THE UNIVERSITY OF MELBOURNE

Department of Economics

Working Paper Series

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Paulo Barelli, Suren Basov,
Mauricio Bugarin and Ian King

April 2010

Research Paper Number 1096

ISSN: 0819-2642
ISBN: 978 0 7340 4449 5

Department of Economics
The University of Melbourne
Parkville VIC 3010
www.economics.unimelb.edu.au

The Inclusiveness of Exclusion*

Paulo Barelli Suren Basov Mauricio Bugarin Ian King†

April 2, 2010

Abstract

We extend Armstrong's (1996) result on exclusion in multi-dimensional screening models in two key ways, providing support for the view that this result is quite generic and applicable to many different markets. First, we relax the strong technical assumptions he imposed on preferences and consumer types. Second, we extend the result beyond the monopolistic market structure to generalized oligopoly settings with entry. We also analyse applications to several quite different settings: credit markets, automobile industry, research grants, the regulation of a monopolist with unknown demand and cost functions, and involuntary unemployment in the labor market.

JEL Codes: C73, D82

Key words: Multidimensional screening, exclusion, regulation of a monopoly, involuntary unemployment.

*We are grateful to Mark Armstrong for helpful comments.

†Paulo Barelli: University of Rochester, Suren Basov: La Trobe University, Mauricio Bugarin: Insper Instituto de Ensino e Pesquisa, Sao Paulo, Ian King: University of Melbourne.

1 Introduction

When considering the problem of screening, where sellers choose a sales mechanism and buyers have private information about their types, it is well known that the techniques used in the multidimensional setting are not as straightforward as in the one dimensional setting. As a consequence, while we have a host of successful applications with one dimensional types, to date we have only a few scattered papers that allow for multidimensional types. This is unfortunate because in most economic applications multidimensional types are needed to capture the basic economics of the environment, and the propositions coming from the one dimensional case do not necessarily generalize to the multidimensional case.[1]

One of the most celebrated results in the theory of multidimensional screening, though, comes from Armstrong (1996) where he shows that a monopolist will find it optimal to not serve some fraction of consumers in equilibrium, even when there is positive surplus associated with those consumers. In settings where consumers vary in at least two different ways, monopolists will choose a sales mechanism that excludes a positive measure of consumers. The intuition behind this result is rather simple: Consider a situation where the monopolist serves all consumers; if she increases the price by $\varepsilon > 0$ she earns extra profits of order $O(\varepsilon)$ on the consumers who still buy the product, but will lose only the consumers whose surplus was below ε. If $m > 1$ is the dimension of the vector of consumers' taste characteristics, then the measure of the set of the lost consumers is $O(\varepsilon^m)$. Therefore, it is profitable to increase the price and lose some consumers. In principle, this result has, potentially, profound implications across a wide range of economic settings. The general belief that heterogeneity of consumer types is likely to be more than unidimensional in nature, for many different commodities, and that these types are likely to be private information, underlines the importance of this result.[2]

However the result itself was derived under a relatively strong set of assumptions that could be seen as limiting its applicability, and subsequent research has identified conditions under which the result does not hold. In particular, in Armstrong's original analysis, he assumed that the utility functions of the agents are

[1] See Rochet and Stole (2003) and Basov (2005) for surveys of the literature.

[2] The type of an economic agent is simply her utility function. If one is agnostic about the preferences and does not want to impose on them any assumptions beyond, perhaps, monotonicity and convexity then the most natural assumption is that the type is infinite dimensional.

homogeneous and convex and in their types, and that these types belong to a strictly convex and compact body of a finite dimensional space. Basov (2005) refers to the latter as the joint convexity assumption and argues that though convexity of utility and types and convexity of the types set separately are not restrictive and can be seen as a choice of parametrization, the joint convexity assumption is technically restrictive.

These assumptions have no empirical foundation, and are nonstandard. For instance, the benchmark case of independent types does not satisfy these assumptions, because the type space, in this case, is a multidimensional box, which is not strictly convex. There is, in general, no theoretical justification for a particular assumption about the curvature of utility functions with respect to types, as opposed to, say, quasi-concavity of utility functions with respect to goods. In the same line, in general, there is no justification, other than analytical tractability, for type spaces to be convex, and for utility functions to be homogeneous in types. Both Armstrong (1999b) and Rochet and Stole (2003) found examples outside of these restrictions where the exclusion set is empty.

As we will argue below these counter-examples constitute knife-edge cases and are not generic. We will also argue that exclusion is generic under more general market structures. i.e. the result is actually quite robust. We then provide examples where we believe it could also be quite important.

We start with relaxing the joint convexity and homogeneity assumptions, and show that they are not necessary for the result. Exclusion is generically optimal in the family of models where types belong to sets of locally finite perimeter (which is a class of sets that includes all of the examples the authors are aware of in the literature) and utility functions are smooth and monotone in types. We show that the examples considered in Armstrong (1999b) and Rochet and Stole (2003) are, themselves, very special cases. We then go on to show that the exclusion results can be generalized to the case of an oligopoly and an industry with free entry. Therefore, the inability of some consumers to purchase the good at acceptable terms is solely driven by the multi-dimensional nature of private information rather than by market conditions or nature or distribution of the consumers' tastes.

To illustrate the generality of the results, we apply them in several quite different settings: credit markets, the automobile industry, and research grants. We also pay particular attention to two applications: the first being the regulation of a monopolist with unknown demand and cost functions, and the second being the existence of equilibrium involuntary unemployment. The former application picks up of the analysis in Armstrong (1999b), where he reviews Lewis and Sappington (1988) and conjectures that exclusion is probably an issue in their analysis. At the time, Armstrong could not prove the point, due to the lack of a more general exclusion result. With our result in hand, we are able to prove Armstrong's conjecture. The latter application is a straightforward way of showing that, when workers have multi-dimensional characteristics, it is generically optimal for employers (with market power in the labor market) to not hire all workers.

Most generally, we believe that the main result of this paper is that private information leads to exclusion in any almost any realistic setting. To avoid it one has either to assume that all allowable preferences lie on a one-dimensional continuum or construct very specific type distributions and preferences.

The remainder of this paper is organized as follows. In Section 2 we present the monopoly problem with consumers that have a type-dependent outside option and the derive conditions under which it is generically optimal to have exclusion. In Section 3 we generalize the results for the case of oligopoly and a market with free entry. In Section 4 we discuss how the results can be generalized without the quasilinearity assumption. Sections 5 and 6 present the application to the regulation of a monopolist with unknown demand and cost functions and to involuntary unemployment. The Appendix presents some relevant concepts from geometric measure theory.

2 The Genericity of Exclusion in a Monopolistic Screening Model

Assume a monopolist faces a continuum of consumers and produces a vector of n goods $x \in R^n_+$. The cost of production is given by a strictly convex twice differentiable cost function $c(\cdot) : R^n_+ \to R$. A consumer's utility is given by:

$$v(\alpha, x) - t(x)$$

where $\alpha \in \Omega$ is her unobservable type, $t(x)$ is the amount of money transferred to the monopolist when the consumer purchases x, and $v : \Omega \times R^n \to R$ is a continuously differentiable function, strictly increasing in both arguments. Moreover, we will assume that $\nu(\cdot, x)$ can be extended by continuity to $\overline{\Omega}$. We assume that $\Omega \subset U \subset R^m$ is a Lebesgue measurable set with locally finite perimeter in the open set U, and that α is distributed according to a density $f(\cdot)$ that is Lipschitz continuous and with $supp(f) = \overline{\Omega}$ compact.[3] Consumers have an outside option of value $s_0(\alpha)$, which is assumed to be continuously differentiable and implementable and extendable by continuity to $\overline{\Omega}$.[4]

The monopolist looks for a selling mechanism that maximizes her profits. The Taxation Principle (Rochet, 1985) implies that one can, without loss of generality, assume that the monopolist simply announces a non-linear tariff $t(\cdot)$.

The above considerations can be summarized by the following model. The monopolist selects a function $t : R^n \to R$ to solve

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha)))f(\alpha)d\alpha$$

where $c(x)$ is the cost of producing x and $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_x v(\alpha, x) - t(x) & \text{if } \max(v(\alpha, x) - t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise} \end{cases}, \tag{1}$$

where $x_0(\alpha)$ is the outside option, which implements surplus $s_0(\alpha)$.

