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Liquidity Constrained Competing Auctions*

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Abstract

When goods are sold through competing auctions, what effect does monetary policy have on the equilibrium allocation? To answer, we extend the competing auctions framework in several ways: buyers choose how much money they bring to an auction, the quantities traded at the auctions are endogenous, and sellers can charge a fee (either positive or negative) to buyers participating in their auction. We present two different specifications of the model. In the first model, sellers post a quantity they wish to sell and a fee, and allow the price to be determined by an auction. In the second model, sellers post a price and a fee and allow the quantity sold to be determined by an auction.

When sellers post a quantity and buyers bid prices, the Friedman rule implements the first best and, in this case, no fee is charged by sellers. Sellers charge buyers a participation fee as soon as the nominal interest rate is positive, and marginal increments in money growth decrease both the posted quantity and buyers' entry. The use of auction fees reduces welfare in this environment.

When sellers post a price and buyers bid quantities, the Friedman rule is optimal but does not yield the first best as agents trade an inefficiently low quantity in multilateral matches and an inefficiently high quantity in pairwise matches. Marginal increments in money growth decrease the posted real price and the quantities traded. When the interest rate is low, sellers pay buyers who participate in their auction, which increases welfare. When the interest rate is high, sellers charge buyers who participate in their auction, which reduces welfare.

Keywords: Competing auctions, money, search, inflation, auction fees.
JEL Classification: C78; D44; E40

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1 Introduction

We study the impact of monetary policy on allocations in an economy where goods are traded through competing auctions. To do that we extend the competing auctions framework in several ways: buyers choose how much money they bring to an auction, trading off the cost of holding money against the expected surplus from participating in an auction; and sellers choose how much of their production good they want to put on auction, trading off the production cost of the advertised quantity against the expected number of potential buyers. We also allow sellers to charge each buyer a fee for participating in their auction. This fee, which can be positive or negative, trades off the additional cost (or revenue) from the fee with the number of buyers taking part in the auction. We use the model to explain how monetary policy affects output, prices and participation, through anticipated inflation. We also derive statements about optimal monetary policy and the welfare implications of using auction fees in this environment.

To build a monetary economy we embed the competing auctions framework into the Lagos and Wright (2005) model of monetary exchange with two-sided divisibility. This model is in the tradition of Kiyotaki and Wright's (1991, 1993) environment in which a role for fiat money emerges endogenously from the frictions of the trading environment, i.e. money is essential for trade (Kocherlakota, 1998; Wallace, 2001). The equilibrium concept we use builds on Peters and Severinov's (1997) limit equilibrium of a competing auctions economy, which we extend to the context of a monetary economy. This limit equilibrium exploits the convergence properties of a competitive matching economy, especially that a deviation by one seller to different terms of trade will not affect buyers' ex ante expected utility. This *market utility property* of the seller's payoff (Peters, 2000) can be used to solve price posting or auction posting models.

We consider two variants of the model. In the first, each seller posts a quantity for sale and a fee, and allows the dollar price of the good to be determined ex post by an auction. This model can be thought of as the monetary version of Peters and Severinov's (1997) competing auctions economy, although some aspects differ (in particular, sellers compete in auction fees rather than reserve prices). In the second model, sellers post a dollar price and a fee, and allow the quantity sold to be determined through the auction. This protocol corresponds to procurement auctions.

The two variants of the model imply different equilibrium outcomes. Both imply an upper bound on the rate of money growth that is required for an equilibrium to exist – but different bounds. In the quantity-posting version of the model, buyers are liquidity constrained and the equilibrium distribution of money holdings is not degenerate. Sellers charge buyers a fee as soon as one departs from the Friedman rule, and marginal increments in money growth decrease both the posted quantity and the participation of buyers. In the price-posting model, by contrast, buyers are not liquidity constrained and the equilibrium distribution of money holdings is degenerate. Moreover, sellers pay buyers who participate in their auction when the nominal interest rate is low, whereas buyers have to pay sellers to participate when the nominal interest rate is high. Marginal increments in money growth decrease the posted real price and the quantities traded.

In the quantity-posting model, the Friedman rule achieves efficiency on both the intensive and extensive margins: the quantity traded in each match maximizes the surplus of the match, and efficient entry by buyers is induced. In the price-posting model, the Friedman rule is also optimal but it fails to give efficiency. Auctions with quantity bidding imply dispersion in production and, hence, not all production can take place at the surplus-maximizing point. This second model is reminiscent of the "second generation" of monetary search models in which goods are divisible but money is not (Shi (1995), Trejos and Wright (1995), Kultti and Riipinen (2003) and Julien, Kennes, and King (2008)). A key difference is that here the fixed, real price posted by sellers is endogenous, while it is exogenously set to 1–the indivisible unit of money–in those models.

Finally, we show that auction fees are welfare deteriorating in the price bidding model unless the central bank implements the Friedman rule, in which case sellers choose not to charge fees and welfare is unchanged. In the quantity bidding model auction fees improve welfare when the nominal interest rate is low and decrease welfare when it is high.

Auctions with monetary exchange have already been studied. Kultti and Riipinen (2003) and Julien, Kennes, and King (2008) studied competing auctions in this second generation of monetary search models. Since money is indivisible in these models buyers can compete only through adjustments in quantity, however. Conversely, Galenianos and Kircher (2008) consider second-price auctions with divisible money and indivisible goods. By way of contrast, here, both money and goods are fully divisible. Also search is not directed in Galenianos and Kircher (2008). Here we allow sellers to post any quantity (or price), and allow buyers to decide which seller to approach so that the matching function is endogenous as in any directed search model.[1] Other papers in the competing auctions literature are McAfee (1993), Julien (1997), Burguet and Sákovics (1999), Schmitz (2003) and Hernando-Veciana (2005). They consider environments with finite numbers of buyers and sellers while we work with an infinitely large economy. Moldovanu, Sela and Shi (2008) have recently constructed a model in which two competing auctioneers can choose the supply of their good as we do here. Their focus, however, is on oligopolistic competition and on the coexistence of two competing auction sites. See also Peters and Severinov (2006) who build a model in which sellers sell a unit of an homogenous good and buyers can bid as often as they like and move between auctions as in e-Bay. None of these competing auctions models are monetary however. Finally, our paper contributes to the literature on the micro-foundations of money by examining an alternative pricing mechanism: competing auctions. In contrast to the bargaining, price-taking, and competitive search pricing mechanisms examined in Rocheteau and Wright (2005), auctions generate terms of trade dispersion. Combined with the divisibility of goods and the fee charged by sellers, this produces interesting trade-offs for both sellers and buyers that have not been previously studied.

The article is organized as follows. Section 2 lays out the general environment. Section 3 characterizes the equilibrium, optimal monetary policy and welfare when sellers advertise

[1] Models of directed search with price posting (or competitive search) include Montgomery (1991), Moen (1997), Acemoglu and Shimer (1999a,b), Burdett, Shi, and Wright (2001), Mortensen and Wright (2002). See also the corresponding sections in the surveys by King (2003) and Rogerson, Shimer and Wright (2005). Directed search is used in monetary models with divisible money by Lagos and Rocheteau (2005), Rocheteau and Wright (2005), Berentsen Menzio and Wright (2008) and Dong (2008). See also Faig and Huangfu (2007) for a competitive search monetary economy in which market markers are allowed to charge fees. Competition with posted auctions has been applied to the labor market by Julien, Kennes, and King (2000).

quantities and buyers bid prices. Section 4 studies the mirror economy in which sellers post a price and buyers bid quantities. Section 5 concludes.

2 The Environment

Time is discrete and goes on forever. Each period is divided into two trading subperiods. In the first subperiod agents participate in a centralized Walrasian market where they can produce and consume any quantity of a single, homogenous consumption good. Then they enter a second, frictional, market where quantities of the same good are allocated via competing auctions. For brevity, we call this the "auction market". We use β to denote the discount factor between the Walrasian and the auction market.

There is a continuum of anonymous, infinitely lived agents who, following Rocheteau and Wright (2005), differ in terms of when they produce and consume the good. In the first subperiod, i.e. in the centralized market, all agents can produce and consume the good. In the second subperiod, i.e. during the auction market, agents are divided into buyers who want to consume the good but cannot produce it, and sellers who want to produce the good but cannot consume it. This assumption generates a temporal double coincidence problem. Combined with the assumption that the good is perishable (no commodity money) and that agents are anonymous (no credit), this ensures money is essential for trade. The number of sellers in this economy is fixed and denoted by s. The number of buyers is equal to $\bar{b}$ but only b buyers participate to the auction market with $b \leq \bar{b}$. This number is determined endogenously by a free-entry condition with parametric outside option $k > 0$. We consider markets with b and s arbitrarily large, but the ratio $\theta = b/s$ is finite.

Money in this economy is a perfectly divisible and storable object whose value relies on its use as a medium of exchange. It is available in quantity M_t at time t, and can be stored in any non negative quantity m_t by any agent. New money is injected or withdrawn via lump-sum transfers by the central bank at rate τ such that $M_{t+1} = (1 + \tau) M_t$. Only sellers receive this transfer but it could be buyers as long as the transfer is not conditional on the entry decision.

In these models, inflation is forecasted perfectly and both the quantity theory and the Fisher effect apply: if the money supply increases at rate τ, so do prices and the nominal interest rate. Denoting r the real interest rate, since $\beta = 1/(1+r)$ the Fisher equation $(1+i) = (1+r)(1+\pi)$ enables to write the nominal interest rate as $i = (1-\beta+\tau)/\beta$ in which we neglect the $r\pi$ term.

In the model with quantity posting and money price bidding, each seller advertises in the Walrasian market a quantity q of his production good that he wishes to sell in the auction market. The final price is determined by an auction. In the model with price posting and quantity bidding, each seller advertises in the Walrasian market a price d for his production good he wishes to sell. The final quantity produced and traded is determined by an auction. In both cases sellers advertise a fee $\delta \in \mathbb{R}$ that applies to each buyer taking part in his auction. If $\delta < 0$ buyers must pay to bid at this auction. If $\delta > 0$ sellers pay each buyer participating in his auction.

To organize the payment of fees, we assume there exists an auction house that centralizes the information posted by sellers and processes the transfer of the auction fees. It is helpful to think of this auction house as an online auction website such as e-Bay. On this website each seller posts his auction. In the price bidding model, for example, each seller uses the website to advertise a quantity q for sale and a participation fee δ. Buyers have free access to this website where they can see all the posted auctions for free. Once they have observed the posted auctions, they choose one seller and pay the corresponding participation fee via the website. The website collects the fees, which, if negative, are then transferred to the sellers minus a commission for the website, assumed equal to zero for simplicity. In exchange for the fee the website delivers to the buyer the physical address of the auction he chose and an access code that will enable the buyer to take part into this particular auction in the coming auction market. Without such information, a buyer cannot take part in an auction. If the fee is positive, then buyers simply receive a payment from the auction house in addition to the location of the auction and the access code. All of this happens at the end of the centralized market before the auction market opens. Finally, to maintain the essentiality of money, we assume the auction house does not keep track of previous transactions.

