17) are

$$\mathcal{L}_i(u;\, b) = \sum_{j=1}^{m} e_j(b)\, v_{i,j}(u)\,, \qquad 0 \le u \le b,\; i \in J\,,$$

where $e_1(b), e_2(b), \ldots, e_m(b)$ are the solutions of the following system of linear equations

$$\sum_{j=1}^{m} e_j(b) v_{i,j}(b) = 1\,, \qquad i \in J\,.$$

5 The probability of hitting the dividend barrier before ruin

For $b > u \ge 0$, define

$$\xi_i(u; b) = P\left\{ \sup_{0\le t\le T_\infty} U(t) < b,\; T_\infty < \infty \;\middle|\; U(0) = u,\; I(0) = i \right\}, \qquad i \in J\,,$$

to be the probability that ruin occurs from the initial surplus u without the surplus process reaching level b prior to ruin, given that the initial environment state is i, where T_∞ is the time of ruin of the risk model (3) without a barrier. Alternatively, $\xi_i(u; b)$ is the probability of ruin in the presence of an absorbing barrier at b, given that the initial environment state is i. Obviously, $\xi_i(u; b) = 0$, for $b \le u$.

Further define $\chi_i(u; b)$ to be the probability that the surplus process attains the given dividend barrier b from the initial surplus u without first falling below zero, given that the initial environment state is i. We have

$$\chi_i(u; b) = 1 - \xi_i(u;\, b)\,, \qquad i \in J\,,$$

since eventually either ruin occurs without the surplus process attaining b or the surplus attains level b.

Next, we show that $\chi_i(u; b)$ satisfies an integro-differential equation with certain boundary conditions.

Theorem 2 For $0 \le u < b$, and $i \in J$,

$$c_i\, \chi_i'(u;\, b) \;=\; (\lambda_i + \alpha_i)\chi_i(u;\, b) - \lambda_i \int_0^u \chi_i(u - x;\, b) dF_i(x) \\ -\alpha_i \sum_{k=1}^m p_{ik}\, \chi_k(u;\, b)\,, \tag{18}$$

with boundary condition

$$\chi_i(b;\, b) = 1\,. \tag{19}$$

Proof: For a small value $h > 0$, conditional on the event occurs in the interval $[0,\, h]$, we have for $0 \le u < b$,

$$\chi_i(u;\, b) = (1 - \alpha_i\, h - \lambda_i\, h)\, \chi_i(u + c_i\, h;\, b) + \lambda_i\, h \int_0^{u+c_i h} \chi_i(u + c_i\, h - x;\, b)\, dF_i(x) \\ + \alpha_i\, h \sum_{k=1}^m p_{ik}\, \chi_k(u + c_i\, h;\, b) + o(h), \qquad i \in J\,.$$

Subtracting $\chi_i(u;\, b)$ from both sides, dividing both sides by h, and letting $h \to 0$, we can prove that equation (18) holds. The boundary condition (19) is from the fact that the surplus hits b at the beginning without ruin occuring when $u = b$. $\square$

Let $v_{i,j}^0(u)$ be the solutions of the integro-differential equations (10) with $\delta = 0$. Then

$$\chi_i(u;\, b) = \sum_{j=1}^m h_j(b)\, v_{i,j}^0(u)\,, \qquad 0 \le u \le b,\ i \in J\,,$$

where $h_1(b), h_2(b), \ldots, h_m(b)$ are the solutions of the following system of equations

$$\sum_{j=1}^m h_j(b)\, v_{i,j}^0(u) = 1\,, \qquad i \in J\,.$$

We remark that when $m = 1$, Dickson and Gray (1984) has shown that $\chi(u;\, b)$, the probability that the surplus process attains the given dividend barrier b from the initial surplus u without first falling below zero in the classical risk model, can be expressed as

$$\chi(u;\, b) = \frac{1 - \Psi(u)}{1 - \Psi(b)}\,, \qquad 0 \le u \le b\,,$$

where $\Psi(u)$ is the probability of ruin of the risk model (3) for $m = 1$ without a barrier.