The cost function $c\,(x)$ is separable across consumers. Moreover, assume that there is a finite solution to the problem of maximizing the joint surplus $v(\alpha, x) - c(x)$ for every consumer α. Let $s(\alpha)$ denote the surplus obtained by type α:

$$s(\alpha) = \max_x (v(\alpha, x) - t(x)) \tag{2}$$

The celebrated result of Armstrong (1996) states that if Ω is strictly convex, $v(\cdot, x) : \Omega \to R$ is a convex, homogenous of degree one function for all x, and

[3] See Evans and Gariepy (1992) and Chlebik (2002) for the relevant concepts from geometric measure theory. For convenience, a brief summary is presented in the appendix.

[4] For conditions of implementability of a surplus function see Basov (2005).

$s_0(\alpha) = 0$ for all α, then the measure of the set $\Omega_0 = \{\alpha \in \Omega : s(\alpha) = 0\}$ is positive. We will replace these assumptions with Assumptions 1 and 2 below.

Assumption 1 *For any $x \in R^n_+$, the net utility gain of consumption $u(\cdot, x)$ defined by*

$$u(\cdot, x) = v(\cdot, x) - s_0(\cdot) \tag{3}$$

is strictly increasing in α.

In the case when the value of the outside option is type independent, Assumption 1 reduces to the standard assumption that the utility is increasing in α.

Let $\partial_e\Omega$ denote the measure theoretic boundary of Ω. Because Ω has locally finite perimeter, $\partial_e\Omega = \bigcup_{i=1}^{\infty} K_i \cup N$, where K_i is a compact subset of a C^1 hypersurface S_i, for $i = 1, 2, \ldots$, and $\mathcal{H}^{m-1}(N) = 0$, where $\mathcal{H}^{m-1}$ is the $m-1$ dimensional Hausdorff measure.

Assumption 2 *For each $i = 1, 2 \ldots$,*

$$K_i = \{\alpha \in \overline{\Omega} : g_i(\alpha, \beta) = 0\}$$

where $g_i : \overline{\Omega} \times R^J \to R$ is smooth, $\beta \in R^J$, $J \geq 1$, are parameters and, for all $x \in R^n_+$ and all $i = 1, 2, \ldots$, there exists $\beta_0 \in R^J$ such that

$$rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_0) \end{pmatrix} = 2. \tag{4}$$

That is, the parameters β determine the underlying set of models that we consider, and there is one model for which the normal of each K_i and of each level curve of the utility function are not colinear. As will be clear in the examples below, Assumption 2 is stronger than what we need: the rank condition has to be met only at the optimal choices $x(\alpha)$. Also, note that an open set $\Omega \subset R^m$ with topological boundary $\partial\Omega$ that is locally Lipschitz (that is, $\partial\Omega$ is the graph of a Lipschitz function near each $\alpha \in \partial\Omega$) is a set with locally finite perimeter.

Let us consider some examples satisfying Assumptions 1 and 2.

Example 1 *Assume that every consumer has an option to buy nothing and pay nothing, i.e.*

$$s_0(\alpha) = u(\alpha, 0).$$

In this case Assumption 1 reduces to a weak single-crossing condition:

$$v(\cdot, x) - v(\cdot, 0)$$

increases in α *for every* x. *Although usually it is assumed that* $v(\cdot, 0) = 0$ *this need not be so. For example, consider a consumer who has wealth* w *in her account and lives for 2 periods. Her second period wealth can take to values* w_H *or* w_L. *Let* p *be the probability that* $w = w_H$, *and let* $\delta \in (0, 1)$ *be the discount factor, so that the private information of the consumer is characterized by a two-dimensional vector* $\alpha = (p, \delta)$. *The individual's preferences are given by:*

$$U(c_1, c_2) = u(c_1) + \delta E u(c_2)$$

where c_1 *and* c_2 *are the consumption levels in periods one and two respectively. We will assume that wealth is not storable between periods. An individual may approach a bank for a loan* X. *If she does so, she will be asked to repay* $t(X)$ *is the next period, provided her wealth is high and default if it is low. If the individual chooses not to take the credit, her expected utility will be:*

$$U_0(p, \delta) = u(w) + \delta(pu(w_H) + (1 - p)u(w_L))$$

which is type dependent. Consider the following change of variables:

$$\gamma = 1 - \delta,\ q = 1 - p,\ \alpha = (\gamma, q), x = X,\ \Delta u = u(w_H) - u(w_H - t(x)). \quad (5)$$

Then,

$$u(\alpha, x) = u(w + x) - u(w) - (1 - \gamma)(1 - q) \triangle u, \quad (6)$$

which is strictly increasing in α. *Therefore, this example satisfies Assumption 1. Define, on* $(0, 1)^2$, *the functions*

$$g_1(\alpha, 1) = \gamma - 1,\ g_2(\alpha, 0) = \gamma,\ g_3(\alpha, 1) = q - 1,\ g_4(\alpha, 1) = q. \quad (7)$$

Then,

$$\nabla_\alpha g_1(\alpha,1)=(1,0)\,,\ \nabla_\alpha g_2(\alpha,0)=(1,0)\,,\ \nabla_\alpha g_3(\alpha,1)=(0,1)\,,\ \nabla_\alpha g_4(\alpha,0)=(0,1)\,. \tag{8}$$

Moreover,

$$\nabla_\alpha u(\alpha,x)=((1-q)\,\Delta u,\ \ (1-\gamma)\,\Delta u)\,, \tag{9}$$

and, for $\beta_0=0,1,\ i=1,2,\ j=3,4,$

$$\begin{array}{c} rank\begin{pmatrix}\nabla_\alpha u(\alpha,x)\\ \nabla_\alpha g_i(\alpha,\beta_0)\end{pmatrix}=rank\begin{pmatrix}(1-q)\,\Delta u & (1-\gamma)\,\Delta u\\ 1 & 0\end{pmatrix}=2\\ rank\begin{pmatrix}\nabla_\alpha u(\alpha,x)\\ \nabla_\alpha g_j(\alpha,1\end{pmatrix}=rank\begin{pmatrix}(1-q)\,\Delta u & (1-\gamma)\,\Delta u\\ 0 & 1\end{pmatrix}=2\end{array}. \tag{10}$$

Therefore, this example also satisfies Assumption 2. Note, however, that in this example the preferences of the agents are not quasilinear. We will discuss this case in Section 4.

The above example is a natural setting to discuss unavailability of credit to some individuals, which is important to justify monetary equilibria in the search theoretic models of money.[5] The next example comes from the theory of industrial organization.

Example 2 *Suppose a monopolist produces cars of high quality. The utility of a consumer is*

$$u(\alpha,x)=A+\sum_{i=1}^{n}\alpha_i x_i \tag{11}$$

where $A>0$ *can be interpreted as utility of driving a car, and the second term in (11) is a quality premium. Suppose a consumer has three choices: to buy a car from the monopolist, to by a car from a competitive fringe, and to buy no car at all. We will normalize the utility of buying no car at all to be zero. Assume the competitive fringe serves low quality cars of quality* $-x_0$*, where* $x_0\in R^n_{++}$ *at price* p*. That is, the consumers experience disutility from the quality of the cars of the competitive fringe, and the higher their type, the higher the disutility. The utility*

[5] See, for example, Lagos and Wright (2005).

of the outside option in this case is given by:

$$s_0(\alpha) = \max(0, A - p - \sum_{i=1}^{n} \alpha_i x_{0i})$$

and is decreasing in α. Therefore, Assumption 1 holds.