The auctions themselves take place in the next subperiod where the double coincidence problem forces buyers to hold money. Because the Walrasian market is closed by that time, buyers are financially constrained by the amount of money they chose to hold while in the Walrasian market. When the auction market closes, all agents proceed to the next Walrasian market, which buyers use to replenish their money holdings while sellers spend any money earned during the auction market. A detailed sequence of events will be provided for each model, below.[2]

Each buyer maximizes $\sum_{t=0}^{\infty} U_t^b$ with

$$U_t^b = x_t + \beta u(q_t). \tag{1}$$

The quantity x_t corresponds to the net utility of consuming and producing x_t units of the good in the Walrasian market and $u(q_t)$ is the utility of consuming q_t units of the good in the auction market.[3] Similarly a seller maximizes $\sum_{t=0}^{\infty} U_t^s$ with

$$U_t^s = x_t - \beta c(q_t), \tag{2}$$

in which $-c(q_t)$ is the disutility of producing q_t units of the good in the auction market. Note that, in the auction market, buyers have no production cost and sellers do not enjoy any utility. We assume buyers are homogenous in preferences (they all have the same valuation for the good) and sellers are homogenous in their production costs. That is, u and c are identical across agents; they also are common knowledge. The nominal price of the good on the centralized market is normalized to 1 and it is the price of money in terms of the good ϕ_t that will adjust to market conditions each period. That is 1 unit of money buys ϕ_t units of the good on the centralized market, or 1 unit of the good costs $1/\phi_t$ units of money. We make standard concavity and

[2] An alternative story to the auction website would be to allow buyers and sellers to pay the auction fees at the opening or closing of each auction. This is interesting because sellers would then have to carry money when posting a negative fee. This, however, creates unnecessary difficulties. Since the number of buyers arriving at any seller is stochastic, some sellers may not have enough funds to pay each participating buyer. Having an auction house organizing the payment of fees before the opening of the auction market gets around this difficulty.

[3] All we need to eliminate the impact of trading histories on money demand is quasi-linearity in either production costs or utility in the Walrasian market (cf. Lagos and Wright 2005). Here we assume linearity in both, via x_t, without loss of generality. Models in which trading history matters can be solved numerically (Molico, 2006; Dressler, 2008) but are not central for the issues examined here.

convexity assumptions for u and c, and let q^* denote the quantity that maximizes the trade surplus, that is $u'(q^*) = c'(q^*)$. We also denote $\hat{q} > 0$ as the quantity such that $u(\hat{q}) = c(\hat{q})$. Finally we assume $u'(0) > c'(0) > 0$.

3 Quantity Posting and Price Bidding

In this section we consider the case where sellers post quantities and buyers bid using money prices. The sequence of events is as follows: first, the Walrasian market opens and all sellers receive the money injection from the central bank. Then buyers make their entry decisions, given the outside option k. Once sellers have observed the entry of buyers, sellers publicly announce a quantity q to be auctioned in the coming auction market, and a fee δ for participating to their auction. On the basis of the posted terms of trade, buyers decide which seller to visit and how much money they are going to bring to the corresponding auction. Then, depending on whether the fee is positive or negative, sellers (resp. buyers) pay the auction fees to the auction house which are then transferred to buyers (resp. sellers) who can spend it on the Walrasian market or hold on to them. In exchange buyers receive information from the auction house about the location of the auction and an access code. Finally, buyers and sellers proceed to the auction market. Buyers submit their bids and the good goes to the buyer that bids the most, who pays the price of the second-highest bid. If a buyer is alone at an auction, he pays a price equal to the seller's reservation value, $c(q)$, assuming he holds enough money (which will be true in equilibrium). At the end of the auction market, buyers and sellers proceed to the next period Walrasian market.[4]

A strategy for a seller is a q and a fee δ he posts for each level of entry by buyers (posting strategy). A strategy for a buyer is a rule that specifies his money holding (monetary strategy) and the probability with which he chooses a particular seller (visit strategy) as a function of the quantity and the fee (q, δ) posted by sellers. We will focus on symmetric equilibria where

[4] There are several types of ascending-bid auctions. We use second-price auctions because they imply a unique optimal bidding strategy for buyers (Riley and Samuelson, 1981) and are therefore easier to work with. Note also that our model differs from the multi-unit auction studied by Hansen (1988). Here it is the size (or quality) of the divisible good q that is chosen by the seller.

sellers post the same (q, δ) and where buyers follow the same monetary and visit strategies. The construction of the equilibrium is discussed in the next two subsections. Finally, note that in contrast to Peters and Severinov (1997), sellers do not compete in reserve prices in our model. This does not mean sellers do not have one. Clearly sellers would reject any bid below $c(q)$ in real terms, at least in the case of production on demand that we examine here.

3.1 The Value Functions

Our model is one of directed search in the sense that buyers direct their search according to the expected terms of trade posted by sellers. Directed search equilibria can be constructed in at least three different manners. One way relies on market makers to organize local submarkets from which buyers can choose, as in Moen (1997). The second explicitly formulates the strategic game of competition between finite numbers of sellers, as in Peters and Severinov (1997). An equilibrium is then a fixed point of the deviating seller's best response correspondence. Here we follow a third approach, discussed in Peters (2000), in which the focus is on the implicit trade-off for buyers between the gross surplus and the probability to trade. When posting (q, δ), the seller realizes that it affects the probability of trade for buyers. A higher posted quantity and a positive fee, for example, are attractive to buyers, but this attraction implies that many buyers will be expected to attend the auction – decreasing the chances of winning the auction for each buyer. Each particular pair (q, δ) then implies for buyers a particular probability of trade and a particular expected utility from this triple. In a competitive economy, however, no seller can beat his competitors by posting a different (q, δ). This is the *market utility property* by which a deviation by one seller to a different (q, δ) will leave buyers' ex ante expected utility unchanged. A simple way to solve for the equilibrium is then to impose a restriction on beliefs, namely that sellers believe the market utility property applies. The combination of (q, δ) and the probability to trade implied by this particular level of utility will then form a fixed indifference curve which sellers take as given when choosing the terms of the posted auctions. In the end sellers effectively compete in the posted (q, δ), and a probability of trade for buyers (summarized by the buyer-seller ratio θ) implied by the market utility property.[5]

From now on we focus on steady state equilibria where aggregate real variables are constant. This includes the real value of the money supply so that $\phi_t M_t = \phi_{t+1} M_{t+1} = \phi_{t+1}(1+\tau) M_t$. Thus, $\phi_t = \phi_{t+1}(1+\tau)$. We start by analyzing the centralized market.

Let $W^b(m)$ and $V^b(m)$ be the value functions for a buyer holding m units of money in the centralized market and the auction market, respectively. We have:

$$\begin{aligned} W^b(m) &= \max_{x,\hat{m}} \left\{ x + \beta V^b(\hat{m}) \right\}, && (3) \\ \text{s.t.} \quad \phi \hat{m} + x &= \phi (m + \delta), && (4) \end{aligned}$$

where $\hat{m}$ corresponds to the money carried from the centralized market to the auction market, x is the net consumption of the good in the centralized market and δ is the nominal participation fee paid to (or received from) the auction house. Again, ϕ corresponds to the price of money in terms of the good on the centralized market. When choosing a quantity of the good to consume and produce in the Walrasian market, x, and a quantity of money to bring to the auction market, $\hat{m}$, buyers take into account that the combined real value of these two quantities must be equal to what they brought to this market and received from (or paid to) the seller via the auction house. Substituting out x,

$$W^b(m) = \phi (m + \delta) + \max_{\hat{m}} \left\{ -\phi \hat{m} + \beta V^b(\hat{m}) \right\}. \quad (5)$$

Let $W^s(m)$ and V^s be the value functions for the seller in the centralized and auction market, respectively. Since sellers have no reason to bring money to the auction market, we have

$$\begin{aligned} W^s(m) &= \max_{x,q,\theta,\delta} \left\{ x + \beta V^s \right\}, && (6) \\ \text{s.t.} \quad x + \phi \Phi(\delta) &= \phi m + \phi T. && (7) \end{aligned}$$

A seller chooses net consumption x in the Walrasian market, a posted quantity q, a queue length θ and a participation fee δ for the auction market that maximizes his payoff. The budget constraint says that the money collected from buyers in the last auction market, ϕm,

[5] See Burdett Shi and Wright (2001) and Shi (2006) for discussions on different approaches.

and that received from the central bank, T, must cover his net consumption of good x and the expected payment to the auction house measured in real terms, that is $\phi\Phi(\delta)$. Substituting out x we obtain

$$W^s(m) = \phi(m+T) + \max_{q,\theta,\delta}\{-\phi\Phi(\delta) + \beta V^s\}. \tag{8}$$

In the auction market, the value function of a buyer holding m units of money is $V^b(m)$. Letting $V^b_n(m)$ be the value function when the buyer faces exactly n competitors at the auction, we have

$$V^b(m) = \sum_{n=0}^{\infty} P[X=n]\, V^b_n(m) - k, \tag{9}$$

where k is the buyer's outside option, X is the random variable equal to the number of competing buyers showing up at the auction, and $P[X=n]$ is the probability measure associated with the event $X=n$.

A buyer holding m units of money facing n competitors wins the auction if he holds the highest money holding. We denote the distribution of money holdings in the population by $F(m)$, which we characterize as part of the equilibrium, and $f(m)$ as the corresponding density. The probability of winning the auction is equal to $F^n(m)$ with density $nF^{n-1}(m)f(m)$. With probability $1-F^n(m)$ the buyer does not win the auction. The support of F is noted $S' = [\underline{\mathrm{m}}, \bar{m}] \subseteq \mathbb{R}_+$ which we define later. Finally, we denote $S = [\underline{\mathrm{m}}, m] \subseteq S'$. At this point, we assume that F is continuous but we show that this is true in equilibrium below. We can now compute $V^b_n(m)$, which is the value function of a buyer holding m units of money, bidding for q units of goods and facing n competitors. Using z to denote how many dollars he will spend if he wins the auction, we have:

$$V^b_n(m) = \int\limits_{z\in S} \left\{u(q) + W^b_{+1}(m-z)\right\} dF^n(z) + [1-F^n(m)]\, W^b_{+1}(m). \tag{10}$$

The first term corresponds to the buyer's expected payoff to winning the auction and paying the second highest amount of money. Precisely, it is equal to the probability that all n competitors have less money than he has, times the payoff to consuming q units of the good and moving to the centralized market with $m-z$ units of money; where we sum over the quantity of money spent, z. This quantity takes value from the lowest money holding $\underline{m}$ to a quantity of money marginally smaller than the buyer's own money holding m, denoted $m-\varepsilon$. Since F is continuous by assumption, it is continuous on the left with $\lim_{\varepsilon\to 0} F(m-\varepsilon)=F(m)$ so that z takes values in S. The second term corresponds to the probability of not winning the auction, multiplied by the value of entering the centralized market with an unchanged amount of money m.

Let $\gamma=\phi\delta/\beta$ denote the discounted real auction fee. Inserting (9) and (10) into (5), using $\phi=(1+\tau)\,\phi_{+1}$, eliminating constant terms and dividing by β, the buyer's monetary strategy is given by

$$\max_{m}\gamma-i\phi_{+1}m+\sum_{n=0}^{\infty}P\left[X=n\right]\left\{u(q)F^{n}(m)-\phi_{+1}\int_{z\in S}zdF^{n}(z)\right\}. \tag{11}$$

Since γ is a parameter for the buyer, he chooses his money holdings m in order to maximize the sum of the opportunity cost of carrying that amount of money, $-i\phi_{+1}m$, and the expected net gain of winning the auction. This expected net gain is composed of the utility of consuming q, multiplied by the probability of winning minus the expected payment associated with holding m units of money.

