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**Notes on the Construction of Geometric Representations of Confidence  
Intervals of Ratios using Stata, Gauss and Eviews**

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# Notes on the Construction of Geometric Representations of Confidence Intervals of Ratios using Stata, Gauss and Eviews

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## Abstract:

These notes demonstrate how one can define optimization problems whose solutions can be interpreted as the Delta and the Fieller confidence intervals for a ratio of normally distributed parameter estimates. Also included in these notes are the details of the derivation of the slope of a constraint ellipse that is common to both optimizations.

In addition, these notes provide an example of how one might generate a graphic representation of both optimization problems using the Stata, Gauss and Eviews statistical computer programs.

**Key words:** Fieller method, Delta method, marginal ellipse

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## 1. Introduction

A statistic defined as the ratio of two normally distributed random variables is often encountered in applied work. The Delta method has been nominated as the most common technique for drawing inferences for such nonlinear combinations. The primary alternative for the computation of the confidence intervals of ratios is the Fieller method (or theorem) (1932, 1944, and 1954) which is derived from the properties of a ratio of bivariate normally distributed random variables (see Marsaglia (1965) and Hinkley (1969) for a detailed discussion of these cases and Zerbe (1978) for an application to the general linear model). In these notes we demonstrate the derivation of the two optimization problems whose solutions are these bounds as well as the slope of the constraint ellipse. We also present an example of how the constraint ellipse may be constructed geometrically with a number of widely available computer programs. This paper provides details for the analysis given in Hirschberg and Lye (2009).

## 2. The Delta confidence interval for a ratio of parameter estimates as the solution to an optimization problem.

It can be shown that the  $100(1-\alpha)\%$  confidence interval for a linear combination of a vector of normally distributed random variables is the solution to the constrained optimization problem as proposed by Durand (1954) and Scheffé (1959 appendix III)

$$L = (\mathbf{a}'\boldsymbol{\beta} + c) - \lambda \left[ (\boldsymbol{\beta} - \mathbf{B})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta} - \mathbf{B}) - z^2 \right] \quad (1)$$

In the general case of a linear combination of a  $k$  dimensional normally distributed random vector:

$$\mathbf{B}_{k \times 1} \sim N(\boldsymbol{\beta}_{k \times 1}, \boldsymbol{\Sigma}_{k \times k}) \quad (2)$$

$\boldsymbol{\Sigma}$  is assumed non-singular and the linear combination is defined as  $\theta = \mathbf{a}'\boldsymbol{\beta} + c$  and  $\mathbf{a}$  is a  $k \times 1$  constant vector and  $c$  is a constant. We propose that the  $(1-\alpha)100\%$  confidence interval for the

estimate of  $\theta$  ( $\hat{\theta} = \mathbf{a}'\boldsymbol{\beta} + c$ ) can be found from the solution to the constrained optimization defined by

(1). Where  $z$  is the appropriate  $z$ -statistic for the  $(1 - \alpha)100\%$  confidence bound (i.e. for

$\alpha = .05$   $z = 1.96$ ), and  $z^2 = \chi_1^2$  the square of which is equivalent to a chi-square distributed random variable with one degree of freedom.

Taking the first derivatives of  $L$  with respect to the parameters and the multiplier and setting them equal to zero we find the following first order conditions which can be solved for the optimal values  $\tilde{\boldsymbol{\beta}}$  and  $\tilde{\lambda}$ :

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \mathbf{a} - \tilde{\lambda} 2 \boldsymbol{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}} - \mathbf{B}) = 0 \quad (3a)$$

$$\frac{\partial L}{\partial \lambda} = (\tilde{\boldsymbol{\beta}} - \mathbf{B})' \boldsymbol{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}} - \mathbf{B}) - z^2 = 0 \quad (3b)$$

Rewriting (3a) we find that:

$$\tilde{\boldsymbol{\beta}} - \mathbf{B} = \frac{1}{2} \tilde{\lambda}^{-1} \boldsymbol{\Sigma} \mathbf{a} \quad (4)$$

Which can then be substituted into (3b) to solve for  $\frac{1}{4} \tilde{\lambda}^{-2}$ :

$$\frac{1}{4} \tilde{\lambda}^{-2} = z^2 (\mathbf{a}' \boldsymbol{\Sigma} \mathbf{a})^{-1} \quad (5)$$