Let us redefine a consumer's utility function by (3). Assumption 1 guarantees that $u(\cdot, x)$ is increasing and we can reformulate the monopolist's problem in the following way: the monopolist selects a function $t : R^n \to R$ to solve

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha \tag{12}$$

where $c(x)$ is the cost of producing a good with quality x and $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_x (u(\alpha, x) - t(x)) & \text{if } \max(u(\alpha, x) - t(x)) \geq 0 \\ x(\alpha) = 0 & \text{otherwise} \end{cases}. \tag{13}$$

We use $t(\cdot)$ to denote the the optimal tariff.[6]

To be able to formulate and prove the main result we have to establish some technical lemmata. For any Lebesgue measurable set $E \subset R^m$ let $\mathcal{L}^m(E)$ denote its Lebesgue measure. Let $\mathcal{K}(R^m)$ be the hyperspace of compact sets in R^m, endowed with the topology induced by the Hausdorff distance d_H, given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset B^{\varepsilon}, B \subset A^{\varepsilon}\}, \tag{14}$$

where

$$A^{\varepsilon} = \bigcup_{\alpha \in A} B_{\varepsilon}(\alpha) \tag{15}$$

and $B_{\varepsilon}(\alpha)$ is the open ball centered at α and with radius $\varepsilon > 0$. Because

$$\lim_{\varepsilon \to 0+} \mathcal{L}^m(E^{\varepsilon}) = \mathcal{L}^m(E), \quad \lim_{\varepsilon \to 0+} \mathcal{H}^s(E^{\varepsilon}) = \mathcal{H}^s(E) \tag{16}$$

for all $s \geq 0$, both $\mathcal{L}^m$ and $\mathcal{H}^s$ are upper semicontinuous functions in $\mathcal{K}(R^m)$ (Beer (1975)). Hence the following lemma holds.

[6] See Basov (2005) for the conditions that ensure the existence of a solution.

Lemma 1 *Let $E \in K(R^m)$ be such that $L^m(E) = H^s(E) = 0$, for some $s \geq 0$, and let $(E_k)_{k\geq 1}$ be a sequence in $K(R^m)$ such that $E_k \to E$. Then $L^m(E_k) \to 0$ and $H^s(E_k) \to 0$.*

Proof. Because $\mathcal{L}^m$ is a non negative upper semicontinuous set function, we have

$$\lim \inf_{E_k \to E} \mathcal{L}^m(E_k) \geq 0 = \lambda(E) \geq \lim \sup_{E_k \to E} \mathcal{L}^m(E_k),$$

so $\mathcal{L}^m(E_k) \to 0$, and analogously for $\mathcal{H}^s$.■

Lemma 1 establishes continuity of Lebesgue and Hausdorff measures at zero. It will be used below, when we prove the main result of this Section.

Lemma 2 *Under Assumption 1, $\mathcal{L}^m(\Omega_0) = 0$ implies $\Omega_0 \subset \partial\Omega$.*

Proof. If $\Omega_0 \nsubseteq \partial\Omega$, there is $\alpha \in \Omega_0$ and an $\varepsilon > 0$ with $B_\varepsilon(\alpha) \subset \Omega$. Then $\mathcal{L}^m(\{\beta \in \Omega : \beta \leq \alpha\} \cap B_\varepsilon(\alpha)) > 0$. But because $s(\cdot)$ is increasing, $\{\beta \in \Omega : \beta \leq \alpha\} \cap B_\varepsilon(\alpha) \subset \Omega_0$, contradicting $\mathcal{L}^m(\Omega_0) = 0$.■

Lemma 2 states that if the exclusion set has Lebesgue measure zero it should be part of the topological boundary of the type set. Assumption 1 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the unidimensional case.

Lemma 3 *Assume $\mathcal{L}^m(\Omega_0) = 0$ and Assumption 2 holds. Then $\mathcal{H}^{m-1}(\Omega_0) = 0$ for almost all β.*

Proof. Let $s(\cdot)$ be the surplus function generated by the optimal tariff via (2). By Lemma 2, $\Omega_0 \subset \partial\Omega$. Because $\mathcal{H}^{m-1}(\partial\Omega \backslash \partial_e\Omega) = 0$, consider $\Omega_0 \cap \partial_e\Omega$, which is given by

$$\Omega_0 \cap \partial_e\Omega = \bigcup_{i=1}^{\infty} \Omega_{0i} \cup (N \cap \Omega_0) \tag{17}$$

where

$$\Omega_{0i} = \{\alpha \in \overline{\Omega} : g_i(\alpha, \beta) = 0, s(\alpha) = 0\}, \tag{18}$$

for $i = 1, 2, ...$ For each Ω_{0i}, Assumption 2 and the Transversality Theorem (see Mas-Colell, Whinston, and Green, 1995, Chapter 17D) imply that

$$rank \begin{pmatrix} \nabla_{\boldsymbol{\alpha}} u(\alpha, x) \\ \nabla_{\boldsymbol{\alpha}} g_i(\alpha, \beta) \end{pmatrix} = 2$$

for almost all β. Therefore, by the Implicit Function Theorem, Ω_{0i} is a manifold of dimension of $(m-2)$ for almost all β, so $\mathcal{H}^{m-1}(\Omega_{0i}) = 0$. Hence $\mathcal{H}^{m-1}(\Omega_0 \cap \partial_e \Omega) \leq \sum_{i=1}^{\infty} \mathcal{H}^{m-1}(\Omega_{0i}) + \mathcal{H}^{m-1}(N \cap \Omega_0) = 0$ for almost all β and we are done.■

For any $a, b \in R^m$ let $(a \cdot b)$ denote the inner product of a and b. The Generalized Gauss-Green Theorem states that for any Ω with locally finite perimeter in $U \subset R^m$, and any Lipschitz continuous vector field $\varphi : U \to R^m$ with compact support in U there is a unique measure theoretic unit outer normal $v_\Omega(\alpha)$ such that

$$\int_\Omega div\varphi d\alpha = \int_U (\varphi \cdot v_\Omega) d\mathcal{H}^{m-1}$$

where

$$div\varphi = \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial \alpha_k} \tag{19}$$

is the divergence of the vector field φ. Because of Assumption 1 we have $\nabla_\alpha u(\alpha, x) \geq 0$ for all $\alpha \in \Omega \subset R_+^m$, so it is only a slight strengthening of Assumption 1 to assume that

$$\sup_{(\alpha,x)\in\overline{\Omega}\times R_+^n} (\alpha \cdot \nabla_\alpha u(\alpha, x)) > 0 \tag{20}$$

Because we can restrict the choices of x to lie in a compact subset $X \subset R_+^n$, since they will never exceed the efficient levels, it is without loss to assume that $u(\cdot, \cdot)$ is bounded on $\overline{\Omega} \times X$. Therefore, we can assume that there exists a number $K > 0$ such that

$$u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$$

for all $(\alpha, x) \in \overline{\Omega} \times X$. The following theorem holds.

Theorem 1 *Consider problem (12)-(13) and assume that $u(\cdot, \cdot)$ is twice continuously differentiable and strictly increasing in both arguments, $c(\cdot)$ is strictly convex and twice continuously differentiable, Ω is a set with locally finite perimeter in an open set $U \subset R^m$, $f(\cdot)$ is Lipschitz continuous with supp$(f) = \overline{\Omega}$ compact and Assumptions 1 and 2 hold. Finally, assume that there exists a number $K > 0$ such that*

$$u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$$

for all $(\alpha, x) \in \overline{\Omega} \times X$. Then, for almost all β, the set of consumers with zero surplus at the equilibrium has positive measure.

Proof. By way of contradiction, assume that $\mathcal{L}^m(\Omega_0) = 0$. For any natural number k, let π_k be the profit obtained by selling to the types in

$$\Omega_k = \{\alpha \in \Omega : s(\alpha) \leq \frac{1}{k}\}. \tag{21}$$

Since $s(\cdot)$ and $c(\cdot)$ are non negative, we must have

$$\pi_k \leq \int_{\Omega_k} u(\alpha, x(\alpha)) f(\alpha) d\alpha \tag{22}$$

Because there exists $K > 0$ such that $u(\alpha, x) \leq K(\alpha \cdot \nabla_{\boldsymbol{\alpha}} u(\alpha, x))$, the envelope condition implies that

$$\pi_k \leq \int_{\Omega_k} K(\alpha \cdot \nabla s(\alpha)) f(\alpha) d\alpha.$$

Applying the Generalized Gauss-Green Theorem to $\varphi(\alpha) = \alpha s(\alpha) f(\alpha)$ we get

$$\pi_n \leq \int_U K s(\alpha) f(\alpha)(\alpha \cdot v_\Omega(\alpha)) d\mathcal{H}^{m-1}(\alpha) - \int_{\Omega_k} K s(\alpha) div(\alpha f(\alpha)) d\alpha.$$