6 Laplace transforms

We now apply Laplace transforms to find the particular solutions $v_{i,j}(u)$ of the system of equations (10). Let $\hat{v}_{ij}$ and $\hat{f}_i$ be the Laplace transforms of $v_{i,j}$ and f_i, respectively, i.e.,

$$\hat{v}_{i,j}(s) = \int_0^\infty e^{-su} v_{i,j}(u) du \,, \qquad \hat{f}_i(s) = \int_0^\infty e^{-sx} f_i(x) dx \,, \qquad i, \, j \in J \,.$$

Taking Laplace transforms on both sides of equation (10) yields

$$\left[s - \frac{\lambda_i + \alpha_i}{c_i} + \frac{\lambda_i}{c_i} \hat{f}_i(s) \right] \hat{v}_{i,j}(s) + \frac{\alpha_i}{c_i} \sum_{k=1}^{m} p_{ik} v_{k,j}(s) = v_{i,j}(0) \,, \qquad i, \, j \in J \,,$$

with $v_{i,j}(0) = I(i = j)$, where $I(\cdot)$ denotes the indicator function, or in a matrix form

$$\mathbf{A}(s) \hat{\mathbf{v}}(s) = \mathbf{I} \,,$$

where

$$\mathbf{A}(s) = \begin{bmatrix} s - \frac{\lambda_1 [1 - \hat{f}_1(s)] + \alpha_1}{c_1} & & \\ & \ddots & \\ & & s - \frac{\lambda_m [1 - \hat{f}_m(s)] + \alpha_m}{c_m} \end{bmatrix} + \begin{bmatrix} \frac{\alpha_1}{c_1} & & \\ & \ddots & \\ & & \frac{\alpha_m}{c_m} \end{bmatrix} Q \,,$$

$\hat{\mathbf{v}}(s) = (\hat{v}_{i,j}(s))_{i,j=1}^m$, $\mathbf{I}$ is an identity matrix of m by m, and Q is given by (1), with $p_{ii} = 0$, for $i \in J$.

Then $\hat{\mathbf{v}}(s)$ can be solved as

$$\hat{\mathbf{v}}(s) = [\mathbf{A}(s)]^{-1} \,.$$

We remark that when the claim sizes are rationally distributed, each element of $\mathbf{A}(s)$ is a rational function, so is each element of $[\mathbf{A}(s)]^{-1}$, therefore, $v_{i,j}(u)$ can be obtained by inverting $\hat{v}_{i,j}(s)$ through partial fractions. This can be shown by the examples in the section that follows.

7 Illustrations for a two-state model

In this section, we derive explicit expressions for $v_{1,1}(u)$, $v_{2,1}(u)$ $v_{1,2}(u)$, and $v_{2,2}(u)$ under some special claim size distributions when $m = 2$, that is, $\{I(t); t \geq 0\}$ is a two-state Markov process, which reflects the random environmental effects due to "normal" vs. "abnormal", or "high risk" vs. "low risk" conditions. The unique stationary probability distribution π_i can be obtained from (2) as

$$\pi_i = \frac{\frac{\lambda_i}{\alpha_i}}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}}, \qquad i = 1, 2,$$

and the positive loading condition (4) becomes

$$d = \frac{\frac{\lambda_1}{\alpha_1}\left(\frac{c_1}{\lambda_1} - \mu_1\right) + \frac{\lambda_2}{\alpha_2}\left(\frac{c_2}{\lambda_2} - \mu_2\right)}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}} > 0\,.$$

7.1 Explicit results for exponential claims

We now consider the case where the claim size distributions f_1 and f_2 are exponentially distributed and their Laplace transformations are of the form:

$$\hat{f}_1(s) = \frac{\beta_1}{s + \beta_1}\,, \qquad \hat{f}_2(s) = \frac{\beta_2}{s + \beta_2}\,,$$

where $\beta_1 > 0,\ \beta_2 > 0$. In this case matrix $\mathbf{A}(s)$ has the form

$$\mathbf{A}(s) = \begin{bmatrix} s - \frac{\lambda_1+\alpha_1+\delta}{c_1} + \frac{\lambda_1\,\beta_1}{c_1(s+\beta_1)} & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s - \frac{\lambda_2+\alpha_2+\delta}{c_2} + \frac{\lambda_2\,\beta_2}{c_2(s+\beta_2)} \end{bmatrix}.$$

For simplicity, define

$$Q_\delta(s) := \left[s - \frac{\lambda_1 + \alpha_1 + \delta}{c_1} + \frac{\lambda_1\,\beta_1}{c_1(s + \beta_1)}\right]\left[s - \frac{\lambda_2 + \alpha_2 + \delta}{c_2} + \frac{\lambda_2\,\beta_2}{c_2(s + \beta_2)}\right].$$