For a seller, because the winner pays the amount of the bidder with the second highest money holding, we need an expression for the distribution of the second highest money holdings among the n buyers whose money holding is distributed according to F. Denoting $x_{(k)}$ as the k^{th} order statistic, its density is given by

$$f_{x_{(k)}}(m)=n\binom{k-1}{n-1}F^{k-1}(m)\left[1-F(m)\right]^{n-k}f(m). \tag{12}$$

Setting $k=n-1$ in the above formula gives the density of the second highest money holding, so that the value function for the seller posting q is

$$V^{s}=\sum_{n=1}^{\infty}P\left[X=n\right]\int_{z\in S'}\left\{-c(q)+W_{+1}^{s}(z)\right\}f_{x_{(n-1)}}(z)dz. \tag{13}$$

Inserting (13) into (8), eliminating constant terms and dividing by β, the seller's objective function is given by:

$$-\Phi(\gamma) - \sum_{n=1}^{\infty} P[X=n]\, c(q) + \phi_{+1} \sum_{n=1}^{\infty} P[X=n] \int\limits_{z \in S'} z f_{x_{(n-1)}}(z) dz \tag{14}$$

with $\Phi(\gamma) = \phi\Phi(\delta)/\beta$. A seller maximizes the difference between the cost of producing the posted q upon a meeting and the expected return of selling this q through a second-price auction, minus the expected payment of (or revenue from) the auction fees.

3.2 Competing Auctions Monetary Equilibrium

Sellers compete against each other via the posted expected terms of trade. By doing so they take into account the buyer's participation constraint. This constraint says that the buyer's payoff from saving m units of money in the Walrasian market for the coming auction market must be no smaller than proceeding directly to the next period's centralized market without money, skipping the auction market. Since the representative buyer's money holding is a random variable, so too is his payoff. Therefore sellers must consider the buyer's *expected* payoff rather than actual payoff by summing over all possible money holdings by buyers. The buyer's participation constraint taken into account by sellers is then

$$\int\limits_{m \in S'} \left[-\phi m + \beta V^b(m)\right] dF(m) \geq \beta W^b_{+1}(0). \tag{15}$$

Using the same simplification technique used to obtain (11), this constraint simplifies into

$$\gamma + E[\chi(m)] \geq k \tag{16}$$

with

$$E[\chi(m)] \equiv \int\limits_{m \in S'} \chi(m) dF(m) \tag{17}$$

in which $\chi(m)$ is given by the expression to the right of γ in (11). A closed-form expression for $E[\chi(m)]$ is derived in section 3.4 below. This participation constraint (16) says that when choosing the terms of the auction sellers make sure that the sum of the fee and the expected gains from trade on the auction market for a representative buyer is no smaller than the cost of entry.

When posting an auction, sellers must also make sure that buyers do not have an incentive to simply cash the fee and walk away from the auction market. This requires that the buyer's gains from trade from carrying money and going to the auction market net of the fee is still positive, which translates into

$$E\left[\chi(m)\right] \geq 0. \tag{18}$$

It is indeed a possibility that the sum $\gamma + E\left[\chi(m)\right]$ is greater than k, yet $E\left[\chi(m)\right]$ is negative. Although this satisfies the buyer's participation constraint, there is an incentive for buyers to cash the fee and not go to the auction market where their expected surplus is negative. Note that no such constraint is required for sellers as production on demand forces buyers to bring at the least sellers' reservation value, that is $c(q)/\phi$, to the auction market (see Lemma 2 below).

Noting $E\left[\varphi(m)\right]$ the seller's expected surplus net of the participation fees, which from (14) is simply

$$E\left[\varphi(m)\right] = -\sum_{n=1}^{\infty} P\left[X=n\right] c(q) + \phi_{+1} \sum_{n=1}^{\infty} P\left[X=n\right] \int\limits_{z \in S'} z f_{x_{(n-1)}}(z) dz, \tag{19}$$

in the end the seller's problem is given by:

$$\max_{q,\theta,\delta} -\Phi\left(\gamma\right) + E\left[\varphi(m)\right] \tag{20}$$

$$\text{s.t.} \quad \gamma + E\left[\chi(m)\right] \geq k, \tag{21}$$

$$\text{s.t.} \quad E\left[\chi(m)\right] \geq 0. \tag{22}$$

A seller chooses a quantity, a queue length and a fee in order to maximize his expected payoff net of the expected auction fees, subject to buyers getting no less than their outside option and to participating buyers not walking away from the auctions.

Let $I \subseteq \mathbb{R}_+$ be the set of possible values for i. Let $Y_P \subseteq \mathbb{R}_+^2 \times \mathbb{R}$ represent the set of possible posted terms of trade with $y = (q, \theta, \delta) \in Y_P$. The seller's optimal posted contract is given by

$$G\left(i\right) = \arg\max_{y \in \Gamma(i)} -\Phi\left(\gamma\right) + E\left[\varphi(m)\right] \tag{23}$$

where $\Gamma : I \to Y_P$ is the constraint correspondence given by

$$\Gamma(i) = \{ y \in Y_P : \gamma + E[\chi(m)] \geq k, \ E[\chi(m)] \geq 0 \}. \tag{24}$$

Lemma 1 *$G(i)$ is non empty, compact valued and upper hemicontinuous.*

Proof. See Appendix. ■

Definition 1 *A **competing auctions monetary equilibrium** is a list $(q, \theta, \delta) \in \Upsilon_P$, $\phi \in \mathbb{R}^+$ and a distribution of money holdings $F \in \mathcal{F}$ such that:*

(i) : *[**Optimal selection by buyers**] Buyers are indifferent between all sellers;*

(ii) : *[**Rational Expectations**]*

· *Buyers' and sellers' beliefs about the relationship between the posted (q, δ), the distribution of money holdings and buyers' entry are correct,*

· $P[X = n] = \frac{\theta^n}{n!} e^{-\theta}$,

· $\Phi(\gamma) = \sum_{n \in \mathbb{N}} P[X = n] n\gamma = \theta\gamma$;

(iii) : *[**Profit maximization**] Sellers maximize the expression in (20) subject to (21) and (22)*;

(iv) :*[**Individual Rationality**]* $E[\chi(m)] \geq 0$;

(v) : *[**Free entry**] Equation (21) is binding due to free-entry on the buyer's side;*

(vi) : *[**Money market clearing**]* $b\phi \int\limits_{m \in S'} m dF(m) = M^S$ *where M^S is the money supply.*

Note that rational expectations imply that each seller forecasts that the probability with which he will be visited by n buyers follows a Poisson process, and that he expects to pay (or receive) $\sum_{n \in \mathbb{N}} \frac{\theta^n}{n!} e^{-\theta} n\gamma = \theta e^{-\theta} \gamma \sum_{n \in \mathbb{N}^*} \frac{\theta^{n-1}}{(n-1)!}$ as participation fees. Setting $n' = n - 1$ this yields $\theta e^{-\theta} \gamma \sum_{n' \in \mathbb{N}} \frac{\theta^{n'}}{n'!} = \theta\gamma$.

3.3 The Distribution of Money Holdings by Buyers in Equilibrium

To prove existence it is necessary to characterize the distribution of money holdings in equilibrium, if any. To do that let us take the first-order condition of the buyer's program in equation

(11) using $P\left[X=n\right]=\frac{\theta^n}{n!}e^{-\theta}$. We obtain

$$i\phi_{+1}=\left[u(q)-\phi_{+1}m\right]f(m)\theta e^{-\theta[1-F(m)]}. \tag{25}$$

Equation (25) equalizes the marginal cost of an additional dollar on the left hand side to its expected marginal return on the right hand side.[6] Rearranging and integrating this expression over $S\subseteq S'$ gives the distribution of money holdings among buyers:

$$F(m)=\frac{1}{\theta}\ln\left\{1-ie^{\theta}\ln\left[\frac{u(q)-\phi_{+1}m}{u(q)-\phi_{+1}\underline{m}}\right]\right\}. \tag{26}$$

The distribution is a function of the price of money ϕ_{+1}, the quantity q posted by sellers, the buyer-seller ratio θ, the nominal interest rate i and the lower support of the distribution $\underline{\mathrm{m}}$. To find $\underline{\mathrm{m}}$ note that the seller is indifferent between producing q for $\underline{\mathrm{m}}$ and doing nothing such that $-c(q)+W^s(\underline{m})=W^s(0)$. Using the linearity of W^s, that is $W^s(\underline{m})=\phi_{+1}\underline{m}+W^s(0)$, we extract $\underline{m}=\frac{c(q)}{\phi_{+1}}$. Dropping the subscript on ϕ it is easy to show that $F(\underline{m})=0$, and $F(\bar{m})=1$ implies:

$$\bar{m}=\frac{u(q)-e^{-\frac{1-e^{-\theta}}{i}}\left[u(q)-c(q)\right]}{\phi} \tag{27}$$

so that

$$S^{'}=\frac{1}{\phi}\left[c(q),u(q)-e^{-\frac{1-e^{-\theta}}{i}}\left[u(q)-c(q)\right]\right] \tag{28}$$

and

$$F(m)=\frac{1}{\theta}\ln\left\{1-ie^{\theta}\ln\left[\frac{u(q)-\phi m}{u(q)-c(q)}\right]\right\}, \tag{29}$$

which is continuous over $S^{'}$ (as assumed, initially).

In the symmetric equilibrium buyers all follow the same monetary strategy given by equation (25). This does not mean that all buyers carry the same amount of money. Stochastic demand at each auction and the resulting uncertainty about the final price means that, despite homogenous preferences, the choice of money holding for the auction market is not trivial. Bringing more money opens the possibility to outbid the leading bidder and win the auction, but it is costly due

[6] $f(m)\theta e^{-\theta[1-F(m)]}$ is the density of the CDF $\sum_{n\in\mathbb{N}}\frac{\theta^n}{n!}e^{-\theta}F^n(m)=e^{-\theta[1-F(m)]}$ which is the probability that a buyer wins the auction in a large market, given m.

to the positive nominal interest rate. As in Galenianos in Kircher (2008), the trade-off between an infinitesimally small marginal cost and a stochastic, discrete increase in the probability to win the auction produces a non degenerate distribution of money holdings represented by $F(m)$. One key difference, here, is that, since sellers choose the terms of the auctions (quantity, auction fee and queue length), they also indirectly shape the distribution of money holdings.

Lemma 2 *(i) : As $i \to 0$, $F(m) \to 0$ for any $m < \bar{m}$ with $\bar{m} \to u(q)/\phi$. As $i \to \infty$, $F(m) \to 1$ for any $m > \underline{m}$ and $\bar{m} \to \underline{m}$;*
(ii) : When q increases or when i falls, the new distribution first-order stochastically dominates the old one. That is, $\frac{\partial F(m)}{\partial q} < 0$ and $\frac{\partial F(m)}{\partial i} > 0$.

Proof. See Appendix. ■

As the nominal interest rate gets closer to zero, the distribution of money holdings concentrate around $u(q)$ in real terms. When $i = 0$ all participating buyers bring exactly $u(q)$ in money at the auction. Conversely, as the interest rate gets higher money holdings concentrate around the seller's outside option, $c(q)$.

3.4 Existence

Let $v(z) = \frac{1-F(z)}{f(z)}$.

Lemma 3 *The seller's program in (20)-(21) simplifies into*

$$\max_{q,\theta,\delta} -\left(1-e^{-\theta}\right)c(q) - \theta\gamma + \theta\phi_{+1} \int\limits_{z\in S'} [z - v(z)]\, f(z) e^{-\theta[1-F(z)]} dz \tag{30}$$

$$s.t. \quad \gamma - i\phi_{+1} \int\limits_{m\in S'} m dF(m) + \frac{1-e^{-\theta}}{\theta} u(q) - \phi_{+1} \int\limits_{z\in S'} [z - v(z)]\, f(z) e^{-\theta[1-F(z)]} dz \geq k. \tag{31}$$

Proof. See Appendix. ■

The seller maximizes the sum of the disutility of producing the posted quantity q when at least one buyer arrives, the amount of fees that he pays to (or receives from) the auction house and the expected payment in real terms coming from the auction. The first term inside the integral in the seller's payoff, $z - v(z)$, corresponds to Myerson's (1981) virtual valuation of a buyer holding z units of money. It is the difference between the money held by the buyer z and the buyer's rent, which is equal to the difference between the first order and the second order statistic $v(z)$. The term $\theta f(z) e^{-\theta[1-F(z)]}$ corresponds to the probability that a buyer wins the auction in a large market. The sum over all possible z gives the expected value.