Because every function above is bounded in Ω_k, choose a common upper bound B. Because $s(\alpha) \leq \frac{1}{k}$ in Ω_k, we have

$$\pi_k \leq \frac{1}{k} B(\mathcal{H}^{m-1}(\Omega_k) + \mathcal{L}^m(\Omega_k)).$$

Now consider increasing the tariff by $\frac{1}{k}$. The consumers in the set Ω_k will exit, and π_k will be lost, but each other consumer will pay $\frac{1}{k}$ more. Since the total number of consumers that exit is bounded by $B\mathcal{L}^m(\Omega_k)$, the change in profit is at least

$$\Delta\pi \geq \frac{1}{k}[(1 - B\mathcal{L}^m(\Omega_k) - B(\mathcal{H}^{m-1}(\Omega_k) + \mathcal{L}^m(\Omega_k))].$$

From Lemma 4, for almost all β we have $\mathcal{H}^{m-1}(\Omega_0) = 0$, and hence from Lemma 1 we have $\mathcal{L}^m(\Omega_k) \to 0$ and $\mathcal{H}^{m-1}(\Omega_k) \to 0$ for almost all β, because of continuity of $s(\cdot)$ and the compact support of $f(\cdot)$. But then for large k, $\Delta\pi$ must be positive, contradicting the optimality of the tariff.■

Therefore, Theorem 1 shows that, generically, the set $\Omega_0 = \{\alpha \in \Omega : s(\alpha) = 0\}$ has positive measure, generalizing Armstrong's result. Strictly speaking, Ω_0 is the set of consumers who have zero surplus, so it is not necessarily the case that a positive measure of consumers will in fact be excluded. That is, a consumer may have zero surplus because she does not consume at all or because she pays for her consumption exactly her opportunity cost. In this latter case she is not excluded from consumption. The following corollary shows that such consumers represent a zero measure subset of Ω_0. As a consequence, the set of excluded consumers contains the interior of Ω_0 and has positive measure.

Corollary 1 *Under the assumptions of Theorem 1 a positive measure of consumers will be excluded at the equilibrium for almost all β.*

Proof. Suppose that a consumer $\alpha \in \Omega_0$ does consume. Moreover, suppose that there exists another consumer $\delta \in \Omega_0$ with $\delta < \alpha$. Then, since u is strictly increasing it must be the case that:

$$t(\alpha) = u(\alpha, x(\alpha)) \geqslant u(\alpha, x(\delta)) > u(\delta, x(\delta)) = t(\delta)$$

But then, $u(\alpha, x(\delta)) - t(\delta) > 0$, which contradicts the optimality of the α-type consumer's choice. Therefore, if $\alpha \in \Omega_0$ does consume, then every $\delta \in \Omega_0$ with $\delta < \alpha$ is excluded from consumption. Hence, the set of agents in Ω_0 who do consume has zero measure.■

Rochet and Stole (2003) provided an example where the exclusion set is empty.[7] In their example $u : \Omega \times R_+ \to R$ has a form

$$u(\alpha, x) = (\alpha_1 + \alpha_2)x$$

and Ω is a rectangle with sides parallel to the 45 degrees and -45 degrees lines. They argued that one can shift the rectangle sufficiently far to the right to have an empty exclusion region. Their result is driven by the fact that they allow only very special collection of type sets, rectangles with parallel sides. Formally, Assumption

[7] Another example along similar lines is provided by Deneckere and Severinov (2009). Though it is a bit more intricate and the authors provide sufficient conditions that ensure full participation in the case of one quality dimension and two-dimensional characteristics, their condition also does not hold generically.

1 fails in this case, since $g_1(\alpha,\beta)=\alpha_1+\alpha_2-\beta=0$ and

$$\begin{pmatrix} \nabla_\alpha u(\alpha,x) \\ \nabla_\alpha g_1(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} x & x \\ 1 & 1 \end{pmatrix} \Rightarrow rank \begin{pmatrix} \nabla_\alpha u(\alpha,x) \\ \nabla_\alpha g_1(\alpha,\beta) \end{pmatrix} = 1.$$

Note that a very small change in the type set changes that result. Suppose, for example, that $g_1(\alpha,\beta)=\alpha_1+(1+\varepsilon)\,\alpha_2-\beta=0$, where ε is a small positive real number. Then, Assumption 1 holds and Ω_0 has positive measure, since, for all $x\neq 0$,

$$\begin{pmatrix} \nabla_\alpha u(\alpha,x) \\ \nabla_\alpha g_1(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} x & x \\ 1 & 1+\varepsilon \end{pmatrix} \Rightarrow rank \begin{pmatrix} \nabla_\alpha u(\alpha,x) \\ \nabla_\alpha g_1(\alpha,\beta) \end{pmatrix} = 2.$$

Note that our results do not guarantee non-empty exclusion region for every multidimensional screening problem. They rather assert that any problem for which the exclusion region is empty can be slightly perturbed is such a way that for the new problem a positive measure of the consumers will be excluded in equilibrium. To understand the results intuitively, assume first that in equilibrium all consumers are served. First, note that at least one consumer should be indifferent between participating and not participating, since otherwise the tariffs can be uniformly increasing for everyone by a small amount, increasing the monopolist's profits. Now, consider increasing the tariff by $\varepsilon>0$, then the consumers who earned surplus below ε will drop out. The measure of such consumers is $o(\varepsilon)$, unless iso-surplus hyper-surfaces happen to be parallel to the boundary of type space. If condition (4) is violated then iso-surplus hyper-surfaces will be parallel to the boundary of Ω by construction. We ruled that case out is our analysis. However, such situation may still occur endogenously, which is the reason why our result holds for *almost* all, rather then for all, screening problems. One class of problems, for which full participation may occur are model with random outside option. They were first considered by Rochet and Stole (2002) for both monopolistic are oligopolistic settings and generalized by Basov and Yin (2010) for the case of risk averse principal(s). Armstrong and Vickers (2001) considered another generalization, allowing for multidimensional vertical types. In this type of model the type consists of a vector of vertical characteristics, $\alpha\in\Omega\subset R^m$, and a parameter $\gamma\in[0,1]$ capturing horizontal preferences. The type space is given by the

Cartesian product $\Omega \times [0,1]$ and γ is assumed to be distributed independently of α. The utility of a consumer is given by:

$$u(\alpha, x; \gamma) = v(\alpha, x) - t\gamma, \tag{23}$$

where t is a commonly known parameter. Let $v(\alpha, 0) = 0$ then iso-surplus hypersurface corresponding to zero quality is $t\gamma =$constant which is parallel to the vertical boundary of type space $\gamma = 0$. Therefore, in such model there is a possibility of full participation. The model was also investigated in oligopolistic setting, where t was interpreted as a transportation cost for the Hotelling model. Conditions for full participation under different assumptions on dimensionality of α and the monopolist's risk preferences were obtained by Armstrong and Vickers (2001), Rochet and Stole (2002), and Basov and Yin (2010). Let us assume that the boundary of set Ω is described by and equation

$$g_0(\alpha) = 0 \tag{24}$$

and embed our problem into a family of problems, for which boundary of the type space is described by an equation

$$g(\alpha, \gamma; \beta) = 0, \tag{25}$$

where $g(\cdot, \beta) : \Omega \times [0,1] \to R$ is a smooth function such that

$$g(\alpha, \gamma; 0) = g_0(\alpha)(g_0(\alpha) - b)\gamma(\gamma - 1), \tag{26}$$

for some constant b, i.e. for $\beta = 0$ the type space becomes the cylinder over the set Ω considered by Armstrong and Vickers (2001). Our result is that for almost all β the exclusion region is non-empty. However, as we saw above, for $\beta = 0$ exclusion region may be empty.

We now consider another class of models, where full participation is possible. The example will also be interesting, since it will allow us to investigate how the relative measure of excluded consumers changes with dimension of Ω.