Then

$$\begin{aligned}
\hat{v}_{1,1}(s) &= \frac{\left[(s - \frac{\lambda_2+\alpha_2+\delta}{c_2})(s+\beta_2) + \frac{\lambda_2\,\beta_2}{c_2}\right](s+\beta_1)}{[Q_\delta(s) - \frac{\alpha_1\,\alpha_2}{c_1\,c_2}](s+\beta_1)(s+\beta_2)}\,,\\
\hat{v}_{1,2}(s) &= -\frac{\alpha_1(s+\beta_1)(s+\beta_2)}{c_1\,[Q_\delta(s) - \frac{\alpha_1\,\alpha_2}{c_1\,c_2}](s+\beta_1)(s+\beta_2)}\,,\\
\hat{v}_{2,1}(s) &= -\frac{\alpha_2(s+\beta_1)(s+\beta_2)}{c_2\,[Q_\delta(s) - \frac{\alpha_1\,\alpha_2}{c_1\,c_2}](s+\beta_1)(s+\beta_2)}\,,\\
\hat{v}_{2,2}(s) &= \frac{\left[(s - \frac{\lambda_1+\alpha_1+\delta}{c_1})(s+\beta_1) + \frac{\lambda_1\,\beta_1}{c_1}\right](s+\beta_2)}{[Q_\delta(s) - \frac{\alpha_1\,\alpha_2}{c_1\,c_2}](s+\beta_1)(s+\beta_2)}\,.
\end{aligned}$$

Since the common denominator of the above formulae is a polynomial of degree 4, it has four zeros, say R_1, R_2, R_3 and R_4. It is easy to show that the four zeros are distinct, then inverting the above Laplace transforms gives

$$v_{i,j}(u) = \sum_{k=1}^{4} r_{i,j,k}\, e^{R_k\, u}\,, \qquad i,j = 1,2\,,$$

where the coefficients, $r_{i,j,k}$, are given, for $k = 1,2,3,4$, by

$$\begin{cases} r_{1,1,k} = \dfrac{\left[\left(R_k - \frac{\lambda_2+\alpha_2+\delta}{c_2}\right)(R_k+\beta_2) + \frac{\lambda_2\,\beta_2}{c_2}\right](R_k+\beta_1)}{\prod_{l=1,l\neq k}^{4}(R_k-R_l)}\,, \qquad r_{1,2,k} = -\dfrac{\alpha_1(R_k+\beta_1)(R_k+\beta_2)}{c_1\prod_{l=1,l\neq k}^{4}(R_k-R_l)}\,, \\ r_{2,2,k} = \dfrac{\left[\left(R_k - \frac{\lambda_1+\alpha_1+\delta}{c_1}\right)(R_k+\beta_1) + \frac{\lambda_1\,\beta_1}{c_1}\right](R_k+\beta_2)}{\prod_{l=1,l\neq k}^{4}(R_k-R_l)}\,, \qquad r_{2,1,k} = -\dfrac{\alpha_2(R_k+\beta_1)(R_k+\beta_2)}{c_2\prod_{l=1,l\neq k}^{4}(R_k-R_l)}\,. \end{cases}$$

Now we get the expected present value of the total dividend payments until ruin, given the initial state i, as follows

$$V_i(u;\, b) = \sum_{j=1}^{2} a_j(b) \left\{ \sum_{k=1}^{4} r_{i,j,k}\, e^{R_k\, u} \right\}, \qquad 0 \le u \le b\,,\ i = 1,2\,,$$

where $a_1(b)$ and $a_2(b)$ are the solutions of the following two equations:

$$\sum_{j=1}^{2} a_j(b) \left\{ \sum_{k=1}^{4} r_{i,j,k}\, R_k\, e^{R_k\, b} \right\} = 1\,, \qquad i = 1,2\,.$$

The higher moment of the present value of all dividend payments prior to ruin, $V_{i,n}(u;\, b)$, can also be derived by solving similar equations.