The constraint (31) describes the average buyer's indifference condition between his outside option payment k and his net gain from taking part into the auctions. Precisely, the first term in the constraint measures the payment from (or to) the auction house. The second term measures the disutility of holding the average amount of money as sellers internalize the effect of inflation on buyers' willingness to carry money to the auction market. The third term corresponds to the utility a buyer gets from consuming the good multiplied by the probability that a buyer gets served in directed search with posted prices. Indeed, from the seller's perspective, the expectation operator on money holdings makes it equivalent to all buyers holding the same amount of money so that the auction environment becomes equivalent to price posting with directed search. Finally, note that given the symmetry of the auction rule, expected seller revenue is just b times the expected payment by buyers divided by the number of sellers s.

Using (31) to substitute γ into (30), equations (30)-(31) simplify into

$$\max_{q,\theta} \left(1 - e^{-\theta}\right) [u(q) - c(q)] - \theta \left[k + i\phi_{+1} \int_{m \in S'} m dF(m) \right]. \tag{32}$$

The seller effectively maximizes the net surplus of the auction in an average trade. This net surplus is made of the sum of the total gross surplus from trade in meetings where at least one buyer arrives, $\left(1 - e^{-\theta}\right) [u(q) - c(q)]$, minus the sum of the buyer's opportunity cost, k, and cost of holding the average amount of cash, $i\phi_{+1} \int_{m \in S'} m dF(m)$, multiplied by the average number of buyers per auction, θ.

As will be clear shortly, when $i = 0$ the seller's program coincides with that of a social planner so that the Friedman rule is the optimal policy. Outside the Friedman rule, however, sellers must subtract the opportunity cost of holding money for buyers, which the social planner does not. Finally, note that substituting $\tilde{m} = \phi_{+1} m$ into $i\phi_{+1} \int\limits_{m \in S'} m dF(m)$ enables to get rid of ϕ_{+1} in (32). This makes it clear that terms of trade are formed in real terms. The value of money on the centralized Walrasian market ϕ_{+1} is determined by the equality between supply and demand for money on the Walrasian market.

Proposition 1 (i) : *When $i = 0$ there exists a unique equilibrium in which sellers charge no fee to buyers, post the efficient q^* and induce efficient entry θ^*; (ii) : For $i \simeq 0$ the equilibrium is unique and inflation reduces both the posted quantity q and entry by buyers θ; (iii) : No monetary equilibrium exists for $i > \bar{\imath}_P$.*

Proof. See Appendix. ■

Comments on this Proposition and the following are delayed to section 3.6.

3.5 Welfare

Welfare is measured by

$$W_P = sV^s + b \int\limits_{m \in S'} V^b (m) \, dF(m). \tag{33}$$

Substituting $V^b(m)$ from (9) and V^s from (13) and simplifying yields

$$W_P = s(1 - e^{-\theta}) \left[u(q) - c(q) \right] - bk. \tag{34}$$

The social planner maximizes the number of trades times the surplus in each trade, minus the cost from buyer's entry. Once divided by s this expression for welfare is equal to the seller's substituted program in (32) with $i = 0$. Clearly the Friedman rule yields full efficiency.

Proposition 2 (i) : *The Friedman rule achieves efficiency on the intensive and extensive margins; (ii) : Sellers would post the same terms of trade at the Friedman rule if not allowed to charge a fee.*

Proof. See Appendix. ■

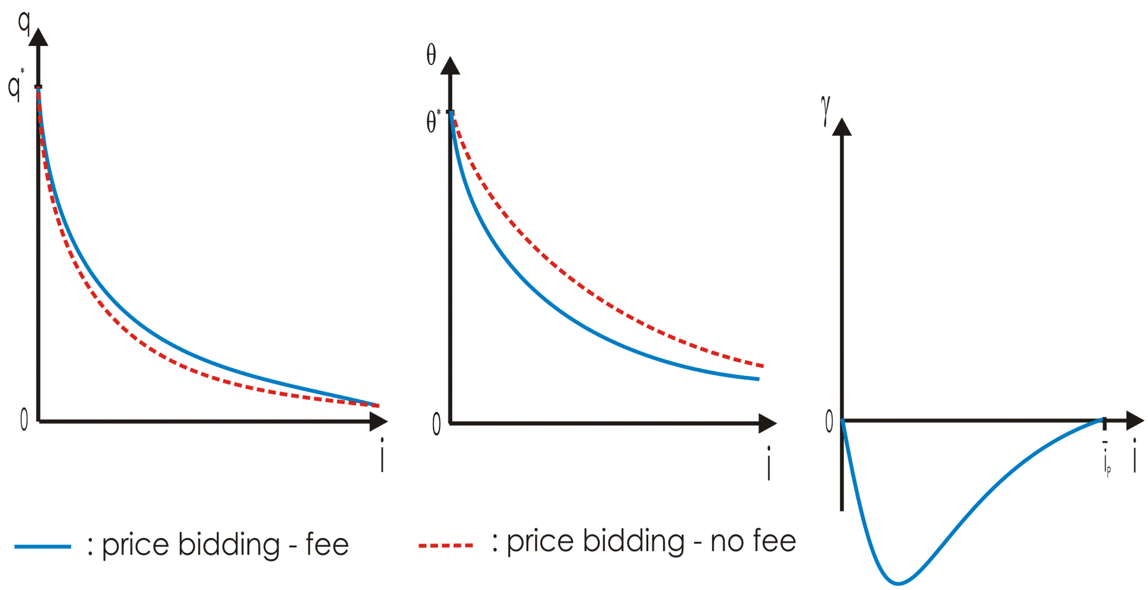


Figure 1: Comparative statics, price bidding.

3.6 Numerical Comparative Statics

In this section we conduct numerical comparative statics to study the impact of inflation on the equilibrium allocation outside the neighborhood of the Friedman rule. We start with a deflation rate equal to the real rate of interest so that the nominal interest rate is zero, and increase inflation until the equilibrium unravels. The outcome is represented in Figure 1 where we represent equilibrium q, θ and γ as the nominal interest rate increases. We run the algorithm when sellers are allowed to charge or pay a fee and when they are not and compare the outcome.[7]

At the Friedman rule, sellers post the efficient quantity and entry is also efficient. As inflation increases, buyers bring less real balances due to the now positive nominal interest rate. This forces sellers to reduce the quantity they put on auction, reducing entry by buyers. On the other side real money holdings among buyers are now disperse rather than concentrated on $u(q)$, offering buyers the opportunity to gain positive surplus in auctions even when facing competitors since the winning bidder will not always pay his reservation value $u(q)$. This favors

[7] We set $u(q) = \sqrt{q}$, $c(q) = q$ and $k = 0.025$. Results are robust to alternative specifications of these two functions and parameter. See Appendix A.6 for details on the numerical analysis.

entry by buyers. In the end the net effect on entry is negative so that both the posted quantity and the buyer-seller ratio fall as inflation increases.

Outside the Friedman rule, the use of fees enables sellers to post a higher quantity and to reduce the number of buyers participating in their auction. This generates an extra surplus for buyers on top of their outside option equivalent surplus, which sellers claim back by charging them an entrance fee. The fee as a function of the nominal interest rate is V-shaped and always negative: its absolute value first increases then decreases as the buyer's extra surplus increases and then falls. The seller's profit is increased thanks to the auction fees (see Appendix A.11).

Regarding welfare, simulation shows that, unless the central bank implements the Friedman rule, welfare is higher when sellers are not allowed to charge fees than when they are. The increased surplus sellers are able to reap through the use of auction fees is more than cancelled by the loss incurred by buyers. From a public policy point of view, as soon as the opportunity cost of holding money is not zero, the model does not recommend the use of auction fees in this environment.

4 Price Posting and Quantity Bidding

In this section we consider the alternative setting in which sellers post a price and buyers bid using quantities. A bid from a buyer is a proposition of a quantity q in exchange for d units of money posted. Buyers observe the posted prices and auction fees and select over sellers. Frictions manifest themselves through ex-post market conditions, more buyers at one seller translating into a smaller quantity for the winning buyer, rather than a higher price as in the previous model.

A key difference with the previous model is that now buyers are no longer liquidity constrained in their bidding strategy. This has an important consequence. Whenever two or more buyers approach a seller (multilateral meetings), because buyers have homogeneous preferences, the outcome of the auction is very simple. Through the bidding process, the quantity falls until the terms of trade leave the winning buyer, chosen at random, indifferent between trading or not–the Bertrand quantity applies. We note this quantity q_m. Alternatively, whenever a seller is visited by only one buyer (pairwise meetings), the terms of trade will be such that the seller is left indifferent between trading or not. We note this quantity q_p. Note that, in contrast to the case with competition in prices, only the first two marginal buyers influence the terms of trade.

The sequence of events is as follows. First, all sellers receive the money injection from the central bank. Buyers make their entry decisions with outside option k. Sellers advertise a dollar price d for their good and an auction fee δ. Then all buyers observe the posted auctions, decide which auction to visit, and produce, trade, consume and pay (receive) the fee to (from) the auction house in the centralized market. The auction house centralizes and allocates the fees, as in the previous model. Subsequently, agents proceed to the auction market where buyers visit the auction of their choosing. If two or more buyers compete, a buyer chosen at random wins the auction and receives q_m for d units of money. If the buyer is alone he gets $q_p > q_m$ with probability 1 and pays d. The procedure for solving in the equilibrium terms of trade is identical to the previous section.

4.1 The Value Functions

Let $W^b(m)$ and $V^b(m)$ be the value functions for a buyer holding m units of money in the centralized and decentralized markets, respectively. We have

$$\begin{aligned} W^b(m) &= \max_{x,\hat{m}} \left\{ x + \beta V^b(\hat{m}) \right\}, &(35)\\ \text{s.t.} \quad \phi\hat{m} + x &= \phi\left(m + \delta\right). &(36) \end{aligned}$$

Let ψ_p denote the probability, for a buyer, of a pairwise match, ψ_m denote the probability of winning the auction in a multilateral match and $1 - \psi_p - \psi_m$ denote the probability of not winning any auction. The Bellman equation for a buyer trading d units of money for q units of the special good is then given by

$$V^b(m) = \psi_p \left\{ u\left(q_p\right) + W^b_{+1}\left(m-d\right) \right\} + \psi_m \left\{ u\left(q_m\right) + W^b_{+1}\left(m-d\right) \right\} \qquad (37)$$
$$+ \left(1 - \psi_p - \psi_m\right) W^b_{+1}(m) - k.$$

With probability ψ_p a buyer is alone and trades with a seller, in which case he purchases and consumes q_p units of the good and enters tomorrow's centralized market with $m - d$ units of money. With probability ψ_m the buyer meets several other buyers but wins the auction, purchasing and consuming q_m and carrying on to the centralized market with $m - d$ units of money as well. In all other cases he proceeds to the centralized market with an unchanged amount of money. In all cases buyers pay k to participate.