Example 3 *Let consumer's preferences be given by:*

$$u(\alpha, x, t) = \sum_{i=1}^{n} \alpha_i x_i, \tag{27}$$

and the monopolist's cost be given by

$$c(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2. \tag{28}$$

The type space is intersection of the region between balls with radii a and $a+1$ with R_+^n, i.e.

$$\Omega = \{\alpha \in R_+^n : a \le \|\alpha\| \le a+1\}, \tag{29}$$

where $\|\cdot\|$ denotes the Euclidean norm

$$\|\beta\| = \sqrt{\sum_{i=1}^{n} \beta_i^2}. \tag{30}$$

To solve for the optimal nonlinear tariff with a fixed number of characteristics start by introducing the consumer surplus by:

$$s(\alpha) = \max(\sum_{i=1}^{n} \alpha_i x_i - t(x)). \tag{31}$$

The symmetry of the problem suggests that we look for a solution in a form

$$s = s(\|\alpha\|)$$

In the "separation region" it solves

$$\begin{cases} \frac{1}{r^{n-1}}\frac{\partial}{\partial r}(r^{n-1}s'(r)) + \frac{s'(r)f'(r)}{f(r)} = n+1+\frac{rf'(r)}{f(r)} \\ s'(a+1) = a+1 \end{cases}, \tag{32}$$

where we introduced notation $r \equiv \|\alpha\|$. To derive system (32) note that from the envelope theorem

$$x = \nabla s(\alpha). \tag{33}$$

The monopolists problem can now be written as

$$\max_s \int [\alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha)] d\alpha \tag{34}$$

$$s.t.\ s(\cdot)\text{-convex},\ s \geq 0. \tag{35}$$

(see Rochet and Chone, 1998). Dropping for a moment the convexity constraint, one obtains the standard calculus of variations problem with free boundary. Therefore, in the participation region (in the points, where $s > 0$) one obtains:

$$\sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} \frac{\partial L}{\partial s_i} = \frac{\partial L}{\partial s} \tag{36}$$

$$\sum_{i=1}^{n} \alpha_i \frac{\partial L}{\partial s_i} = 0 \tag{37}$$

(see, Basov (2005)), where s_i denotes i^{th} partial derivative of surplus and

$$L = \alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha) \tag{38}$$

But this is exactly system (32). Let us assume that types are distributed uniformly on Ω, so the derivative of type distribution vanishes. Then, solving (32) one obtains:[8]

$$x_i(\alpha) = \max(0, \frac{\alpha_i}{n}(n + 1 - (\frac{a+1}{r})^n)). \tag{39}$$

Corresponding iso-surplus hyper-surfaces are given by intersection of a sphere of appropriate dimension with R_+^n They are parallel to the boundary, hence we have a possibility of an empty exclusion region. To investigate this possibility further note that the exclusion region is given by

$$\Omega_0 = \{\alpha \in \Omega : \|\alpha\| \leq \frac{a+1}{\sqrt[n]{1+n}}\}. \tag{40}$$

It is non-empty if

$$\frac{a+1}{\sqrt[n]{1+n}} > a. \tag{41}$$

[8] It is easy to check that the surplus function, corresponding to allocation (39) is convex, therefore (39) solves the complete problem.

Note that if $n = 1$ the exclusion region is empty if and only if $a > 1$, if $n = 2$ it is empty if and only if $a > 1/(\sqrt{3} - 1) \approx 1.36$, and since

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{1+n}} = 1, \tag{42}$$

the exclusion region is non-empty for any $a > 0$ for sufficiently large n. The relative measure of the excluded consumer's (the measure of excluded consumers if we normalize the total measure of consumers to be one for all n) is:

$$\zeta = \frac{(a+1)^n/(n+1) - a^n}{(a+1)^n - a^n}. \tag{43}$$

It is easy to see that as $n \to \infty$ the measure of excluded consumers converges to zero as $1/n$, i.e. exclusion becomes asymptotically less important. This accords with results obtained by Armstrong (1999a). The convergence, however, is not monotone. For example, if $a = 1.3$ the measure of excluded customers first rises from zero for $n = 1$ to 11.6% for $n = 5$, and falls slowly thereafter. For $a = 2$ maximal exclusion of 8.3% occurs for $n = 11$ and for $a = 0.7$ maximal exclusion of 19.7% of consumers obtains when $n = 2$.

Note also that though asymptotically higher fraction of consumers gets served as $n \to \infty$, this does not mean that the consumers become better off. Indeed, as $n \to \infty$ the radius of the exclusion region converges to $(a + 1)$, therefore almost all served consumers are located near the upper boundary. This means that the trade-off between efficient provision of quality and minimization of information rates disappears. The monopolist provides asymptotically efficient quality but is able to appropriate almost the entire surplus.

3 The Genericity of Exclusion in an Oligopolistic Screening Model

Consider a framework similar to the one of the previous section but assume that the market is served by K producers. The production cost is identical among the producers, who play a one shot-game. A pure strategy of a producer k is a non-linear tariff, i.e. a measurable mapping $t^k : R^n_+ \to R$. Consider a symmetric pure strategy Nash equilibrium at which all producers charge the same tariff. We will argue that at such an equilibrium a positive measure of the consumers are not served.

Assume that, in equilibrium, producer k charges tariff $t^k(\cdot) : R^n_+ \to R$. Then $t^k(\cdot)$ solves

$$\max_{t^k(\cdot)} \int_\Omega (t^k(x^k(\alpha)) - c(x^k(\alpha)))f(\alpha)d\alpha,$$

subject to:

$$\begin{cases} x(\alpha) \in \arg\max_{\mathbf{x}\in R^n_+} v(\alpha, x) - t(x) & \text{if } \max_{\mathbf{x}\in R^n_+}(v(\alpha, x) - t(x)) \geq s_0\,(x) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

where

$$\begin{cases} t(x) = \min \sum_{j=1}^{K} t^j(x^j) \\ s.t. \sum_{j=1}^{K} x^j = x, \ x^j \geq 0 \end{cases}, \tag{44}$$

and

$$s_0(\alpha) = \max\{s_0^*(\alpha), \max_{\mathbf{x}\in R^n_+, \mathbf{x}^k=0} (v(\alpha, x) - t^{-k}(x))\} \tag{45}$$

and $t^{-k}(\mathbf{x})$ solves problem (44) subject to an additional constraint $\mathbf{x}^k = 0$. Equation (45) states that the outside option of a consumer seen from the point of view of producer k is determined by her best opportunity outside the market and the best bundle she may purchase from the competitors.

Let us define

$$u(\alpha, x^k) = v(\alpha, x^k + \sum_{j=1, j\neq k}^{K} x^j(\alpha)) - \sum_{j=1, j\neq k}^{K} t^j(x^j(\alpha)) - s_0(\alpha),$$

where $x^j(\alpha)$ is the equilibrium quantity purchased by the consumer of type α from the producer j and $s_0(\alpha)$ is defined by equation (45). Then the problem of producer k becomes:

$$\max_{t^k(\cdot)} \int_{\Omega} (t^k(x^k(\alpha)) - c(x^k(\alpha)))f(\alpha)d\alpha,$$

subject to:

$$\begin{cases} x^k(\alpha) \in \arg\max_{\mathbf{x}^k \in R^n_+} u(\alpha, x^k) - t^k(x^k) & \text{if } \max_{\mathbf{x}^k \in R^n_+} (u(\alpha, x^k) - t^k(x^k)) \geq 0 \\ x^k(\alpha) = 0 & \text{otherwise} \end{cases} \quad (46)$$

We impose the following form of single-crossing property:

$$\frac{\partial^2 v}{\partial \alpha_i \partial x_j} \geq 0$$

for all i, j. Then $u(\alpha, x^k)$ is strictly increasing in α for all $x^k \in R^n_+$.

Theorem 2 *Under the assumptions of Theorem 1 a positive measure of consumers will be excluded in any symmetric equilibrium of the oligopolistic market.*

Proof. Consider oligopolist 1. Given the behavior of her competitors, her problem is isomorphic to the problem of the monopolist, with appropriately redefined utility. Therefore, Theorem 1 implies that she will find it optimal to exclude a positive measure of consumers. By symmetry, so will the other oligopolists. Finally, by symmetry again, each oligopolist will exclude the same set of consumers, so the intersection the sets of excluded consumers has positive measure.■

Champsuar and Rochet (1989) note that the profit functions of the oligopolists can become discontinuous when there are bunching regions. Even though Basov (2005) shows that bunching in the multidimensional case is not as typical as suggested by Rochet and Chone (1998), existence of an equilibrium is not a trivial matter in the oligopoly game above. We now show that under some conditions a symmetric equilibrium exists.