To illustrate the results numerically, set $c_1 = 110,\ c_2 = 84,\ \lambda_1 = 100,\ \lambda_2 = 40,$ $\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{3}{4},\ \beta_1 = 1,\ \beta_2 = 0.5$ and $\delta = 0.1$. Then we get $\pi_1 = \frac{15}{17},\ \pi_2 = \frac{2}{17},\ \eta_1 = \frac{3}{4},\ \eta_2 = \frac{1}{4}$, the positive loading $d = 0.1$, and $R_1 = -0.121,\ R_2 = -0.065,\ R_3 = 0.010$ and $R_4 = 0.074$. The upper rows in Table 1 give the expected present value of the total dividend payments until ruin $V(u;\, b)$ in the stationary case, given by $\eta_1\, V_1(u;\, b) + \eta_2\, V_2(u;\, b)$, and the lower rows give the standard deviation of the present value of the total dividend payments until ruin in the stationary case, given by $SD(u;\, b) = \sqrt{[\eta_1\, V_{1,2}(u;\, b) + \eta_2\, V_{2,2}(u;\, b)] - V(u;\, b)^2}$.

7.2 Comparison with the associated averaged compound Poisson model

We now compare some dividend related quantities in the Markov-modulated Poisson risk process with that in the associated averaged compound Poisson model.

Table 1: $V(u;\,b)$ and $SD(u;\,b)$ for $b = 10, \ldots, 80$ and $u = 10, \ldots, 50$

$u \setminus b$	10	20	30	40	50	60	70	80
10	16.590	26.625	34.310	37.577	37.364	35.327	32.588	29.712
	15.757	30.816	39.178	41.297	40.143	37.744	35.006	32.275
20		39.151	50.469	55.287	54.977	51.980	47.951	43.719
		31.832	40.259	41.664	40.018	37.512	34.893	32.365
30			61.683	67.593	67.220	63.558	58.631	53.456
			40.050	40.665	38.545	36.004	33.603	31.379
40				77.969	77.552	73.329	67.645	61.674
				40.288	37.674	35.061	32.837	30.876
50					87.476	82.718	76.306	69.570
					37.446	34.711	32.627	30.891

Asmussen et al. (1995) compared ruin functions for these two risk processes with respect to the stochastic ordering, stop-loss ordering and ordering of adjustment coefficients. In their paper the arrival rate is obtained by averaging over time the arrival rate in the Markov-modulated model and the distribution of the claim size is obtained by averaging the ones over consecutive claim sizes.

As pointed by Asmussen et al. (1995), if the arrival rates λ_i, the premium rates c_i, and the claim size distributions F_i in a Markov-modulated Poisson model do not fluctuate too much from the corresponding average values λ^*, c^* and F^*, one can see the model as a perturbation of a classical compound Poisson risk process $\{U^*(t);\, t \geq 0\}$ with arrival rate λ^*, premium rate c^* and claim size distribution F^*. The rigorous definition of λ^*, c^* and F^* in this paper for the two-state case is as follows: $\lambda^* = \eta_1\,\lambda_1 + \eta_2\,\lambda_2$, $c^* = \eta_1\,c_1 + \eta_2\,c_2$, and

$$F^*(x) = \frac{1}{\lambda^*}\left[\eta_1\,\lambda_1\,F_1(x) + \eta_2\,\lambda_2\,F_2(x)\right]\,, \qquad x \geq 0\,.$$

Then the associated averaged compound Poisson surplus process U^* is defined by

$$U^*(t) = u + c^*\,t - \sum_{n=1}^{N^*(t)} X_n^*\,, \qquad t \geq 0\,,$$

where $\{N^*(t);\, t \geq 0\}$ is Poisson with rate λ^* and X_n^*, for $n \geq 1$, are i.i.d. with distribution F^*. Moreover, one can prove that the risk processes U, given by (3), and U^* have the same positive safety loading d, given by (4).

Table 2: $V(u;b)$ (upper rows) and $V^*(u;b)$ (lower rows) for $b = 10, \ldots, 80$ and $u = 10, \ldots, 50$

$u \setminus b$	10	20	30	40	50	60	70	80
10	16.590	26.625	34.310	37.577	37.364	35.327	32.588	29.712
	16.322	26.613	35.174	38.817	38.366	35.938	32.893	29.823
20		39.151	50.469	55.287	54.977	51.980	47.951	43.719
		39.298	51.940	57.320	56.653	53.068	48.572	44.038
30			61.683	67.593	67.220	63.558	58.631	53.456
			63.275	69.829	69.017	64.650	59.172	53.649
40				77.969	77.552	73.329	67.645	61.674
				80.211	79.278	74.262	67.970	61.625
50					87.476	82.718	76.306	69.570
					89.157	83.515	76.439	69.304