Turning now to sellers, they solve the following program

$$W^s(m) = \max_{x,d,\theta,\delta} x + \beta V^s \qquad (38)$$

$$\text{s.t.} \quad x + \phi\Phi\left(\delta\right) = \phi\left(m+T\right), \qquad (39)$$

in which $\Phi\left(\delta\right)$ is what sellers expect to pay to (or receive from) the auction house. In the auction market, the probability for a seller of a pairwise match is noted ξ_p while ξ_m is the probability of a multilateral match (at least two buyers are present) and $1 - \xi_p - \xi_m$ that of no buyer showing up. Since there is no incentive for sellers to carry money into the auction market, m is not a state variable for sellers in the next submarket and we have

$$V^s = \xi_p \left\{ -c\left(q_p\right) + W^s_{+1}\left(d\right) \right\} + \xi_m \left\{ -c(q_m) + W^s_{+1}\left(d\right) \right\} \qquad (40)$$
$$+ \left(1 - \xi_p - \xi_m\right) W^s_{+1}(0)$$

with similar interpretation as (37). If a symmetric equilibrium exists we will have $\psi_p = e^{-\theta}$ and $\psi_m = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\theta}$.[8] Similarly, $\xi_p = \theta e^{-\theta}$ and $\xi_m = 1 - e^{-\theta} - \theta e^{-\theta}$.

[8] For instance, the probability for a buyer of getting served in a multilateral match is $\sum_{n\in\mathbb{N}^*} \frac{\theta^n}{n!} e^{-\theta} \frac{1}{n+1} = \frac{1}{\theta} \sum_{n\in\mathbb{N}^*} \frac{\theta^{n+1}}{(n+1)!} e^{-\theta} = \frac{1}{\theta} \sum_{n\geq 2} \frac{\theta}{n!} e^{-\theta} = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\theta}$

4.2 Competing Auctions Monetary Equilibrium

Let $z = \phi_{+1}d$ denote the real value of the posted price. Because bidding under multilateral matches yields no surplus for a buyer and no surplus for the seller under pairwise matches, we have the following lemma.

Lemma 4 *The seller's program is given by*

$$\max_{z,\theta,\gamma} -\Phi(\gamma) + \xi_m(\theta)\left\{-c\left[u^{-1}(z)\right] + z\right\}, \tag{41}$$

$$s.t. \quad \gamma - iz + \psi_p(\theta)\left\{u\left[c^{-1}(z)\right] - z\right\} \geq k, \tag{42}$$

$$s.t. \quad -iz + \psi_p(\theta)\left\{u\left[c^{-1}(z)\right] - z\right\} \geq 0. \tag{43}$$

where buyers bring exactly $m = d = z/\phi_{+1}$.

Proof. See Appendix. ■

Effectively, sellers maximize the expected sum of the fees they receive from or pay to buyers via the auction house, plus the surplus they get out of multilateral meetings, subject to the constraint that buyers' net gains from participating is no smaller than their outside option, and that participating buyers do not walk away from the auction market. By doing so sellers acknowledge that the real value of the posted price is equal to both their production cost in pairwise meetings and the buyer's utility in multilateral meetings. See Figure 2. Note that buyers gain only in pairwise meetings, and that the distribution of money holdings is degenerate and equal to z in real terms.

Let $I \subseteq \mathbb{R}_+$ be the set of possible values for i. Let $Y_Q \subseteq \mathbb{R}^2_+ \times \mathbb{R}$ represent the set of possible posted terms of trade with $y = (z, \theta, \delta) \in Y_Q$. Denoting $\chi(z,\theta) = -iz + \psi_p(\theta)\left\{u\left[c^{-1}(z)\right] - z\right\}$ the seller's optimal posted contract is given by

$$H(i) = \arg\max_{y \in \Omega(i)} -\Phi(\gamma) + \xi_m(\theta)\left\{-c\left[u^{-1}(z)\right] + z\right\} \tag{44}$$

where $\Omega : I \to Y_Q$ is the constraint correspondence given by

$$\Omega(i) = \left\{y \in Y_Q : \gamma + \chi(z,\theta) \geq k,\ \chi(z,\theta) \geq 0\right\}. \tag{45}$$

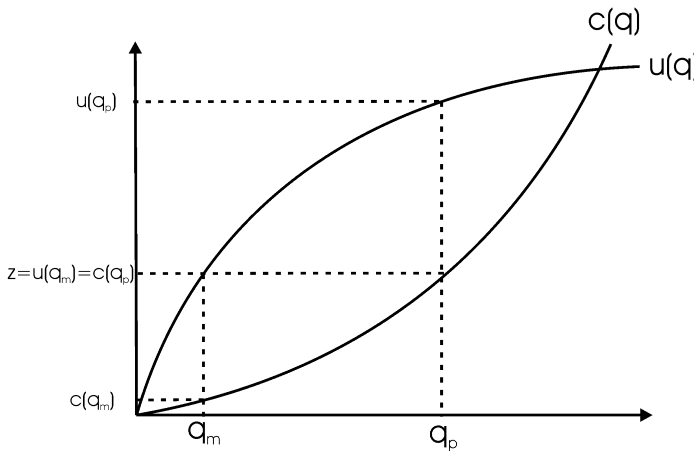


Figure 2: z, q_p and q_m.

Lemma 5 *$H(i)$ is non empty, compact valued and upper hemicontinuous.*

Proof. See Appendix. ■

Definition 2 *When buyers bid quantities, a **competing auctions monetary equilibrium** is a list $(z, \theta, \delta) \in \Upsilon_Q = [0, u(\hat{q})] \times \left[0, \hat{\theta}\right] \times [k - \{u(q^*) - c(q^*)\}, k]$ and $\phi \in \mathbb{R}_+$ such that:*

(i) : *[**Optimal selection by buyers**] Buyers are indifferent between all sellers;*

(ii) : *[**Rational Expectations**]*

· *Buyers' and sellers' beliefs about the relationship between the posted (z, θ, δ) and buyers' entry is correct,*

· $P[X = n] = \frac{\theta^n}{n!} e^{-\theta}$,

· $\Phi(\gamma) = \sum_{n \in \mathbb{N}} P[X = n] n\gamma = \theta\gamma$;

(iii) : *[**Profit maximization**] Sellers maximize (41) subject to (42) and (43)*;

(iv) :*[**Individual Rationality**]* $\chi(z, \theta) \geq 0$;

(v) : *[**Free entry**] Equation (42) is binding due to free-entry on the buyer's side;*

(vi) : *[**Money market clearing**] $b\phi m = M^S$ where M^S is the money supply.*

4.3 Existence

Inserting $\Phi(\gamma) = \theta\gamma$ into (41) and using the constraint (42) to substitute γ into (41), equations (41)-(42) in the sellers program become

$$\max_{z,\theta} \left(1 - e^{-\theta} - \theta e^{-\theta}\right) \left\{-c\left[u^{-1}(z)\right] + z\right\} + \theta e^{-\theta} \left\{u\left[c^{-1}(z)\right] - z\right\} - \theta\,(iz + k)\,. \tag{46}$$

As in the previous model, the seller maximizes the net surplus from an average trade. It is made of the gross surplus, which is equal to the weighted average of the buyer's gain from trade in a pairwise match, $u\left[c^{-1}(z)\right] - z$, and the seller's gain from trade in a multilateral match, $-c\left[u^{-1}(z)\right] + z$, minus the cost per buyer $iz + k$ multiplied by the average number of buyers at one auction, θ.

Proposition 3 (i) : *When* $i = 0$ *there exists a unique equilibrium in which sellers post a real price* z *that implies agents trade* $q_m < q^*$ *in multilateral matches,* $q_p > q^*$ *in pairwise matches, and sellers attract buyers by paying them* $\gamma = \theta e^{-\theta}\left\{[u(q_m) - c(q_m)] - [u(q_p) - c(q_p)]\right\}$; (ii) : *No monetary equilibrium exists for* $i > \bar{\imath}_Q$.

Proof. See Appendix. ■

4.4 Welfare

Welfare is measured by

$$W_Q = sV^s + b \int\limits_{m \in S'} V^b\,(m)\,dF(m). \tag{47}$$

Substituting $V^b\,(m)$ by (37) and V^s by (40) we obtain after simplification

$$W_Q = s\theta e^{-\theta}\left[u(q_p) - c(q_p)\right] + s\left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left[u(q_m) - c(q_m)\right] - bk. \tag{48}$$

The social planner maximizes the number of pairwise trades times the corresponding surplus, plus the number of multilateral trades times the corresponding surplus, minus the cost from buyer's entry. Dividing by s, the first-order conditions yield

$$u'(q_p) = c'(q_p), \tag{49}$$

$$u'(q_m) = c'(q_m), \tag{50}$$

$$\theta e^{-\theta} \left[u(q_m) - c(q_m)\right] + e^{-\theta} \left(1 - \theta\right) \left[u(q_p) - c(q_p)\right] = k. \tag{51}$$

On the intensive margin the first best requires marginal utility to be equal to marginal cost in pairwise and multilateral meetings so that agents trade $q_p = q_m = q^*$. On the extensive margin the first-best requires the buyer's expected marginal contribution to a match to be equal to his participation cost k. Inserting $q_p = q_m = q^*$ into the first order condition with respect to θ (51) becomes

$$\theta e^{-\theta} \left[u(q^*) - c(q^*)\right] + e^{-\theta} \left(1 - \theta\right) \left[u(q^*) - c(q^*)\right] = k, \tag{52}$$

and

$$\theta^* = - \ln \left[\frac{k}{u(q^*) - c(q^*)}\right] \tag{53}$$

as in the price bidding case. That is, the social planner would choose the same quantity traded and the same level of entry as in the price bidding case.

Proposition 4 (i) : *The Friedman rule is the optimal monetary policy;* (ii) : *The Friedman rule does not implement the first best as both production and entry are inefficient;* (iii) : *Sellers would not post the same terms of trade at the Friedman rule if not allowed to charge a fee.*

Proof. See Appendix. ■

The inefficiency at the Friedman rule contrasts with the results in the price bidding case. In both cases stochastic matching in an auction environment translates into ex post terms of trade dispersion. The difference is that in the price bidding case sellers can post the efficient quantity and let the auction process determine how the surplus is shared. With quantity bidding, it works the other way round: sellers post a contingent quantity mechanism with a fixed price ex ante and quantities determined ex post by the auctions. Note that even though the first best

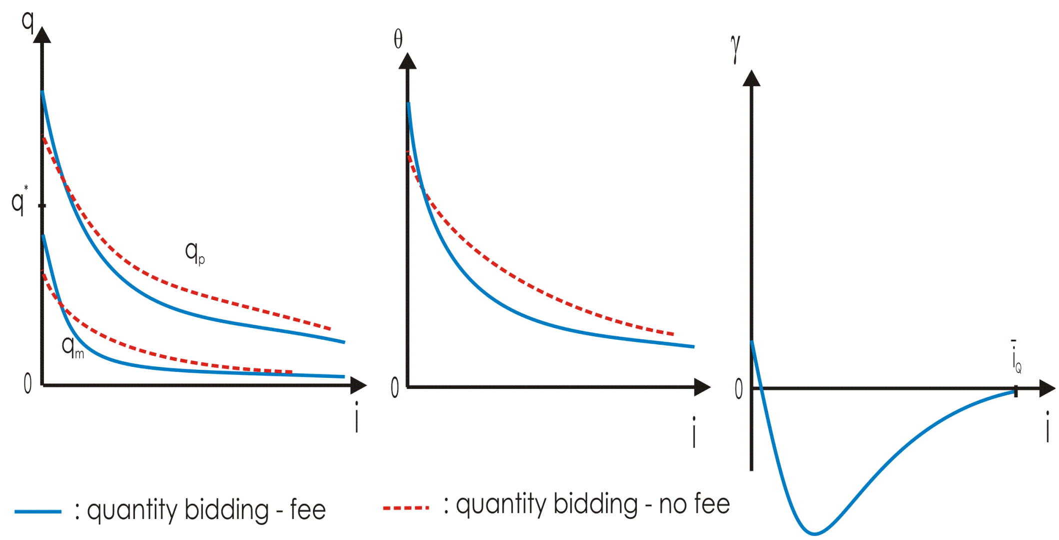


Figure 3: Comparative statics, quantity bidding.

cannot be implemented, directed search implies that entry is "efficient" in the sense that the buyer's expected marginal contribution to a match is equal to his participation cost k.