Let M be a bound on the utility function, and note that it is also a bound on the tariffs, and hence it must be that no producer will ever produce x with $c(x) > M$. So it is without loss of generality to restrict attention to tariffs $t : C \to [0, M]$ where $C \subset R^m_+$ is compact. Assume that producers choose Lipschitz continuous tariffs, so that the strategy space of each producer k is

$$T^k = T = \{t : C \to [0, M] \text{ s.t. } t \text{ is Lipschitz continuous}\}. \quad (47)$$

Using the *sup* norm, it follows from the Arzela-Ascoli's theorem that T is compact. Assume that when producers choose a symmetric profile $(t, ..., t)$ of tariffs, the solution to the maximization problem of the consumers is also symmetric: $x^1 = \cdots = x^K$. It follows that profits are symmetric: $\pi^k(t, ..., t) = \pi(t, ..., t)$ for $k = 1, ..., K$. Hence the game played by the producers, $(T \times \cdots \times T, \pi)$, is symmetric.

Let $P(t, ..., t) = \{s \in T : \pi(t, ..., s, ..., t) > \pi(t, ..., t)\}$ denote a producer's strict upper contour set when others choose the same tariff t (we use $\pi(t, ..., s, ..., t)$ to denote the profit of a given producer when he/she chooses s and all the others choose t).

Definition 1 *A symmetric game $(T \times \cdots \times T, \pi)$ is diagonally quasiconcave if*

$$t \notin coP(t, ..., t) \quad (48)$$

for each $t \in T$, where co denotes the convex hull of a set.

Definition 2 *A symmetric game $(T \times \cdots \times T, \pi)$ is continuously secure if for every symmetric profile $(t, .., t)$ that is not an equilibrium there exists a continuous function*

$$f_{(t,...,t)} : (s, ..., s) \mapsto z \in T \quad (49)$$

such that

$$f_{(t,...,t)}(s, ..., s) \in P(s, ..., s), \quad (50)$$

for each s in an open neighborhood of t.

Applying the argument in Theorem 2.2 in Barelli and Soza (2009) we have:

Lemma 5 *A symmetric game $(T \times \cdots \times T, \pi)$, where T is compact and convex, has a symmetric pure strategy Nash equilibrium whenever it is diagonally quasiconcave and continuously secure.*

Proof. See appendix.■

Theorem 3 *Assume that the game played by the producers satisfies the assumptions above, so it is a compact, convex, symmetric. Assume further that it is diagonally quasiconcave game. Then there exists a symmetric pure strategy Nash equilibrium.*

Proof. We show that, although discontinuous, the game is continuously secure. For each non equilibrium profile $(t, ..., t)$, there exists a profile $(\bar{t}, ..., \bar{t})$ with $\bar{t} \in P(t, ..., t)$. Put

$$f_{(t,...,t)}(s, ..., s) = \bar{t} + (s - t) \tag{51}$$

for any s in a neighborhood of t. Then $f_{(t,...,t)}(\cdot)$ is continuous and

$$f_{(t,...,t)}(s, ..., s)(x) \gtreqless s(x) \tag{52}$$

if and only if $\bar{t}(x) \gtreqless t(x)$, so if consumers choose a given producer offering $\bar{t}$ at x when all the others offer t, they also choose the same producer when she offers $f_{(t,...,t)}(s, ..., s)$ and the others offer s. This means that the discontinuities arising due to either Bertrand-like competition or bunching are avoided, and $\pi(s, ..., f_{(t,...,t)}(s, ..., s), ..., s)$ is continuous in s, for s close to t. By assumption, consumers choose

$$x^1(s) = \cdots = x^K(s) = \frac{x(s)}{K} \tag{53}$$

when faced with the profile $(s, ..., s)$ of tariffs, where $x(s)$ is the optimal solution if there was only one firm offering tariff s. It follows that $\pi(s, ..., s)$ is also continuous in s, so we have $f_{(t,...,t)}(s, ..., s) \in P(s, ..., s)$ for every s in a neighborhood of t.■

The assumption of diagonal quasiconcavity restricts some of the allowed densities $f(\cdot)$. Alternatively, we can work with the mixed extension of the game, where quasiconcavity obtains, and use the argument above to conclude that a symmetric mixed strategy equilibrium exists. Note that the argument in Theorems 1 and 2 remain valid in the mixed extension.

Let us now assume that the number of producers is not fixed but there is a positive entry cost $F > 0$. It is easy to see that this problem can be reduced to the previous one, since equilibrium number of the producers is always finite. Indeed, with K producers the profits of an oligopolist in a symmetric equilibrium are bounded by π^m/K, where π^m are the profits of a monopolist. Therefore, at equilibrium $K \leq \pi^m/F$ and a positive measure of the consumers will be excluded from the market.

4 The Genericity of Exclusion Without the Quasilinearity Hypothesis

In this Section we relax the quasilinearity assumption. It is not difficult to find economically interesting examples, where the most natural formulation leads to a consumer's utility, which is not quasilinear in money. Consider, for example, the following model of grant allocation (Bardsley and Basov, 2004). Risk averse institutions compete for grants for completing a research project. A project, if successful, will result in the provision of a public good whose value to the society is equal to one. Different institutions have projects that differ in the cost of completion and the probability of success. The government can choose an up-front payment and the prize in the case of success and is interested in maximizing the benefits of the society minus the completion costs. The institutions are assumed to be politically small, so there expected profits do not enter the government's objective. If one denotes the cost of the project c, the probability of success q, the up-front payment t and the prize for success x, the utility of the institution conditional on participation in the government's scheme will be

$$v(c, a; x, t) = qv(t + x - c) + (1 - q)v(t - c), \tag{54}$$

which is not quasilinear in the up-front payment. Another example is an insurance company, which faces customers that differ in their loss probability and the degree of risk-aversion. The competitive variant of this model was first considered by Smart (2000) and Villeneuve (2003).

Let $\Omega \subset R_+^m$ be a convex, open, bounded set and the utility of consumer of type $\boldsymbol{\alpha}$ who obtains good of quality $\mathbf{x}$ and pays t is

$$v(\boldsymbol{\alpha}, \mathbf{x}, t), \tag{55}$$

where v is twice continuously differentiable in $\mathbf{x}$ and $\boldsymbol{\alpha}$ and continuously differentiable in t . Moreover, we assume that v is strictly increasing in the consumer's type and quality and strictly decreasing in the tariff paid. Given a tariff $t(x)$ define the consumer's surplus, $s(\cdot)$ by:

$$s(\alpha) = \max_{x} v(\boldsymbol{\alpha}, \mathbf{x}, t(x)). \tag{56}$$

The monopolist selects a function $t : R^n \to R$ to solve

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha \tag{57}$$

where $c(x)$ is the cost of producing a good with quality x and $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x}(v(\alpha, x, t(x)) & \text{if } \max(u(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise} \end{cases}, \tag{58}$$

where $x_0(\alpha)$ is the outside option, which implements surplus $s_0(\alpha)$.

For any continuous function $\varphi(\cdot)$, let $\tau(\boldsymbol{\alpha}, \mathbf{x}, \varphi(\boldsymbol{\alpha}))$ be the unique solution of the equation

$$\varphi = v(\cdot, \cdot, \tau) \tag{59}$$

and let

$$u(\alpha, x) = \tau(\boldsymbol{\alpha}, \mathbf{x}, s_0(\boldsymbol{\alpha})) \tag{60}$$

In the quasilinear case equation (60) reduces to equation (3). Note that, since the function $\tau(\boldsymbol{\alpha}, \mathbf{x}, \cdot)$ is strictly decreasing, and the optimal surplus satisfies $s(\alpha) \geq s_0(\alpha)$ the optimal tariff paid by type α satisfies

$$t(x(\alpha)) \leq u(\alpha, x(\alpha)). \tag{61}$$

Assumptions 1 and 2 should be modified to read:

Assumption 3 *For any $x \in R^n_+$, the net utility gain of consumption $u(\cdot, x)$ defined by (60)is strictly increasing in α.*

Note that the preferences described in Example 2 satisfy this assumption.