By definition the covariance matrix is positive semi-definite thus  $\mathbf{a}' \boldsymbol{\Sigma} \mathbf{a} \geq 0$  and we can take the square root of both sides of (5) to obtain two values for  $\frac{1}{2} \tilde{\lambda}^{-1}$ :  $\frac{1}{2} \tilde{\lambda}^{-1} = \pm z (\mathbf{a}' \boldsymbol{\Sigma} \mathbf{a})^{-\frac{1}{2}}$ . By adding  $\mathbf{B}$  to both sides of (4) we find the optimal value of  $\boldsymbol{\beta}$  from:  $\tilde{\boldsymbol{\beta}} = \mathbf{B} + \frac{1}{2} \tilde{\lambda}^{-1} \boldsymbol{\Sigma} \mathbf{a}$ . By pre-multiplying both sides of this equation by  $\mathbf{a}'$  and then substituting for the optimal value of  $\frac{1}{2} \tilde{\lambda}^{-1}$  we can now find an expression for the optimal value of the constrained linear combination defined as:  $\tilde{\theta} = \mathbf{a}' \tilde{\boldsymbol{\beta}} + c$  as:

$$\tilde{\theta} = (\mathbf{a}' \mathbf{B} + c) \pm z (\mathbf{a}' \boldsymbol{\Sigma} \mathbf{a})^{\frac{1}{2}} \quad (6)$$

Which is the usual expression for the  $(1 - \alpha)100\%$  confidence interval of a linear combination of multivariate normally distributed random variables.

In the case of the Delta approximation applied to the ratio of parameters we have that the

approximation is given by the linear combination:

$$\begin{aligned}\tilde{\psi} &= \hat{\psi} + \left(\frac{1}{b_2}\right)(\beta_1 - b_1) - \left(\frac{b_1}{b_2^2}\right)(\beta_2 - b_2) \\ &= \hat{\psi} + \left(\frac{1}{b_2}\right)\beta_1 - \left(\frac{\hat{\psi}}{b_2}\right)\beta_2\end{aligned}\quad (7)$$

Where  $\hat{\psi} = \left(\frac{b_1}{b_2}\right)$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\right)$  we then obtain an optimization in two dimensions by

application of equation (1) as:

$$\mathbf{L} = \left(\left(\frac{1}{b_2}\right)\beta_1 - \left(\frac{\hat{\psi}}{b_2}\right)\beta_2 + \hat{\psi}\right) - \lambda \left( \begin{bmatrix} (b_1 - \beta_1)(b_2 - \beta_2) \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - z^2 \right) \quad (8)$$

### 3. The Fieller confidence interval for a ratio of parameter estimates as the solution to an optimization problem.

Following the form of the discussion in Von Luxburg and Franz (2004) the bounds of the ratio of the means where the restriction is defined by the confidence ellipsoid of the two parameters can be found from the solution to the following constrained optimization problem:

$$\mathbf{L} = \frac{\beta_1}{\beta_2} - \lambda \left( \begin{bmatrix} (b_1 - \beta_1)(b_2 - \beta_2) \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - z^2 \right) \quad (9)$$

If we substitute  $\omega_{ij}$  as the elements of the inverse of the covariance and use  $\psi = \beta_1/\beta_2$  and  $\beta_1 = \beta_2\psi$

(9) can be written as:

$$\mathbf{L} = \psi - \lambda \left( \begin{array}{l} (b_2 - \beta_2)(\omega_{22}(b_2 - \beta_2) + \omega_{12}(b_1 - \psi\beta_2)) + \\ (b_1 - \psi\beta_2)(\omega_{12}(b_2 - \beta_2) + \omega_{11}(b_1 - \psi\beta_2)) - z^2 \end{array} \right) \quad (10)$$

The first order partial derivatives of  $\mathbf{L}$  with respect to  $\beta_2$ ,  $\psi$ , and  $\lambda$  evaluated at the optimal values defined as  $\tilde{\beta}_2$ ,  $\tilde{\psi}$ , and  $\tilde{\lambda}$ :