For the numerical example showed in Section 7.1, the upper rows of Table 2 give the values of $V(u;b)$, while the lower rows give the values of the corresponding values of $V^*(u;b)$ in the associated average classical risk model. Figure 1 gives the curves of $V(u;b)$ (solid line) and $V^*(u;b)$ (dashed line) as functions of b for $u = 10, 20, 30, 40, 50$ (from bottom to top). It can be observed that the expected present values of the total dividend payments until ruin in the Markov-modulated risk model is overall smaller than those in the associated averaged compound Poisson model, which is consistent with the result, obtained in Asmussen et al. (1995), that the ruin functions related to two surplus processes $\{U^*(t);\, t \geq 0\}$ and $\{U(t);\, t \geq 0\}$ have the stochastic ordering relationship $\Psi^* \prec_{\mathrm{so}} \Psi$ under some naturally fulfilled conditions.

Next, we compare $\mathcal{L}^*(u;\, b)$, the Laplace transform of the first time that the surplus $U^*(t)$ reaches barrier b without ruin ever occuring, with $\mathcal{L}(u;\, b) = \eta_1\, \mathcal{L}_1(u;\, b) + \eta_2\, \mathcal{L}_2(u;\, b)$, where

$$\mathcal{L}_i(u;\, b) = \sum_{j=1}^{2} e_j(b) \left\{ \sum_{k=1}^{4} r_{i,j,k}\, e^{R_k\, u} \right\}, \qquad 0 \leq u \leq b\,,\ i = 1, 2\,,$$

with $e_1(b)$ and $e_2(b)$ being the solutions of the following two equations:

$$\sum_{j=1}^{2} e_j(b) \left\{ \sum_{k=1}^{4} r_{i,j,k}\, e^{R_k\, b} \right\} = 1\,, \qquad i = 1, 2\,.$$

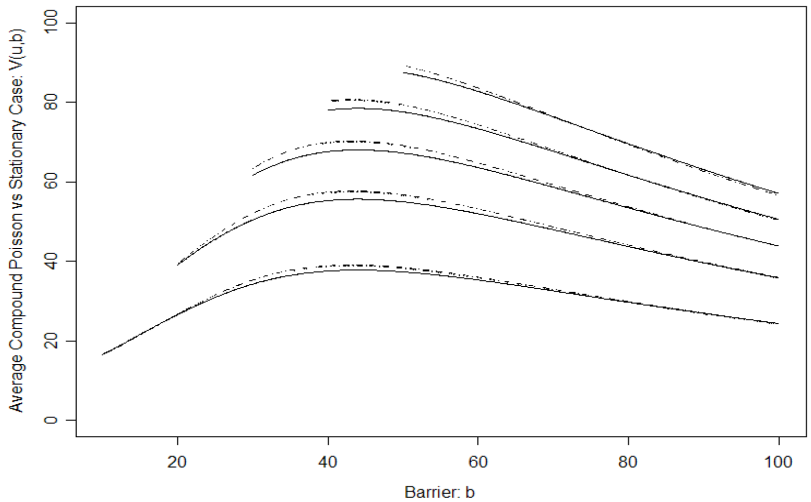

Figure 1: $V(u;\, b)$ (solid lines) and $V^*(u;\, b)$ (dashed lines) for $u = 10, \ldots, 50$ (from bottom to top)

From Figure 2, we can see that both $\mathcal{L}^*(u; b)$ and $\mathcal{L}(u; b)$ are increasing in u and decreasing in b. Furthermore, $\mathcal{L}^*(u; b)$ is slightly bigger than the corresponding value of $\mathcal{L}(u; b)$.