Assumption 4 *For each $i = 1, 2...,$*

$$K_i = \{\alpha \in \overline{\Omega} : g_i(\alpha, \beta) = 0\},$$

where $g_i : \overline{\Omega} \times R^J \to R$ is smooth, $\beta \in R^J$, $J \geq 1$, are parameters and, for all

$x \in R^n_+$ and all $i = 1, 2, \ldots,$ there exists $\beta_0 \in R^J$ such that

$$rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_0) \end{pmatrix} = 2.$$

Now one can prove the analogs of lemmata 1 to 4, where Assumptions 1 and 2 are replaced by Assumptions 3 and 4 respectively. The previous proofs apply verbatim and are omitted. This observation together with (61) allow us to formulate a following theorem:

Theorem 4 *Consider problem (57)-(58) and assume that $v(\cdot, \cdot)$ is twice continuously differentiable and strictly increasing in both arguments, $c(\cdot)$ is strictly convex and twice continuously differentiable, Ω is a set with locally finite perimeter in an open set $U \subset R^m$, $f(\cdot)$ is Lipschitz continuous with $supp(f) = \overline{\Omega}$ compact and Assumptions 3 and 4 hold. Finally, assume that there exists a number $K > 0$ such that*

$$u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$$

for all $(\alpha, x) \in \overline{\Omega} \times X$. Then for almost all β the set of consumers with zero surplus at the equilibrium has positive measure.

Proof. The proof is similar to that of Theorem 1. For any natural number k, let π_k be the profit obtained by selling to the types in

$$\Omega_k = \{\alpha \in \Omega : s(\alpha) \leq \frac{1}{k}\}. \tag{62}$$

Since $c(\cdot)$ is non-negative, we must have

$$\pi_k \leq \int_{\Omega_k} t(x(\alpha)) f(\alpha) d\alpha, \tag{63}$$

but now formula (61) implies

$$\pi_k \leq \int_{\Omega_k} u(\alpha, x(\alpha)) f(\alpha) d\alpha. \tag{64}$$

The rest of the proof is identical to Theorem 1 and is omitted.■

5 An Application to the Regulation of a Monopolist with Unknown Demand and Cost Functions

Armstrong (1999b) reviews Lewis and Sappington (1988) study of the optimal regulation of a monopolist firm when the firm's private information is two dimensional. In this study, a single product monopolist faces a stochastic demand function given by $q(p) = a + \theta - p$, where p is the product's price, a is a fixed parameter and θ is a stochastic component to demand, taking values in an interval $\left[\underline{\theta}, \overline{\theta}\right] \subset R_{+}$. The firm's cost is represented by the function $C(q) = (c_0 - c)\, q + K$, where q is the quantity produced, c_0 and K are fixed parameters and c is a stochastic component to the cost, taking values in an interval[9] $[-\overline{c}, -\underline{c}] \subset R_{-}$. The firm observes both the demand and the cost functions, but the regulator only knows that $\alpha = (\theta, c)$ is distributed according to the strictly positive continuous density function $f(\theta, c)$ on the rectangle $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [-\overline{c}, -\underline{c}]$. For the sake of feasibility we assume that $a + \theta > c_0 - c$ for all $\alpha = (\theta, c) \in \Omega$, i.e., the highest demand exceeds marginal costs, for all possible realizations of the stochastic components of demand and costs.

The regulator wants to maximize social welfare and presents to the monopolist a menu of contracts $\{p, T(p)\}$. If the firm chooses contract $(p, T(p))$ it sells its product at price p and receives subsidy $T(p)$ from the regulator.

Therefore, the regulator's problem is to select a continuous subsidy schedule $T(\cdot) : R_{+} \rightarrow R$ to solve:

$$\max_{T(\cdot)} \int_{\Omega} \left[\frac{1}{2}\left(a + \theta - p(\alpha)\right)^2 - T(\alpha)\right] f(\alpha) d\alpha$$

where $p(\alpha)$ satisfies

[9] In the original model $C(q) = (c_0 + c)\, q + K$ with $c \in [\underline{c}, \overline{c}] \subset R_{+}$. We substitute c by its negative for convenience.

$$p(\alpha) \in \begin{cases} \arg\max_{p} \begin{array}{l} \{(a+\theta-p)(p-c_0+c) \\ -K+T(p)\} \end{array} & \text{if } \max\{((a+\theta-p)(p-c_0+c) -K+T(p)\} \geq 0 \\ \{a+\theta\} & \text{otherwise} \end{cases}$$

The first term in the regulator's objective function, $\frac{1}{2}(a+\theta-p(\alpha))^2$, corresponds to the consumer's surplus while the second term, $T(\alpha)$, is the subsidy cost. The choice of $p(\alpha)$ by the monopolist depends on whether she can derive nonnegative returns when producing. If that is not possible, she will choose $p(\alpha) = a+\theta$ and there will be zero demand, i.e., the firm "shuts down".

A fundamental hypothesis in Lewis and Sappington's analysis is that the parameter a can be chosen sufficiently large relative to parameters K and c_0 so that a firm will always find it in its interest to produce, even for the very small values of θ. However, Armstrong (1999b) shows that such a hypothesis cannot be made when Ω is the square $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [-\overline{c}, -\underline{c}] = [0,1] \times [-1,0]$. Furthermore, when Ω is a strictly convex subset of that square, Armstrong (1999b) uses the optimality of exclusion theorem in Armstrong (1996) to show that some firms will necessarily shut down under the optimal regulatory policy, in equilibrium. Armstrong (1999b) adds "... I believe that the condition that the support be convex is *strongly* sufficient and that it will be the usual case that exclusion is optimal, even if a is much larger than the maximum possible marginal cost." That insight could not be pursued further due to a lack of a more general result, and Armstrong (1999b) switched to a discrete-type model in order to check the robustness of the main conclusions in Lewis and Sappington (1988).

Let us review this problem in the language of the present paper. Consider the following change of variables. Change p with x and define the following functions:

$$\begin{aligned} u(\alpha, x) &= (a+\theta-x)(x-c_0+c)-K \\ t(x) &= -T(x) \\ c(x) &= -\frac{1}{2}(a+\theta-x)^2 \end{aligned}$$

Then the regulator's problem can be rewritten as:

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha)))f(\alpha)d\alpha,$$

where $x(\alpha)$ satisfies

$$x(\alpha) \in \begin{cases} \arg\max_x u(\alpha, x) - t(x) & \text{if } \max(u(\alpha, x) - t(x)) \geq 0 \\ \{a + \theta\} & \text{otherwise} \end{cases} \tag{65}$$

Note that this is essentially the standard problem solved in our original model.

In order to apply Theorem 1, first note that it is sufficient that the conditions of Assumptions 1 and 2 hold at the agents's optimal choice of x, given his type α, i.e., at the equilibrium $x(\alpha)$.

Now notice that $u(\alpha, x)$ is strictly increasing in c, as long as $a + \theta - x > 0$. But this is always the case for $x(\alpha)$, since $a + \theta - x(\alpha)$ is a demand curve. Moreover, $u(\alpha, x)$ is strictly increasing in θ, as long as $x - c_0 + c > 0$. This is again the case for $x(\alpha)$ since this is the difference between price and marginal cost. Therefore, $u(\alpha)$ is strictly increasing in α for the relevant choice of price.

Define g by:

$$g_1(\alpha, 0) = \theta, g_2(\alpha, 1) = \theta - 1, g_3(\alpha, -1) = c + 1, .g_4(\alpha, 0) = c. \tag{66}$$

Then we can define

$$\begin{array}{c} \Sigma_1 = \{\alpha \in \Omega : \ g_1(\alpha, \underline{\theta}) = 0\} \\ \Sigma_2 = \left\{\alpha \in \Omega : \ g_2\left(\alpha, \overline{\theta}\right) = 0\right\} \\ \Sigma_3 = \{\alpha \in \Omega : \ g_3(\alpha, -\underline{c}) = 0\} \\ \Sigma_4 = \{\alpha \in \Omega : \ g_4(\alpha, -\overline{c}) = 0\} \end{array}. \tag{67}$$

Therefore, the boundary of Ω can be expressed as:

$$\partial\Omega = \bigcup_{i=1}^{4} \Sigma_i.$$

Moreover, the gradient of function u is

$$\nabla_\alpha u(\alpha, x) = (x - c_0 + c, a + \theta - x) \tag{68}$$

and

$$\nabla_\alpha g_i(\alpha,\beta) = (1,0)\,, i=1,2, \nabla_\alpha g_j(\alpha,\beta) = (0,1)\,, j=1,2. \tag{69}$$

Therefore, for all possible values of β, for $i = 1, 2$ and for $j = 1, 2$,

$$\begin{pmatrix} \nabla_{\boldsymbol{\alpha}} u(\alpha,x) \\ \nabla_\alpha g_i(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} x - c_0 + c & a + \theta - x \\ 1 & 0 \end{pmatrix} \tag{70}$$

$$\begin{pmatrix} \nabla_{\boldsymbol{\alpha}} u(\alpha,x) \\ \nabla_\alpha g_j(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} x - c_0 + c & a + \theta - x \\ 0 & 1 \end{pmatrix} \tag{71}$$

In particular, the rank of these matrices is 2[10]. Therefore, Assumptions 1 and 2 are satisfied in this model, as well as all remaining hypothesis of Theorem1. Hence we may conclude that a set of positive firms will generically be "excluded" from the regulated market, i.e., will not produce at all. This example confirms Armstrong's (1999) conjecture.