4.5 Numerical Comparative Statics

As in the previous model we conduct comparative statics analysis via computer simulation. We vary the nominal interest rate from 0% (the Friedman rule) until the equilibrium unravels, plot equilibrium q_p, q_m, θ and γ as functions of i. We run the algorithm when sellers are allowed to charge a fee and when they are not and compare the outcome. See Figure 3.

As inflation increases buyers are not ready to bring as much money so that sellers decrease the posted real price, translating into smaller quantities traded in pairwise and multilateral matches, and reduced entry.

For low values of the nominal interest rate, the fee is positive, which means sellers pay participating buyers. To understand this result, first recall from Proposition 4 that terms of trade at the Friedman rule are different whether sellers are allowed to charge fees or not. Introducing fees translates into more buyers entering the auction market than would be the case without fee, and a higher posted price (equal to q_p on Figure 3 since $z = c(q_p)$ and

$c(q) = q$), translating into greater quantities traded. This means both the probability to trade in pairwise meetings, equal to $e^{-\theta}$, and the surplus in pairwise meetings, equals to $u(q_p) - c(q_p)$, fall following the introduction of auction fees, which brings the buyer's surplus below his outside option. The only option for sellers is to compensate participating buyers by paying each one of them. The fee is funded by sellers' increased profit (see Appendix A.11).

As inflation increases, the fall in entry and in q_p increases buyers' expected surplus back to his outside option, and eventually above, allowing sellers to claim back this excess surplus by charging buyers with an entrance fee ($\delta < 0$). Note that the equilibrium fee is equal to zero when the surplus in pairwise matches is equal to the surplus in multilateral matches, that is $u(q_p) - c(q_p) = u(q_m) - c(q_m)$. In this case agents' payoffs and welfare are the same whether sellers post fees or not. Indeed, sellers always have the option to set $\delta = 0$. If they do so that means it is optimal for them not to charge any fee and sellers' program with and without fee coincide. From simulation, when $k = 0.025$, the nominal interest rate that induces sellers to charge a zero fee in equilibrium is roughly equal to 4%. Note also that there exists an even greater $i > 0$ such that agents trade the efficient quantity in pairwise meetings, i.e. $q_p = q^*$, yet this policy reduces welfare even further.

Finally, simulation shows that the use of auction fees increases welfare as long as sellers pay buyers, that is for low nominal interest rates (below 4% in our calibration, see Appendix A.11). In this case the loss by buyers is more than compensated by the gain for sellers even after entry is taken into account. As soon as sellers start charging buyers, however, fees are welfare deteriorating. From a public policy point of view, the model recommends the use of auction fees by sellers only if this translates into sellers paying buyers.

5 Conclusion

In this paper we have considered a monetary competing auctions economy by allowing buyers to choose how much money they bring to an auction, trading off the cost of holding money with the expected surplus from the auction. Combined with endogenous output and buyers free entry this makes it possible to study intensive and extensive margin effects of inflation in this environment. A central finding is that the Friedman rule is the optimal monetary policy, whether buyers bid prices or quantities. As for auction fees, although they increase the seller's profit, they decrease welfare most of the time. Auction fees are welfare improving only in procurement auctions when inflation is low.

The analysis of competing auctions in a monetary environment is still incomplete. We examine only the case of homogenous preferences and do not allow sellers to extract surplus via publicized reserve prices. We hope that incorporating money into auction models will shed new light on some old issues in the literature, however, and raise new questions. For example, does the revenue equivalence theorem still hold when holding money is costly? Introducing heterogenous preferences back into the model, how does the distribution of preferences interact with that of money holdings in equilibrium? Having divisibility on both sides, would it be possible to consider mechanisms in which buyers bid a combination of a price and a quantity?

Appendix

A.1. Proof of Lemma 1

Since the objective function is continuous, we only need to show compactness and continuity for Γ. First note that $q \in [0, \hat{q}]$ otherwise gains from trade would be negative. Second note from (11) that if $\theta \to \infty^+$ then $F^n(m) \to 0$ and $\chi(m) = -i\phi m$ so that $E[\chi(m)] < 0$. We can then restrict $\theta \in [0, \bar{\theta}]$ where $\bar{\theta}$ is an upper bound on θ. Finally, (21) implies $\gamma \geq k - E[\chi(m)]$. Since $E[\chi(m)] \leq u(q^*) - c(q^*)$, which corresponds to the maximum gains from trade, we have $\gamma \geq k - [u(q^*) - c(q^*)]$. Combining (21) with (22) implies $\gamma \leq k$ so that $\gamma \in [k - \{u(q^*) - c(q^*)\}, k]$. From the Theorem of the Maximum, all maximizers lie in a compact, upper hemicontinuous set $\Upsilon_P = [0, \hat{q}] \times [0, \bar{\theta}] \times [k - \{u(q^*) - c(q^*)\}, k]$ and the maximum is continuous.

A.2. Proof of Lemma 2

Part (i) : As $i \to 0$, $\phi\bar{m} \to u(q)$ and as $i \to \infty^+$, $\phi\bar{m} \to c(q)$.

Part (ii) : We note $A = -ie^{\theta} \ln\left[\frac{u(q)-\phi m}{u(q)-c(q)}\right] \in [0, \infty^+)$. We have

$$\frac{\partial F(m)}{\partial q} = \frac{-ie^{\theta}\left[(u - \phi m)\, c' + (\phi m - c)\, u'\right]}{\theta\,(A+1)\,(u - \phi m)\,(u - c)} < 0 \tag{54}$$

and

$$\frac{\partial F(m)}{\partial i} = \frac{A/i}{\theta\,(1+A)} > 0. \tag{55}$$

A.3. Proof of Lemma 3

The seller's objective and the buyer's constraint are given by

$$\max_{q,\theta,\delta} -\left(1 - e^{-\theta}\right) c(q) - \theta\gamma + \phi_{+1} \sum_{n\in\mathbb{N}^*} P[X = n] \int_{z\in S'} z f_{x_{(n-1)}}(z) dz \tag{56}$$

$$\text{s.t.} \quad \gamma - i\phi_{+1} \int_{m\in S'} m dF(m) + \tag{57}$$

$$\sum_{n\in\mathbb{N}} P[X = n] \left\{ u(q) \int_{m\in S'} F^n(m) dF(m) - \phi_{+1} \int_{\underline{m}}^{\bar{m}} \int_{\underline{m}}^{m} z dF^n(z) dF(m) \right\} \geq k.$$

Let us start with the seller. Using the definition of $f_{x_{(n-1)}}$ the integral in (56) simplifies into

$$\int\limits_{z\in S'} zn\,(n-1)\,F^{n-2}(z)\,[1-F(z)]\,f(z)dz = n\int\limits_{z\in S'} z\,[1-F(z)]\,dF^{n-1}(z) \tag{58}$$

$$= \quad n\left[z\,[1-F(z)]\,F^{n-1}(z)\right]_{\underline{m}}^{\bar{m}} - n\int\limits_{z\in S'}[1-zf(z)-F(z)]\,F^{n-1}(z)dz \tag{59}$$

$$= \quad n\int\limits_{z\in S'}[zf(z)+F(z)-1]\,F^{n-1}(z)dz. \tag{60}$$

Taking the sum over $n\in\mathbb{N}^*$ of the above expression multiplied by $P\,[X=n]=\frac{\theta^n}{n!}e^{-\theta}$ we obtain

$$\int\limits_{z\in S'}[zf(z)+F(z)-1]\sum_{n\in\mathbb{N}^*}\frac{\theta^n}{n!}e^{-\theta}nF^{n-1}(z)dz \tag{61}$$

$$= \quad e^{-\theta}\theta\int\limits_{z\in S'}[zf(z)+F(z)-1]\sum_{n\in\mathbb{N}^*}\frac{\theta^{n-1}}{(n-1)!}F^{n-1}(z)dz \tag{62}$$

$$= \quad \theta\int\limits_{z\in S'}[zf(z)+F(z)-1]\,e^{-\theta[1-F(z)]}dz. \tag{63}$$

Factorizing by $f(z)$ yields the expression in (30).

As for equation (57), integrating by part the third term simplifies according to:

$$u(q)\sum_{n\in\mathbb{N}}P\,[X=n]\int\limits_{m\in S'}F^n(m)dF(m) \tag{64}$$

$$= \quad u(q)\sum_{n\in\mathbb{N}}P\,[X=n]\left[\frac{F^{n+1}(m)}{n+1}\right]_{\underline{m}}^{\bar{m}} \tag{65}$$

$$= \quad u(q)\sum_{n\in\mathbb{N}}\frac{\theta^n}{(n+1)!}e^{-\theta} \tag{66}$$

$$= \quad u(q)\frac{1}{\theta}\sum_{n'\in\mathbb{N}^*}\frac{\theta^{n'}}{n'!}e^{-\theta} = u(q)\frac{1-e^{-\theta}}{\theta}. \tag{67}$$

Finally, carefully reversing the order of integration, the double integral in the last term can be rewritten

$$\int_{\underline{m}}^{\bar{m}} \int_{z}^{\bar{m}} z dF^n(z) dF(m) \tag{68}$$

$$= \int_{\underline{m}}^{\bar{m}} z dF^n(z) \left[1 - F(z)\right] \tag{69}$$

$$= \int_{z \in S'} \left[z f(z) + F(z) - 1\right] F^n(z) dz. \tag{70}$$

Using the same technique as above it is straightforward to see that the sum over $n \in \mathbb{N}$ of the above expression multiplied by $P\left[X = n\right]$ is equal to

$$\int_{s \in S'} \left[z f(z) + F(z) - 1\right] e^{-\theta[1-F(z)]} dz, \tag{71}$$

which is the seller's expected gross revenue divided by θ. Factorizing by $f(z)$ yields the expression in (31).

A.4. Proof of Proposition 1

To begin, rewrite (32) as

$$\Psi\left(i, k\right) = \max_{q,\theta} g(q, \theta; i, k) \tag{72}$$

with

$$g(q, \theta; i, k) = \left(1 - e^{-\theta}\right)\left[u(q) - c(q)\right] - \theta\left[k + iE(m)\right] \tag{73}$$

in which

$$E(m) = \phi \int_{m \in S'} m dF(m). \tag{74}$$

Note that for all $i \in I$ such that $\Psi\left(i, k\right) > 0$ and $E[\chi\left(m\right)] \geq 0$ an equilibrium exists.

Part (i) : First note that $\Psi\left(0, k\right) > 0$. To see this set $i = 0$ and take the first-order condition with respect to q. This implies $u'(q) = c'(q)$ so that sellers post the efficient q^*. The first-order condition with respect to θ (also when $i = 0$) yields $e^{-\theta}\left[u(q) - c(q)\right] = k$ so that

$$\theta^* = -\ln\left[\frac{k}{u(q^*) - c(q^*)}\right]. \tag{75}$$

Inserting q^* and θ^* into Ψ we obtain $\Psi(0,k) = u(q^*) - c(q^*) - k + k \ln\left(\frac{k}{u(q^*)-c(q^*)}\right)$. Since $\lim_{k \to 0} \Psi(0,k) = u(q^*) - c(q^*) > 0$ and $\Psi[0, u(q^*) - c(q^*)] = 0$ and also $\partial \Psi(0,k) / \partial k = \ln\left(\frac{k}{u(q^*)-c(q^*)}\right) < 0$, we conclude $\Psi(0,k) > 0$ for all $k \in (0, u(q^*) - c(q^*)]$.