$$\frac{\partial \mathbf{L}}{\partial \psi} = 2\tilde{\lambda}(\tilde{\beta}_2^2\omega_{12} - b_2\tilde{\beta}_2\omega_{12} - b_1\tilde{\beta}_2\omega_{11} + \tilde{\psi}\tilde{\beta}_2^2\omega_{11}) + 1 \quad (11a)$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda} = & 2b_1b_2\omega_{12} + b_1^2\omega_{11} + b_2^2\omega_{22} - z^2 - \tilde{\beta}_2(2b_1\omega_{12} + 2b_2\omega_{22} - \tilde{\beta}_2\omega_{22}) \\ & - 2\tilde{\psi}\tilde{\beta}_2(b_1\omega_{11} + b_2\omega_{12} - \tilde{\beta}_2\omega_{12}) + \tilde{\psi}^2\tilde{\beta}_2^2\omega_{11} \end{aligned} \quad (11b)$$

$$\frac{\partial L}{\partial \beta_2} = 2\tilde{\lambda}(\beta_2(\omega_{22} + 2\tilde{\psi}\omega_{12} + \tilde{\psi}^2\omega_{11}) - b_2\omega_{22} - b_1\omega_{12} - \tilde{\psi}(b_1\omega_{11} + b_2\omega_{12})) \quad (11c)$$

The first order conditions for an optimum are given by setting these partial derivatives to zero. First we can solve (11c) for  $\tilde{\beta}_2$ . Then we substitute for  $\tilde{\beta}_2$  in (11b) which results in a quadratic equation in  $\tilde{\psi}$ . The roots of this quadratic are given by:

$$\tilde{\psi}_i = \frac{\left( -b_1b_2\omega_{11}\omega_{22} - z^2\omega_{12} + b_1b_2\omega_{12}^2 \pm z \left( z^2(\omega_{12}^2 - \omega_{11}\omega_{22}) + 2b_1b_2\omega_{11}\omega_{12}\omega_{22} + b_2^2\omega_{11}\omega_{22}^2 - 2b_1b_2\omega_{12}^3 - b_1^2\omega_{11}\omega_{12}^2 + b_1^2\omega_{11}^2\omega_{22} - b_2^2\omega_{12}^2\omega_{22} \right)^{1/2} \right)}{z^2\omega_{11} - b_2^2\omega_{11}\omega_{22} + b_2^2\omega_{12}^2} \quad (12)$$

Alternatively, the Fieller method is defined as the solution for the values of  $\tilde{\psi}$  as:

$$\tilde{\psi}_i = \left( z^2\sigma_{22} - b_2^2 \right)^{-1} \left( z^2\sigma_{12} - b_1b_2 \pm z \left( \begin{array}{c} -z^2\sigma_{11}\sigma_{22} - 2b_1b_2\sigma_{12} \\ + z^2\sigma_{12}^2 + b_2\sigma_{11}^2 + b_1^2\sigma_{22} \end{array} \right)^{1/2} \right) \quad (13)$$

Using the correspondence between the covariance matrix and its inverse defined as:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \left( \sigma_{12}^2 - \sigma_{11}\sigma_{22} \right)^{-1} \begin{bmatrix} -\sigma_{22} & \sigma_{12} \\ \sigma_{12} & -\sigma_{11} \end{bmatrix} \quad (14)$$

We can show that the roots for the constrained optimization problem solution in (12) are equal to the expression (13).

#### 4. The determination of the slope of the constraint ellipse evaluated at the estimated values.

Following Marks (1982) we derive the slope of tangents to the constraint ellipse defined by:

$$0 = \begin{bmatrix} (b_1 - \beta_1) & (b_2 - \beta_2) \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - z^2 \quad (15)$$

Solving this quadratic for  $\beta_1$  we obtain:

$$\beta_1 = \frac{1}{\omega_{11}} \left( b_1\omega_{11} + b_2\omega_{12} - \beta_2\omega_{12} \pm \sqrt{2b_2\beta_2\omega_{11}\omega_{22} + z^2\omega_{11} - 2b_2\beta_2\omega_{12}^2 - b_2^2\omega_{11}\omega_{22} - \beta_2^2\omega_{11}\omega_{22} + b_2^2\omega_{12}^2 + \beta_2^2\omega_{12}^2} \right) \quad (16)$$

where  $b_1$  and  $b_2$  are the estimated parameters  $\omega_{ij}$  are elements of the inverse of the covariance of the regression parameters, and  $z^2$  is the critical value of the Normal distribution for a two tailed test.

