ACHIEVING SMOOTH ASYMPTOTICS FOR THE PRICES OF EUROPEAN OPTIONS IN BINOMIAL TREES

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Abstract. A new binomial approximation to the Black–Scholes model is introduced. It is shown that for digital options and vanilla European call and put options that a complete asymptotic expansion of the error in powers of n^{-1} exists. This is the first binomial tree for which such an asymptotic expansion has been shown to exist.

1. Introduction

There are three main approaches to developing the prices of derivative contracts: Monte Carlo, PDE methods and tree methods. The last are conceptually appealing in that they have a natural financial interpretation, are easy to explain and converge in the limit to the Black–Scholes value. They are also well-adapted to the pricing of derivatives with early exercise features.

Whilst it follows from a suitably modified version of the Central Limit theorem that tree prices converge to the Black–Scholes price, one would also like to know in what way the convergence occurs. In particular, one would like to have an asymptotic expansion for the error. Whilst there has been a reasonable of numerical work, there has not been a large amount of theoretical work on the asymptotics of convergence for the prices of European options. The main results are due to Diener and Diener, [4], and independently Walsh, [13], who show that the lead error term is of the form $\theta(n)/n$ with n a bounded function for the Cox–Ross–Rubinstein (CRR) tree from [3]. They show that θ is not constant and that its value varies according to the distance between the strike and the neighbouring tree nodes at maturity. This,

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in particular, shows that a conventional asymptotic expansion cannot exist.

Since the oscillations are determined by the distance between the strike and the node below divided by the distance between the node below and the node above, which we call $\kappa(n) \in [0, 1)$, this suggests that controlling $\kappa(n)$ can improve the asymptotics. Indeed, in [14] Widdicks, Andricopoulos, Newton and Duck suggest restricting to a subsequence of values of n such that $\kappa(n) \to \frac{1}{2}$ and applying Richardson extrapolation. They achieve good numerical results but do not carry out a theoretical investigation.

Leisen suggests in [10] using a tree restricted to even numbers of steps with the central node placed on the strike. He calls this SMO. He presents numerical results suggesting that the lead asymptotic error is of the form C/n and the error is $\mathcal{O}(n^{-2})$. Again the asymptotics are not justified theoretically. Leisen and Reimer suggest in [9] three new approaches to binomial trees each using an odd number of steps with the tree centred on the strike. Despite the fact that the Leisen and Reimer tree deforms probabilities by moving the centre of the tree possibly a long distance from the strike, they numerically achieve second order convergence for European options but are not able to develop asymptotic expansions. They also demonstrate that many well-known binomial trees such as Tian [12], Jarrow–Rudd [5] and Cox–Ross–Rubinstein [3] will not achieve better than order one convergence. This corresponds to the later work in [4] where they are all demonstrated to have oscillatory first order terms. The Leisen–Reimer tree is the tree currently in most common use for the pricing of American options, see for example [11].

Recent work by Chang and Palmer, [2], rigorously examines a tree where the nodes have been moved a small amount so that the strike is half-way between two neighbouring nodes in log-space. They study the problem of "smooth convergence" by which they mean how to achieve a non-oscillatory lead order term. This is different from the usage in this paper where "smooth" is taken to mean a complete asymptotic expansion in powers of $1/n$ with each term non-oscillatory. They use Uspensky's theorem to show that in this case the lead error is of the form

$$A/n + o(1/n),$$

and give a formula for A. Let CP_n denote the price in their model for n steps, and let BS denote the Black–Scholes price. Their result is equivalent to the statement

$$n(\mathrm{CP}_n - \mathrm{BS} - A/n) \to 0 \quad \text{as} \quad n \to \infty,$$

and says nothing about the next term. This means that if we apply Richardson extrapolation to their tree then we will again have an error of size $o(1/n)$. Although the authors do not address this question, it seems likely from the work of Diener and Diener, [4], that the next term is order $n^{-3/2}$ and that that order will be obtained by Richardson extrapolation.

Here we present a new tree which is restricted to odd numbers of steps, and we require the strike to be half-way between the two central nodes in log-space. As one would expect from the results of Diener and Diener, and Walsh, this results in non-oscillatory asymptotics. We are able to establish that a complete asymptotic expansion of the error in powers of n^{-1} exist. We are also to able to show that even for digital options (i.e. cash or nothing options) the rate of convergence is $1/n$, and the same expansion exists. This contrasts to the CRR tree where the error is of order $n^{-\frac{1}{2}}$, [4]. An immediate consequence of our work is that after two-point Richardson extrapolation, the error is of order n^{-2} and after k point extrapolation it will be order n^{-k}.