Let us define $h = g(q, \theta; 0, k)$ and compute

$$h''_{qq} = \left(1 - e^{-\theta}\right) \left[u''(q) - c''(q)\right], \tag{76}$$

$$h''_{\theta\theta} = -e^{-\theta} \left[u(q) - c(q)\right], \tag{77}$$

$$h''_{q\theta} = e^{-\theta} \left[u'(q) - c'(q)\right]. \tag{78}$$

Inserting $q = q^*$ we have $h''_{qq} < 0$, $h''_{\theta\theta} < 0$ and $h''_{q\theta} = 0$. The Hessian is definite negative so g is strictly concave in the neighborhood of $i = 0$.

Finally, to see that $E[\chi(m)] > 0$ when $i = 0$, we show that $\gamma^* = 0$ so that $E[\chi(m)] = k$. First recall from Lemma (2) that when $i = 0$ the distribution of money holdings is degenerate and equal in real terms to $u(q)$. This implies that a buyer gets some positive surplus only when alone, in which case he pays the reserve price $c(q)$ and enjoys $u(q)$. As for the seller, he pays $c(q)$ only when he meets a buyer and gets $c(q)$ in payment if there is only one buyer and $u(q)$ if there is at least two of them. Then at the Friedman rule the seller solves

$$\max_{q,\theta,\gamma} -\theta\gamma - \left(1 - e^{-\theta}\right) c(q) + \theta e^{-\theta} c(q) + \left(1 - e^{-\theta} - \theta e^{-\theta}\right) u(q) \tag{79}$$

$$\text{s.t.} \quad \gamma + e^{-\theta}\left[u(q) - c(q)\right] = k, \tag{80}$$

which simplifies into

$$\max_{q,\theta,\gamma} -\theta\gamma + \left(1 - e^{-\theta} - \theta e^{-\theta}\right) \left[u(q) - c(q)\right] \tag{81}$$

$$\text{s.t.} \quad \gamma + e^{-\theta}\left[u(q) - c(q)\right] = k. \tag{82}$$

Substituting γ into the seller's objective, the first-order condition with respect to θ yields $e^{-\theta}\left[u(q) - c(q)\right] = k$ which from (82) implies $\gamma^* = 0$.

Since $\Psi(i,k) > 0$ and $E[\chi(m)] > 0$ when $i = 0$ an equilibrium exists at the Friedman rule. Since g is strictly concave in the neighborhood of $i = 0$, the equilibrium is unique.

Part (ii) : Given that g is strictly concave and $E[\chi(m)] > 0$ for i equal or close to 0, the seller's program (72) is unconstrained at and around the Friedman rule and the first-order conditions with respect to q and θ from (72) describe the unique equilibrium. The first-order conditions are given by, respectively

$$\Omega_1(q;i) = \left(1 - e^{-\theta(q)}\right)\left[u'(q) - c'(q)\right] - \theta(q)\, iE_q'(m) = 0, \tag{83}$$

$$\Omega_2(\theta;i) = e^{-\theta}\left\{u\left[q(\theta)\right] - c\left[q(\theta)\right]\right\} - k - i\left[E(m) + \theta E_\theta'(m)\right] = 0. \tag{84}$$

In (83) $\theta(q)$ denotes the implicit continuous relationship between θ and q given by (84) and $q(\theta)$ in (84) denotes the implicit continuous relationship between q and θ given by (83). Computing $d\Omega_1(q;i) = 0$ enables to extract

$$\frac{dq}{di}_{|i=0} = \frac{\theta(q^*)\, E_q'(m)}{\left(1 - e^{-\theta(q)}\right)\left[u''(q) - c''(q)\right]} < 0 \tag{85}$$

since $u''(q) - c''(q) < 0$ and $E_q'(m) > 0$ by $\frac{\partial F(m)}{\partial q} < 0$ in Lemma 2.

Similarly calculating $d\Omega_2(\theta;i) = 0$ enables to extract

$$\frac{d\theta}{di}_{|i=0} = \frac{E(m) + \theta(q^*)\, E_\theta'(m)}{-e^{-\theta}\left\{u\left[q(\theta^*)\right] - c\left[q(\theta^*)\right]\right\}}. \tag{86}$$

Using the Leibniz integral rule and noting that the lower bound is not a function of θ we have

$$E_\theta'(m) = \int_{\underline{m}}^{\bar{m}} \frac{\partial}{\partial\theta} m f(m) dm + \bar{m} f(\bar{m}) \frac{\partial \bar{m}}{\partial\theta} \tag{87}$$

$$= -\frac{1}{\theta}\int_{\underline{m}}^{\bar{m}} m f(m)\left[1 - \theta e^{-\theta F(m)}\right] dm + \bar{m} f(\bar{m}) \frac{\partial \bar{m}}{\partial\theta} \tag{88}$$

$$= -\frac{E(m)}{\theta} + \frac{1}{\theta}\int_{\underline{m}}^{\bar{m}} m\left[\theta f(m) e^{-\theta F(m)}\right] dm + \bar{m} f(\bar{m}) \frac{\partial \bar{m}}{\partial\theta} \tag{89}$$

in which $f(\bar{m})\frac{\partial \bar{m}}{\partial\theta} = \theta^{-1} e^{-\theta}$ so that the numerator in (86) is equal to

$$\int_{\underline{m}}^{\bar{m}} m\left[\theta f(m) e^{-\theta F(m)}\right] dm + \bar{m} e^{-\theta} > 0. \tag{90}$$

This implies $\frac{d\theta}{di}_{|i=0} < 0$.

Part (iii) : First, note that $\lim_{i\to\infty^+} E(m) = c(q)$ as the distribution of money holdings concentrates around $c(q)$ for high i, see Lemma 2.[9] Second, note that in equation (30)

$$\lim_{i\to\infty^+} \theta\phi_{+1} \int\limits_{z\in S'} [z - v(z)]\, f(z) e^{-\theta[1-F(z)]} dz = \left(1 - e^{-\theta}\right) c(q), \tag{91}$$

which is seller's expected gross revenue when all buyers hold $c(q)$ in real terms.[10] Therefore as $i \to \infty^+$ the seller solves

$$\max_{q,\theta,\gamma} -\theta\gamma \tag{92}$$

$$\text{s.t.} \quad \gamma - ic(q) + \frac{1-e^{-\theta}}{\theta}\left[u(q) - c(q)\right] = k, \tag{93}$$

$$\text{s.t.} \quad - ic(q) + \frac{1-e^{-\theta}}{\theta}\left[u(q) - c(q)\right] \geq 0. \tag{94}$$

This requires that sellers would have to charge buyers for participating to their auctions ($\gamma < 0$). Assume now $\lim_{i\to\infty^+} c(q) = c(\tilde{q}) > 0$. Since $\gamma < 0$, from (93) no equilibrium exists as $i \to \infty^+$. Assume now $\lim_{i\to\infty^+} c(q) = 0$ implying $\lim_{i\to\infty^+} u(q) = 0$. Then from (93) no equilibrium exists either.

A.5. Proof of Proposition 2

Part (ii) : To see that sellers post the same q and θ whether allowed to charge fees or not, simply set $\gamma = 0$ into (81) and (82) and use (82) to plug θ into (81). The first-order condition with respect to q gives $q = q^*$ which once inserted into (82) yields again $\theta^* = -\ln\left[\frac{k}{u(q^*)-c(q^*)}\right]$ as in the model with fee.

A.6. Numerical comparative statics - price bidding

From (29), the density $f(m)$ is given by

$$\frac{\partial F(m)}{\partial m} = \frac{ie^{\theta}\phi}{\theta\left[u(q) - \phi m\right]\left[1 - ie^{\theta}\ln\left(\frac{u(q)-\phi m}{u(q)-c(q)}\right)\right]}. \tag{95}$$

[9] This can also be seen by evaluating equation (96) below as $i \to \infty^+$.

[10] This can also be seen by evaluating $\Pi^S_{nf} + (1 - e^{-\theta})c(q)$ in equation (98) below as $i \to \infty^+$.

Substituting $v = 1 - ie^{\theta} \ln\left(\frac{u(q)-\phi m}{u(q)-c(q)}\right)$ and $dv = \frac{ie^{\theta}\phi}{u(q)-\phi m} dm$ and $m = \frac{u(q)-e^{\frac{1-v}{ie^{\theta}}}[u(q)-c(q)]}{\phi}$ into $E(m) = \phi \int\limits_{m\in S'} m dF(m)$ we obtain

$$E(m) = \frac{1}{\theta} \int_{1}^{e^{\theta}} \left\{ u(q) - e^{\frac{1-v}{ie^{\theta}}} \left[u(q) - c(q)\right] \right\} \frac{dv}{v}, \tag{96}$$

so that (32) transforms into

$$\max_{q,\theta} \left(1 - e^{-\theta}\right) [u(q) - c(q)] - \theta k - i\theta \left[u(q) - \frac{u(q) - c(q)}{\theta} \int_{1}^{e^{\theta}} e^{\frac{1-v}{ie^{\theta}}} \frac{dv}{v} \right]. \tag{97}$$

When sellers are not allowed to charge a fee, we note Π_{nf}^{S} the expression in (30) in which we have set $\gamma = 0$. Using the same substitution as above we obtain

$$\Pi_{nf}^{S} = \left(1 - e^{-\theta}\right) [u(q) - c(q)] + e^{-\theta} [u(q) - c(q)] \int_{1}^{e^{\theta}} e^{\frac{1-v}{ie^{\theta}}} \left[\frac{v \ln v - \theta v}{ie^{\theta}} - 1 \right] dv. \tag{98}$$

From (30)-(31) note that the buyer's payoff in the no-fee case, denoted Π_{nf}^{B}, and given by the LHS of (31) in which we have set $\gamma = 0$, is given by

$$\Pi_{nf}^{B} = -\frac{\Pi_{nf}^{S} + \left(1 - e^{-\theta}\right) c(q)}{\theta} + \frac{\left(1 - e^{-\theta}\right) u(q)}{\theta} - iE(m) \tag{99}$$

where $E(m)$ is given by (96). Equations (97)-(99) are the ones we use in the simulation.

A.7. Proof of Lemma 4

If a buyer faces no competitor he is able to impose terms of trade that leave the seller indifferent between not trading or producing and trading q_p against z, i.e. $z = c(q_p)$. Similarly competition between two or more buyers leads to $z = u(q_m)$. Inserting (40) into (38), using the linearity of W^s and getting rid of constant terms, the seller's objective (38) becomes

$$\max_{z,\theta,\gamma} -\Phi(\gamma) + \beta\xi_p \left[-c(q_p) + z\right] + \xi_m \left[-c(q_m) + z\right] \tag{100}$$

in which inserting $z = c(q_p)$ yields (41). As for buyers, the payoff from saving m units of money for the coming auction market must be no smaller than proceeding directly to the next period's centralized market, skipping the auction market. Algebraically,

$$-\phi m+\beta V^{b}(m)\geq\beta W_{+1}^{b}(0). \tag{101}$$

Substituting x into (35) using (36), inserting (37) into (35), using the linearity of W^b and $\phi_{+1}(1+\tau)=\phi$, (101) transforms into

$$\phi\delta/\beta-iz+\psi_{p}\left[u(q_{p})-z\right]+\psi_{m}\left[u(q_{m})-z\right]\geq k. \tag{102}$$

Inserting $z=u(q_m)$ into (102) yields (42). Finally, equation (43) makes sure buyers go to the auction once they have cashed the fee.