Define the elements of the inverse of the covariance matrix in terms of the correlation coefficient

$(\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}})$  given as:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_{11}} & \frac{\rho}{\sigma_{12}} \\ \frac{\rho}{\sigma_{12}} & \frac{1}{\sigma_{22}} \end{bmatrix}$$

And take the first derivative of  $\beta_1$  with respect to the value of  $\beta_2$  we define the slope of the ellipse as:

$$\frac{\partial\beta_1}{\partial\beta_2} = \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} \left( \rho \pm (b_2 - \beta_2) \sqrt{\frac{\rho^2 - 1}{b_2^2 - 2b_2\beta_2 + \beta_2^2 - t^2\sigma_{22}}} \right) \quad (17)$$

Which when evaluated at the estimate  $\beta_2 = b_2$  results in:  $\frac{\partial\beta_1}{\partial\beta_2} = \rho\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}$ .

## 5. An example of the construction of the constraint ellipse using Stata, Gauss, and Eviews.

The data for this example is the file *californian.dta* and from Stock and Watson's text (2007, p. 114). The data is for 420 school districts in the year 1998 on the average fifth grade test scores ( $y$ ) and the average annual per capita income in the school district measured in tens of thousands of 1998 dollars ( $z$ ). The regression of interest is:

$$y = \beta_0 + \beta_1 z + \beta_2 z^2 + \varepsilon \quad (18)$$

The ratio of interest in this example is the turning point of the quadratic function which determines the level of income per capita at which the relationship between test scores and income changes sign.

This level of income is defined as:

$$\psi = \frac{\beta_1}{-2\beta_2} \quad (19)$$

The standard  $100(1-\alpha)\%$  joint confidence ellipse produced in most packages that plot ellipses is specified as

$$(\hat{\delta} - \delta)' \left[ \text{cov}(\hat{\delta}) \right]^{-1} (\hat{\delta} - \delta) = 2F_{\alpha}(2, T - K) \quad (20)$$

where  $\delta = \begin{bmatrix} \beta_1 \\ -2\beta_2 \end{bmatrix}$  and  $\hat{\delta}$  is the corresponding OLS estimate. However, the marginal 100(1- $\alpha$ )%

confidence ellipse as defined for the optimizations defined in (needed to obtain the Fieller interval for  $\psi$ ) is given by:

$$(\hat{\delta} - \delta)' [\text{cov}(\hat{\delta})]^{-1} (\hat{\delta} - \delta) = F_{\alpha}(1, T - K) \quad (21)$$

The marginal 100(1- $\alpha$ )% confidence ellipse can be obtained from (20) by specifying an equivalent confidence level such that  $F_{\alpha}(1, T - K) = 2F_{\tilde{\alpha}}(2, T - K)$ . The Table below lists the appropriate confidence levels to use in order to construct the appropriate marginal ellipse when the computer package is designed to plot only joint ellipses (such as the case of Eviews).

<i>df</i>	$(1 - \tilde{\alpha})$ when $(1 - \alpha) = .95$	$(1 - \tilde{\alpha})$ when $(1 - \alpha) = .90$
10	0.867	0.758
20	0.860	0.750
30	0.858	0.747
40	0.856	0.746
50	0.856	0.745
100	0.855	0.743
200	0.854	0.742
5000	0.854	0.742

**Table 1.** The correspondence between  $(1 - \tilde{\alpha})$  and  $(1 - \alpha)$  for various sample sizes.

From this table it can be seen that in a moderately sized sample (>120) to obtain a 95% marginal confidence ellipse (ie corresponding to  $\alpha = 0.05$  in (21)) using (20), the value of  $\tilde{\alpha}$  in (20) needs to be set to 0.15, which corresponds to an 85% joint confidence ellipse.

### 5.1 Stata Program

In the Stata program we rewrite the regression equation defined in (17) as

$$y = \gamma_0 + \gamma_1 z + \gamma_2 (-0.5z^2) + \varepsilon \quad (22)$$

So that the turning point defined in (19) becomes

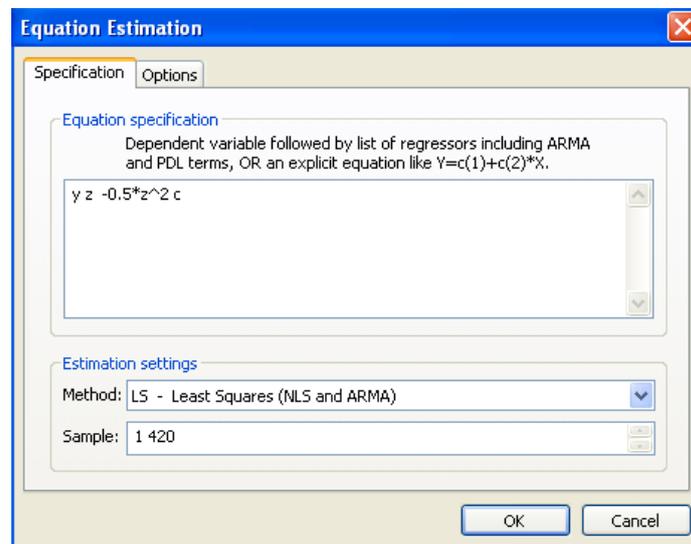
$$\psi = \frac{\gamma_1}{\gamma_2} \quad (23)$$