Note that this result is substantially stronger than the one in [2], in that our choice of node placement mean that we are able to prove a whole higher order of convergence for two-point Richardson extrapolation and of arbitrarily high order after k-point extrapolation as opposed to $o(1/n)$ convergence.

As with the Leisen–Reimer tree, the placement of the strike in the centre of the tree results in some deformation of the probabilities. However, the ability to eliminate the first order term via Richardson extrapolation more than outweighs any disadvantages arising from this.

Our approach is closest to that of [4]; the main difference being that our careful choice of tree allows the application of symmetries to simplify and improve the analysis. We use very little theory but instead work through an analysis of explicit integrals describing the asymptotics. In this paper, we do not attempt to gain explicit formulas for the error terms. These computations are addressed together with adjustments to remove the error terms in the follow-up paper, [7].

2. A short review of pricing on binomial trees

Whilst the binomial tree is a well-known technique for pricing derivatives, we provide a short review of the techniques to improve readability. For a more expository account we refer the reader to [6] or indeed virtually any elementary book on option pricing.

In a binomial tree, one typically has two underlyings: the stock and the bond. The bond is assumed to grow at a riskless rate r, whilst the stock is stochastic and follows a process designed to approximate geometric Brownian motion. Time is divided into n steps, and at each step the stock can move to new values S_+ and S_-. Since there are only two possible new states and there are two underlyings, this implies that the market is complete, and that for each choice of numeraire there is a unique martingale measure. A measure is defined by the probability of an up move, and since the tree is generally defined in such a way that each step is the same in relative terms, a single probability number (depending on n) defines the measure.

In order to minimize computational cost, the steps are generally chosen to have the same properties in such a way that an up move followed by a down move is the same as a down move followed by an up move, making the tree recombine. We therefore have n random variables $X_j^{(n)}$ which are identically distributed and Bernouilli, such that the stock after k steps is equal to

$$S_j = S_0 X_1^{(n)} X_2^{(n)} \dots X_k^{(n)}. \tag{2.1}$$

The properties of the variables $X_j^{(n)}$ are chosen so as to make the ratio of every asset with the numeraire a martingale, and so that the process converges to geometric Brownian motion in the limit. The convergence to the Black–Scholes price then follows from a suitable version of the Central Limit Theorem (e.g. the Lindeberg–Feller extension.) There are many choices for the variables $X_j^{(n)}$ and these lead to differing convergence properties. Note that conventionally the bond is taken to be numeraire but the arguments work equally with the stock as numeraire, and this will simplify our computations.

Note that if a European derivative pays a unit of the numeraire for $S_T > K$, and zero otherwise, then its value in the binomial model will be N_0 times the probability that the stock finished above K. This in particular means that the price is given by N_0 times a cumulative binomial probability. We shall use this fact extensively.

3. Binomial probabilities

In this section, we develop the asymptotics for the probability of an up move in both the risk-neutral measure (taking the bond as numeraire) and the stock measure (taking the stock as numeraire.) We assume that we are approximating a real-world process with

$$d \log S_t = \mu dt + \sigma dW_t.$$

With n steps, we find the nodes of step j from the nodes of step $j-1$ via

$$\log(S_{j,\pm}) = \log(S_{j-1}) + \mu\frac{T}{n} \pm \sigma\sqrt{\frac{T}{n}}, \tag{3.1}$$

where S_{j-1} could be any of the nodes on the previous step, or equivalently

$$S_{j,\pm} = S_{j-1}e^{\mu\frac{T}{n}\pm\sigma\sqrt{\frac{T}{n}}}.$$

In the risk-neutral measure, we must have

$$\mathbb{E}(S_{j,\pm}e^{-r\frac{T}{n}}) = S_{j-1}.$$

If we let p_n denote the probability of an up-move in the risk-neutral measure, then this implies

$$p_n = \frac{e^{r\frac{T}{n}} - e^{\mu\frac{T}{n}-\sigma\sqrt{\frac{T}{n}}}}{e^{\mu\frac{T}{n}+\sigma\sqrt{\frac{T}{n}}} - e^{\mu\frac{T}{n}-\sigma\sqrt{\frac{T}{n}}}}. \tag{3.2}$$

In the stock-measure, we must have

$$\mathbb{E}(S_{j,\pm}^{-1}e^{r\frac{T}{N}}) = S_{j-1}^{-1}.$$

If we let q_n denote the probability of an up-move in the stock measure, then we have

$$q_n = \frac{e^{-r\frac{T}{n}} - e^{-\mu\frac{T}{n}+\sigma\sqrt{\frac{T}{n}}}}{e^{-\mu\frac{T}{n}-\sigma\sqrt{\frac{T}{n}}} - e^{-\mu\frac{T}{n}+\sigma\sqrt{\frac{T}{n}}}}. \tag{3.3}$$