A.8. Proof of Lemma 5

First $z\in[0,u(\hat{q})]$ otherwise gains from trade would be negative. Second, note from (43) that if $\theta\to\infty^{+}$ then $\chi(z,\theta)\to-iz<0$. We can then restrict $\theta\in\left[0,\hat{\theta}\right]$ where $\hat{\theta}$ is an upper bound on θ. Finally, (42) implies $\gamma\geq k-\chi(z,\theta)$. Since $\chi(z,\theta)\leq u(q^{*})-c(q^{*})$ we have $\gamma\geq k-\left[u(q^{*})-c(q^{*})\right]$. Combining (42) with (43) implies $\gamma\leq k$ so that $\gamma\in\left[k-\left\{u(q^{*})-c(q^{*})\right\},k\right]$. From the Theorem of the Maximum, all maximizers lie in a compact, upper hemicontinuous set $\Upsilon_{Q}=[0,u(\hat{q})]\times\left[0,\hat{\theta}\right]\times\left[k-\left\{u(q^{*})-c(q^{*})\right\},k\right]$ and the maximum is continuous.

A.9. Proof of Proposition 3

To begin, rewrite (46) as

$$\Lambda\left(i,k\right)=\max_{z,\theta}\lambda\left(z,\theta;i,k\right) \tag{103}$$

with

$$\lambda\left(z,\theta;i,k\right)=\left(1-e^{-\theta}-\theta e^{-\theta}\right)\left\{-c\left[u^{-1}(z)\right]+z\right\}+\theta e^{-\theta}\left\{u\left[c^{-1}(z)\right]-z\right\}-\theta\left(iz+k\right),$$

and note that for all $i\in I$ such that $\Lambda\left(i,k\right)>0$ and $\chi\left(z,\theta\right)\geq0$ an equilibrium exists.

Part (i) : Set $i=0$ and assume an interior solution exists. The first-order condition with respect to θ yields

$$e^{-\theta}\left\{u\left[c^{-1}(z)\right]-z\right\}+\theta e^{-\theta}\left\{\left[z-c\left[u^{-1}(z)\right]\right]-\left[u\left[c^{-1}(z)\right]-z\right]\right\}=k, \tag{104}$$

which using $u(q_m) = c(q_p) = z$ yields

$$e^{-\theta}(1-\theta)\{u(q_p) - c(q_p)\} + \theta e^{-\theta}\{u(q_m) - c(q_m)\} = k. \tag{105}$$

Then note that

$$\lim_{k\to 0} \partial\lambda(z,\theta;0,k)/\partial\theta = e^{-\theta}(1-\theta)\{u(q_p) - c(q_p)\} + \theta e^{-\theta}\{u(q_m) - c(q_m)\} > 0. \tag{106}$$

This implies that $\theta^* \to \infty^+$ when $k \to 0$. Inserting this into (103) yields

$$\lim_{k\to 0} \Lambda(0,k) = \max_z -c\left[u^{-1}(z)\right] + z, \tag{107}$$

$$= \max_{q_m} u(q_m) - c(q_m), \tag{108}$$

$$= u(q^*) - c(q^*) > 0 \tag{109}$$

so that $z^* \to c(q^*)$. Since $\Lambda[0, u(q^*) - c(q^*)] = 0$ and $\partial\Lambda(0,k)/\partial k = -\theta < 0$, we conclude $\Lambda(0,k) > 0$ for all $k \in (0, u(q^*) - c(q^*)]$. Finally, since $\chi(z,\theta) \geq 0$ when $i = 0$ an equilibrium exists.

To show agents trade $q_m < q^* < q_p$, take the first-order condition in (46) with respect to z when $i = 0$,

$$\theta e^{-\theta}\left[\frac{u'\left[c^{-1}(z)\right]}{c'\left[c^{-1}(z)\right]} - 1\right] + \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left[1 - \frac{c'\left[u^{-1}(z)\right]}{u'\left[u^{-1}(z)\right]}\right] = 0, \tag{110}$$

which using $u(q_m) = c(q_p) = z$ yields

$$\theta e^{-\theta}\left[\frac{u'(q_p) - c'(q_p)}{c'(q_p)}\right] = \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left[\frac{c'(q_m) - u'(q_m)}{u'(q_m)}\right] = 0. \tag{111}$$

Equation (111) imposes that $u'(q_p) - c'(q_p)$ is of the same sign as $c'(q_m) - u'(q_m)$. Since $q_p > q_m$ these two differences must be negative implying $u'(q_p) < c'(q_p)$ and $c'(q_m) < u'(q_m)$ so that $q_m < q^* < q_p$. To compute the value of the fee, simply rewrite (105) as

$$e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\} + \theta e^{-\theta}\{[u(q_m) - c(q_m)] - [u(q_p) - c(q_p)]\} = k \tag{112}$$

which by identification using (42) yields γ in Proposition 3.

Finally, for a given $\theta > 0$ there exists a unique pair (q_m, q_p) that satisfies (111) and then a unique equilibrium z. Given $z \in [0, u(\hat{q})]$, there exists a unique θ that satisfies (104). The equilibrium is unique. To see it is a maximum, note that $\lambda(0, \theta; 0, k) = \lambda(u(\hat{q}), \theta; 0, k) < 0$. Similarly $\lambda(z, 0; i, k) = \lambda(z, \infty^+; i, k) < 0$.

Part (ii) : Assume $z > 0$ as $i \to \infty^+$. Since γ is bounded, the participation constraint (42) is violated as $i \to \infty^+$ and no equilibrium exists. Assume $z = 0$ as $i \to \infty^+$. Then $u\left[c^{-1}(z)\right] = 0$. Again participation constraint (42) is violated and no equilibrium exists.

A.10. Proof of Proposition 4

Part (i) : Setting $i = 0$ in (46) and multiplying by s yields (48). From the Envelope Theorem $\partial \Lambda(i, k) / \partial i = -\theta z < 0$ so that the maximum is decreasing in i.

Part (ii) : We have seen in Proposition 3 that $q_m < q^* < q_p$. To see that the equilibrium is inefficient on the extensive margin as well, comparing (52) with (105) shows that equilibrium θ at the Friedman rule cannot be the same as θ^* in (53) chosen by the central planner since $q_m \neq q_p \neq q^*$.

Part (iii) : To show that sellers would not post the same terms of trade at the Friedman rule when not allowed to charge a fee, set $i = 0$ and $\gamma = 0$ in the seller's program which gives

$$\max_{z,\theta} \left(1 - e^{-\theta} - \theta e^{-\theta}\right) \left\{-c\left[u^{-1}(z)\right] + z\right\} \tag{113}$$

$$\text{s.t.} \quad e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\} = k. \tag{114}$$

Using (114) to plug θ into (113), taking the first-order condition with respect to z and using $u(q_m) = c(q_p) = z$ yields

$$\theta e^{-\theta}\left[\frac{u'(q_p) - c'(q_p)}{c'(q_p)}\right] = B\left[\frac{c'(q_m) - u'(q_m)}{u'(q_m)}\right], \tag{115}$$

with

$$B = e^{-\theta}\left[u(q_p) - z\right]\frac{u(q_p) - z - k(1 + \theta)}{k\left[z - c(q_m)\right]}. \tag{116}$$

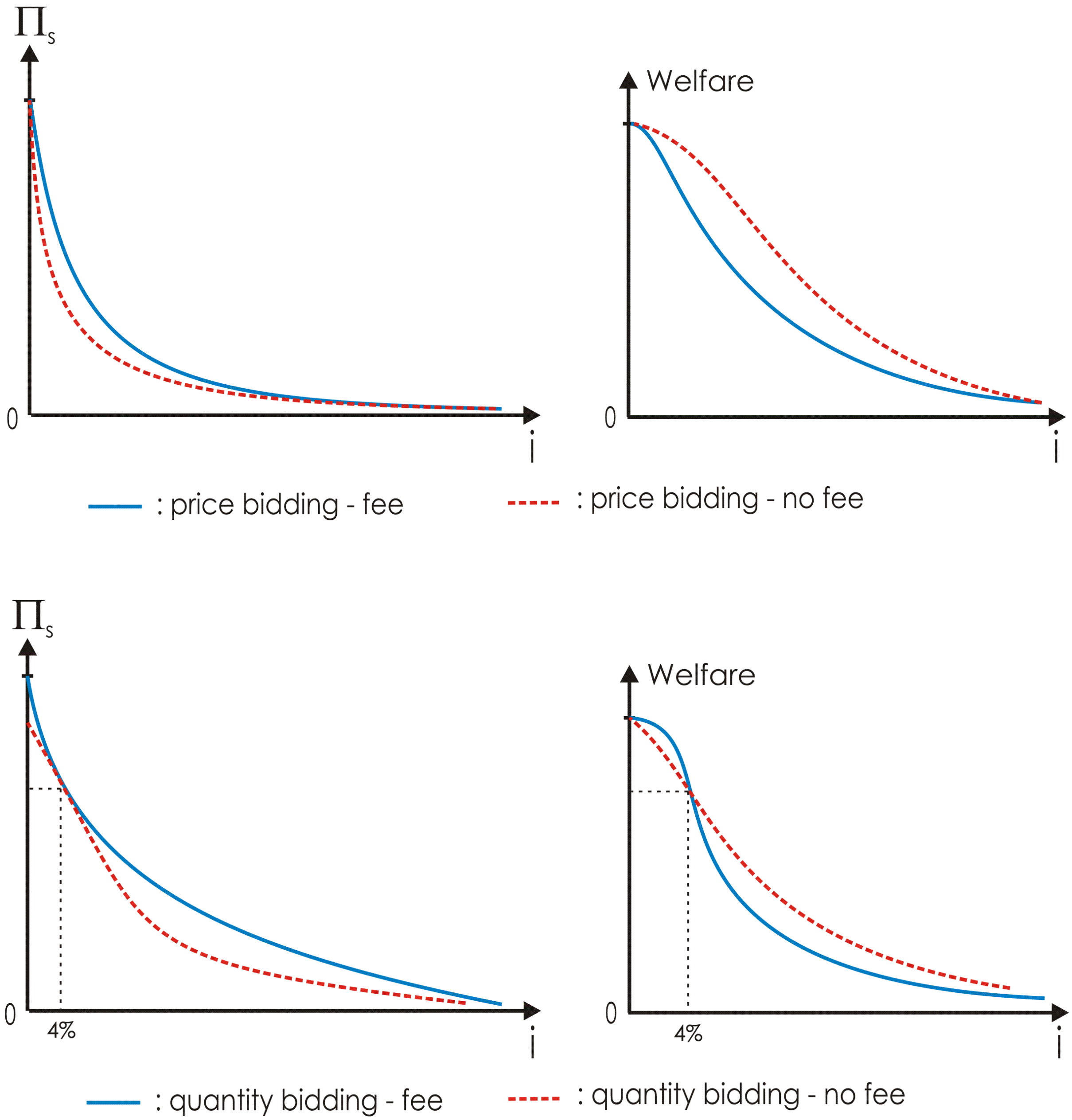
Using (114) to replace $u(q_p) - z$ by $e^{\theta}k$ in B yields

$$
\begin{aligned}
B &= e^{-\theta}\left[u(q_p) - z\right] \frac{k\left(e^{\theta} - 1 - \theta\right)}{k\left[z - c(q_m)\right]} && (117) \\
&= \left(1 - e^{-\theta} - \theta e^{-\theta}\right) \frac{u(q_p) - z}{z - c(q_m)}. && (118)
\end{aligned}
$$

From (111), (115) and (118), for the first-order condition with respect to z to yield the same z with and without a fee requires $u(q_p) - z = z - c(q_m)$. Inserting this equality into the expression of the fee implies $\gamma = 0$, a contradiction.

A.11. Comparative statics - Sellers' profit and welfare in both models

See next page.


Π_s
0
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Welfare
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: price bidding - fee
: price bidding - no fee
Π_s
0
4%
i
Welfare
0
4%
i
: quantity bidding - fee
: quantity bidding - no fee

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