which can be estimated using the OLS estimates  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  from (22). The program contains 4 lines to generate a confidence ellipse. The ellipse is generated by calling upon the program *ellip* from Alexandersson (2004). For large samples, to obtain the appropriate dimensions of the marginal confidence ellipse the appropriate boundary constant is a chi square with 1 degree of freedom. The options *xlab* and *ylab* are used to plot the ellipse over appropriate values.

```
use "c:\cubic\californian.dta"
generate z2 = -0.5*z^2
regress y z z2
ellip z z2, coefs c(chi2 1) ylab(0 5 10 15 20 25 30 35 40 45) xlab(0 1 2 3 4 5 6 7
8 9 10 11)
```

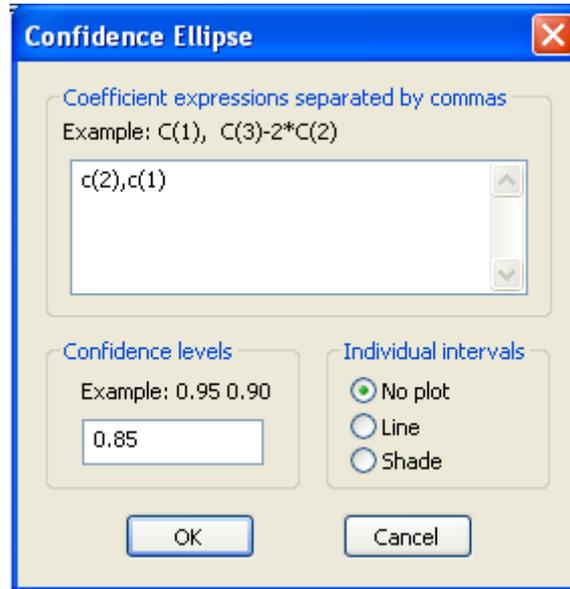
## 5.2 Eviews program (versions 5 and 6)

Once the data have been read into Eviews, the first step is to estimate the regression specified in (22) as follows:



**Figure 1** The regression specification dialogue window in Eviews

Then to obtain the 95% marginal confidence ellipse, we use *Confidence Ellipse* available under the *view/Coefficient Tests option* and specify the confidence level as 0.85. The estimated coefficients correspond to  $c(2)$ , the estimate of the denominator of the ratio (ie  $-0.5*z^2$ ) and  $c(1)$ , the estimate of the numerator of the ratio (ie  $z$ ). Note that the scales on both the  $x$  and  $y$  axes can be altered in the default graph by clicking on them.



**Figure 2** The ellipse dialogue window for the analysis of regression results in Eviews.

### 5.3 Gauss Program

Gauss is a general purpose computer program for the computation of linear algebra with a similar syntax to MATLAB, Proc IML in SAS and R. In the Gauss program listed below we estimate (22) where we refer to the OLS estimates of  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  as  $g1$  and  $g2$  respectively. *fstat* uses the function *invf* to compute the critical value from the F distribution. Note to obtain the 95% critical value for the appropriate confidence ellipse this is required to be set to 0.15. The program to compute the confidence ellipse is modified from Hill and Adkins (2001, pages 47 & 62). Note that *scale* is used in the plotting routine to plot the ellipse over the appropriate values.

```

/* Read in data and generate data for regression*/
num = 420;k=3; df=num-k;
load data[num,2]= c:\cubic\ellipse.txt;

y = data[.,1];
z = data[.,2];
z2 = -0.5*z.^2;
/* Run Regression*/
__con=1;
{nam,m,b,stb,vc,std,sig,cx,rsq,resid,dbw} = ols(0,y,z~z2);

/* Obtain estimates and variances and covariances */
g1 = b[2,1];
g2 = b[3,1];

cov = zeros(2,2);
cov[1,1] = vc[2,2];
cov[1,2] = vc[2,3];
cov[2,1] = cov[1,2];