Finally, we compare $\chi^*(u;\, b)$, the probability that $U^*(t)$ attains the dividend barrier b from the initial surplus u without first falling below zero, with $\chi(u;\, b) = \eta_1 \chi_1(u;\, b) + \eta_2 \chi_2(u;\, b)$, where

$$\chi_i(u;\, b) = \sum_{j=1}^{2} h_j(b) \left\{ \sum_{k=1}^{4} r^0_{i,j,k}\, e^{R^0_k\, u} \right\}, \qquad 0 \le u \le b\,,\ i = 1, 2\,,$$

with $r^0_{i,j,k}$ and R^0_k being the corresponding values of $r_{i,j,k}$ and R_k for $\delta = 0$, and $h_1(b)$ and $h_2(b)$ being the solutions of the following system of equations:

$$\sum_{j=1}^{2} h_j(b) \left\{ \sum_{k=1}^{4} r^0_{i,j,k}\, e^{R^0_k\, b} \right\} = 1\,, \qquad i = 1, 2\,.$$

Figure 3 shows that $\chi(u; b)$ is overall smaller than the corresponding value of $\chi^*(u; b)$, this is consistent with the fact that the Markov-modulated risk model is risky and therefore the probability of the surplus attains level b without ruin

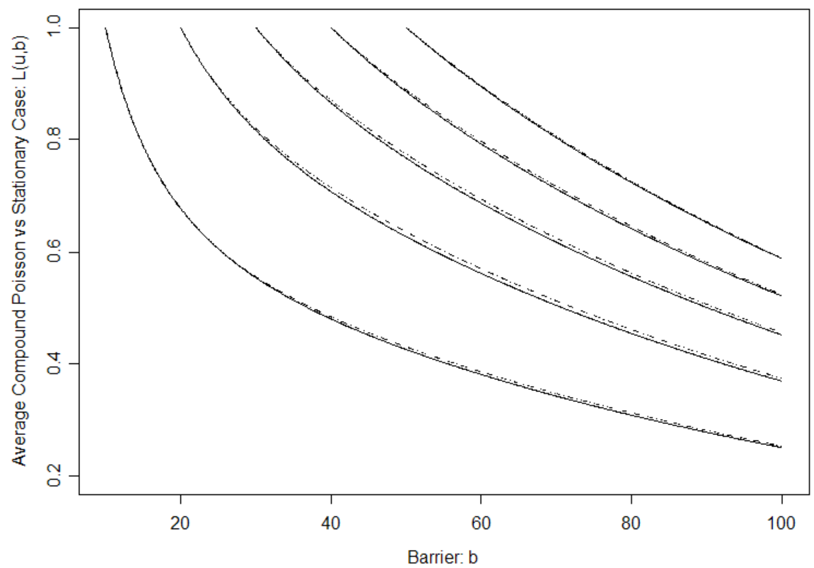

Figure 2: $\mathcal{L}(u;\, b)$ (solid lines) and $\mathcal{L}^*(u;\, b)$ (dashed lines) for $u = 10, \ldots, 50$ (from bottom to top)

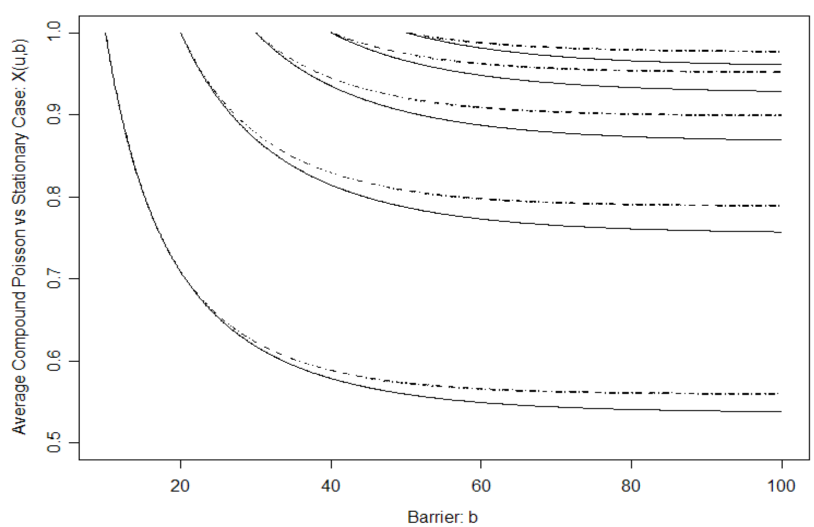

Figure 3: $\chi(u;\, b)$ (solid lines) and $\chi^*(u;\, b)$ (dashed lines) for $u = 10, \ldots, 50$ (from bottom to top)

occuring is smaller than that for the associated averaged compound Poisson risk model. Furthermore, we note that the two probabilities depart farther as the dividend barrier b becomes bigger.

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