6 An Application to Involuntary Unemployment

Consider a firm in an industry that produces n goods captured by a vector $x \in R^n_+$. The firm hires workers to produces these goods. A worker is characterized by the cost she bears in order to produce goods $x \in R^n_+$, which is given by the effort cost function $e(\alpha, x)$. The parameter $\alpha \in \Omega \subset R^m$ is the worker's unobservable type distributed on an open, bounded, set $\Omega \subset R^m$ according to a strictly positive, continuous density function $f(\cdot)$.

Therefore, if a worker of type α is hired to produce output x and receives wage $\omega(x)$, her utility is $\omega(x) - c(\alpha, x)$. If the worker is not hired by the firm, she will receive a net utility $s_0(\alpha)$, either by working on a different firm, or by receiving unemployment compensation.

Suppose the firm sells its product for competitive international prices, $p(x)$.

[10]Indeed, it cannot be the case that $x = p = c_0 - c = a + \theta$ since the price cannot be, at the same time, the marginal cost (prefect competitive price) and the price that makes demand vanish.

Then, the firm's problem is to select a wage schedule $\omega(\cdot) : R_+^n \to R$ to solve:

$$\max_{\omega(\cdot)} \int_{\Omega} \left[p\left(x\left(\alpha\right)\right) x\left(\alpha\right) - \omega\left(x\left(\alpha\right)\right)\right] f(\alpha) d\alpha$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max\limits_{\mathbf{x} \in R_+^n} \omega(x) - e(\alpha, x) & \text{if } \max\limits_{\mathbf{x} \in R_+^n} \omega(x) - e(\alpha, x) \geq s_0\left(x\right) \\ x(\alpha) = 0 & \text{otherwise} \end{cases} \tag{72}$$

Consider the following change in variables: $t\left(x\right) = -\omega\left(x\right)$, $v(\alpha, x) = -e(\alpha, x)$, $c(x) = -p\left(x\right)x$, then the firm's problem can be rewritten as:

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies:

$$\begin{cases} x(\alpha) \in \arg\max\limits_{\mathbf{x} \in R_+^n} v(\alpha, x) - t(x) & \text{if } \max\limits_{\mathbf{x} \in R_+^n} (v(\alpha, x) - t(x)) \geq s_0\left(x\right) \\ x(\alpha) = 0 & \text{otherwise} \end{cases} \tag{73}$$

Therefore, the same arguments that have been presented for the monopolist can also be extended for the hiring decision of the firm. In particular, the firm will generically find it optimal not to hire a set of positive measure. If the firm is a monopsonist in the region in the sense that agents can only work at that firm, then Theorem 1 presents a new explanation for involuntary unemployment. Note that, according to Theorem 2, the result can be extended to a region with several firms hiring for the production of goods $x \in R_+^n$, so that there is an oligopsony for workers, as long as the corresponding industry is the only source of formal work. This is true even in the case of free entry in that industry, according to the comment following Theorem 2. Finally, if one includes the category of informal work (underemployment) as unemployment, the present model suggests that an informal sector will generically exist in equilibrium.

This application is, to the knowledge of the authors, the first explanation of involuntary unemployment based on the adverse selection problem, whereby firms decide to offer a wage schedule that excludes some less productive workers so they can require higher output levels from the more productive ones.

7 Conclusions

Armstrong's (1996) exclusion result applies quite widely to a diverse set of markets in the economy and, as such, offers a deep insight into the workings of market economies. In general, outside of the very special cases of perfect competition, complete and perfect information, or unidimensional private information, we should expect to see exclusion operating in markets. We have explored, in this paper, five diverse settings where we believe this result applies: credit markets, automobiles, research grants, monopoly regulation, and labor markets. Further applications, and further depth on these applications, seem warranted for future research. [11]

A Appendix

A set $\Omega \subset R^m$ has *finite perimeter* in an open set $U \subset R^m$ if $A \cap U$ is measurable and there exists a finite Borel measure μ on U and a Borel function $v : U \to S^{m-1} \cup \{0\} \subset R^m$ with

$$\int_\Omega div\varphi dx = \int_U \varphi \cdot v d\mu$$

for every Lipschitz continuous vector field $\varphi : U \to R^m$ with compact support U, where S^{m-1} is the $m-1$ dimensional unit sphere. The perimeter of Ω in U is given by $\mu(V)$. A set $\Omega \subset R^m$ is of *locally finite perimeter* if $\mu(V) < \infty$ for every open proper subset of U. The *measure theoretic boundary* of Ω is given by

$$\partial_e(\Omega) = \{x \in R^m : 0 < \mathcal{L}^m(\Omega \cap B_\varepsilon(x)) < \mathcal{L}^m(B_\varepsilon(x)),\ \forall \varepsilon > 0\}$$

[11] Another interesting extension is the auction-theoretic setting considered in Monteiro, Svaiter, and Page, (2001).

where $\mathcal{L}^m$ is the $m$dimensional Lebesgue measure and $B_\varepsilon(x)$ is the open ball centered at x with radius $\varepsilon > 0$. When Ω has locally finite perimeter we have $\partial_e\Omega = \bigcup_{i=1}^{\infty} K_i \cup N$, where K_i is a compact subset of a C^1 hypersurface S_i, for $i = 1, 2, ...$, and $\mathcal{H}^{m-1}(N) = 0$ where $\mathcal{H}^{m-1}$ is the $m-1$ dimensional Hausdorff measure, and a C^1 hypersurface $S \subset R^m$ is a set for which ∂S is the graph of a smooth function near each $x \in \partial S$. The *measure theoretic unit outer normal* $v_\Omega(x)$ of Ω at x is the unique point $u \in S^{m-1}$ such that $\theta^m(O,x) = \theta^m(I,x) = 0$, where $O = \{y \in \Omega : (y-x)\cdot u > 0\}$ and $I = \{y \notin \Omega : (y-x)\cdot u < 0\}$, and $\theta^m(A,x)$ is the $m$dimensional density at x. The reduced boundary $\partial^*\Omega$ is the set of points x for which Ω has a measure theoretic unit outer normal at x. For a set of locally finite perimeter Ω the three boundaries $\partial\Omega$, $\partial_e\Omega$ and $\partial^*\Omega$ are up to $\mathcal{H}^{m-1}$ null-sets the same.

Proof of Lemma 5 Assume to the contrary, and let Δ be the diagonal in $T \times \cdots \times T$. Then for each $(t,..,t) \in \Delta$ there exists an open neighborhood U_t as in the definition of continuous security. The family $\{V_{(t,...,t)}\}_{(t,...,t)\in\Delta}$ with

$$V_{(t,...,t)} = U_t \times \cdots \times U_t \cap \Delta \tag{74}$$

forms an open cover of the compact set Δ. There is, therefore, a partition of unity $g_i : \Delta \to [0,1]$ subordinated to a finite subcover $\{V_i\}$ of $\{V_{(t,...,t)}\}_{(t,...,t)\in\Delta}$, which allows us to define $f : \Delta \to \Delta$ as

$$f(s,...,s) = (\sum_i g_i(s,...s) f_i(s,...,s), ..., \sum_i g_i(s,...s) f_i(s,...,s)). \tag{75}$$

Now, f is continuous and must have a fixed point

$$(z,...z) = f(z,...z). \tag{76}$$

But

$$f(z,...z) \in coP(z,...,z) \times \cdots \times coP(z,...,t), \tag{77}$$

contradicting diagonal quasiconcavity.■

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