```

```

cov[2,2] = vc[3,3];

/* Choose critical value for ellipse*/
fstat = invf(2,df,.15);
fn invf(df1,df2,alpha) = 0.95 + 0.05*
minindc( abs (cdfFc( seqa(1,.05,2000),df1,df2)- alpha ));

/* Generate ellipse */
a = inv(cov);
q = a[1,1]*a[2,2] - a[1,2]*a[1,2];
lb = g2 - sqrt(2*fstat*a[1,1]/q);
ub = g2 + sqrt(2*fstat*a[1,1]/q);
beta2 = seqa(lb,(ub-lb)/100,101);
csq = (g2 - beta2)^2*( - q/a[1,1]^2)
+ fstat*2/a[1,1];
c = sqrt(abs(csq));
betala = g1 + (g2-beta2)*a[1,2]/a[1,1] + c;
betalb = g1 + (g2-beta2)*a[1,2]/a[1,1] - c;

d1 = beta2|rev(beta2);
d2 = betala|rev(betalb);

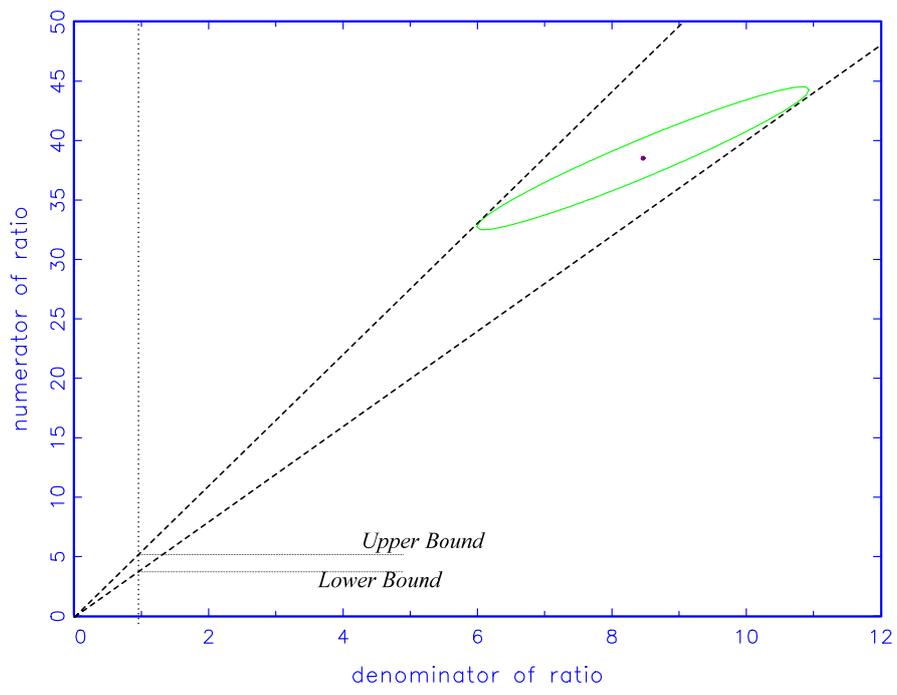
/* Plot the ellipse and estimated values of the numerator and denominator of
ratio*/
library pgraph;
_paxes = 1;
_pdate = 0;
_pmcolor=9;
let xx = 0 12;
let yy = 0 48;
scale(xx,yy);

xlabel("denominator of ratio");
ylabel("numerator of ratio");
xy(d1,d2);
_psym = zeros(1,7);
_psym[1] = g2;
_psym[2] = g1;
_psym[3] = 8;
_psym[4] = 1;
_psym[5] = 5;
_psym[6] = 1;
_psym[7] = 2;

```

#### 5.4 Construction of Fieller intervals.

Once the ellipse has been drawn it can be pasted into a standard drawing package such as Microsoft Visio or one can construct the bounds using a ruler on the printed version. For the figure shown below we imported the default graph from Gauss into Microsoft Visio and added in the additional lines to obtain the lower and upper bounds of the Fieller interval. Note it is necessary to have the origin (0,0) included in the graph to construct the intervals. See Hirschberg and Lye (2009) for details as to the steps for construction of the interval.



**Figure 3** Example of ellipse generated by GAUSS routine with lines added with Microsoft VISIO.

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