Our principal result in this section is

Theorem 1. *Each of p_n and q_n has an asymptotic expansion in powers of $\sqrt{\frac{1}{n}}$. We also have*

$$p_n = \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2 - \mu}{2\sigma}\sqrt{\frac{T}{n}} + \mathcal{O}(n^{-3/2}),$$
$$q_n = \frac{1}{2} - \frac{\mu - \frac{1}{2}\sigma^2 - r}{2\sigma}\sqrt{\frac{T}{n}} + \mathcal{O}(n^{-3/2}).$$

In fact, all positive even powers of $n^{-1/2}$ have zero coefficient. This implies that there exist smooth functions g, h on $[0, \epsilon]$ for some $\epsilon > 0$, such that

$$p_n = \frac{1}{2} + \sqrt{\frac{1}{n}}f\left(\frac{1}{n}\right),$$
$$q_n = \frac{1}{2} + \sqrt{\frac{1}{n}}g\left(\frac{1}{n}\right).$$

Importantly, there are no terms of order n^{-1} and n^{-2}. We also have that the term of order $-\frac{1}{2}$ will only vanish for one choice of μ and will never vanish for both p_n and q_n when volatility is non-zero. This term affects the value converged to, rather than the rate of convergence, and for our tree its vanishing will correspond to the case where the value of the digital divided by the numeraire is $\frac{1}{2}$.

Proof. The result for q_n can be deduced from that for p_n if one replaces r, μ, and σ by their negatives, we therefore focus on p_n. For notional simplicity, we write

$$\Delta t = \frac{T}{n}.$$

We can write

$$p_n = \frac{e^{(r-\mu)\Delta t} - e^{-\sigma \Delta t^{1/2}}}{e^{\sigma \Delta t^{1/2}} - e^{-\sigma \Delta t^{1/2}}}.$$

First, observe that the numerator is an analytic function of $\Delta t^{1/2}$. The denominator is equal to $2\sinh(\sigma \Delta t^{1/2})$, which is equal to $\Delta t^{1/2} f(\Delta t^{1/2})$ for some analytic function f with $f(0) \neq 0$. It is now obvious that an asymptotic expansion in powers of $\Delta t^{1/2}$ exists. We will explicitly compute to determine the first few terms.

Expanding, the numerator equals

$$\begin{aligned} 1 + (r-\mu)\Delta t - \left(1 - \sigma \Delta t^{\frac{1}{2}} + \frac{1}{2}\sigma^2 \Delta t - \frac{1}{6}\sigma^3 \Delta t^{3/2}\right) + \mathcal{O}(\Delta t^2) \\ = \sigma \Delta t^{1/2} + \left(r - \mu - \frac{1}{2}\sigma^2\right)\Delta t + \frac{\sigma^3 \Delta t^{3/2}}{6} + \mathcal{O}(\Delta t^2). \quad (3.4) \end{aligned}$$

Expanding the denominator, we get

$$2\sigma \Delta t^{1/2}\left(1 + \frac{\sigma^2 \Delta t}{6} + \mathcal{O}(\Delta t^2)\right),$$

we therefore have that p_n is equal to

$$\begin{aligned} \frac{1}{2}\frac{1}{\sigma \Delta t^{1/2}}\left(\sigma \Delta t^{1/2} + \left(r - \frac{1}{2}\sigma^2 - \mu\right)\Delta t + \frac{\sigma^3 \Delta t^{3/2}}{6} + \mathcal{O}(\Delta t^2)\right) \\ \times \left(1 - \frac{\sigma^2 \Delta t}{6} + \mathcal{O}(\Delta t^2)\right). \end{aligned}$$

Multiplying, we get

$$p_n = \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2 - \mu}{2\sigma}\Delta t^{1/2} + \mathcal{O}(\Delta t^{3/2}),$$

as required.

To complete the proof we need to show that all positive even powers of $\Delta t^{\frac{1}{2}}$ vanish. The term $e^{(r-\mu)\Delta t}$ will only contribute to odd powers, so we can take $r = \mu$ without affecting the result. Since the functions are analytic, the result is then equivalent to the statement that

$$h(s) = \frac{1 - e^{-s}}{e^{s} - e^{-s}} - \frac{1}{2}$$

is odd. This is clear from an elementary computation. □

4. Choosing the tree

We know from the work of Diener and Diener, [4], and Walsh, [13], that the oscillations in the expansion arise from the movements of nodes relative to the strike price of the option. We remove these oscillations by choosing our tree in such a way that the strike is always half-way between two nodes. This will allow a clean expansion.

We proceed by making the tree be centred on the strike price. We also only consider odd number of steps. Thus we let n the number of steps satisfy

$$n = 2N + 1. \tag{4.1}$$

This guarantees that there is an even number of nodes in the final layer and that the strike is in the middle so there will be $N + 1$ nodes above the strike and the same number below.

In order to make the strike be in the middle, we pick μ appropriately. In particular, if today's stock price is S_0, the strike is K, and expiry is T, we set $\mu = \frac{\log(K) - \log(S_0)}{T}$. After $2N + 1$ steps, if we have j up moves, the log of the stock price is

$$\log K + (2j - 2N - 1)\sigma\sqrt{T};$$

clearly this is above the strike if

$$j > N + \frac{1}{2},$$

and below otherwise. In the final layer, the strike will be equidistant from nodes N and $N + 1$ in log coordinates. Note that we can pick μ arbitrarily since the drift of the stock in the real-world measure does not affect the price in the Black–Scholes model.

We can write the pay-off of a call option as

$$S_T I_{S_T > K} - K I_{S_T > K},$$

where I_A denotes the indicator function of the set A. We value the first of these using the stock as numeraire and the second using the riskless bond. We therefore have in any complete market that the value is

$$S_0\mathbb{P}_S(S_T > K) - Ke^{-rT}\mathbb{P}_B(S_T > K),$$

with $\mathbb{P}_S$ denoting probability in the stock measure, and $\mathbb{P}_B$ probability in the bond measure (i.e. the risk-neutral measure.) The division of the pay-off of a call option into two contracts valued using different numeraires is a standard technique, see for example [1] or [6].

If we use our binomial tree with $2N+1$ steps, then we can write this as

$$C(2N+1) = S_0\Phi(2N+1, N+1, q_{2N+1}) - e^{-rT}K\Phi(2N+1, N+1, p_{2N+1}), \tag{4.2}$$

where Φ is the incomplete binomial sum

$$\Phi(n, k, p) = \sum_{j=k}^{n} \begin{pmatrix} n \\ j \end{pmatrix} p^j(1-p)^j.$$

In contrast to the work in [4], we show that each of these terms has a nice asymptotic expansion. Thus our tree differs from the CRR tree in that the asymptotic expansion is a true expansion rather than one with bounded coefficients, and in that there are fewer terms present. Our main theorem is

Theorem 2. *Each of $\phi_q(N) = \Phi(2N+1, N+1, q_{2N+1})$ and $\phi_p(N) = \Phi(2N+1, N+1, p_{2N+1})$ has an asymptotic expansion in powers of N^{-1}, and in particular, we have that there exist constants C_0 and C_1 such that*

$$\phi(N) = C_0 + \frac{1}{N}C_1 + \mathcal{O}(N^{-2}).$$

The terms of order $-1/2$ and $-3/2$ are therefore, unlike for the CRR tree, not present. In [4], Diener and Diener show that the term of order $-1/2$ is present for the CRR tree for a digital option, but disappears via cancellation for a call option. The key to the disappearance of the odd powers is Theorem 1.

5. The proof

Our objective in this section is to prove Theorem 2. The only difference between the two terms in the theorem is that one involves p_{2N+1} and the other q_{2N+1}. Let s_N denote one of p_{2N+1} and q_{2N+1}. By Theorem 1, we have that

$$s_N = \frac{1}{2} + \sqrt{\frac{1}{N}}h\left(\frac{1}{N}\right) \tag{5.1}$$

with h a smooth function on $[0, \epsilon]$, for some $\epsilon > 0$. From here on, we focus on $\Phi(2N+1, N+1, s_N)$ with s_N as in (5.1).

Recall from [4],

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^j = k \binom{n}{k} \int_0^p y^{k-1}(1-y)^{n-k} dy. \tag{5.2}$$

We can therefore write

$$\Phi(2N+1, N+1, s_N) = (N+1) \binom{2N+1}{N+1} \int_0^{s_N} y^N (1-y)^N dy. \tag{5.3}$$

Note the nice feature that the integrand is symmetric around $y = \frac{1}{2}$.

By assumption, we have that the lead term of s_N is $\frac{1}{2}$ so setting

$$t_N = s_N - \frac{1}{2},$$

and using the above-mentioned symmetry, we obtain

$$\Phi(2N+1, N+1, s_N) = \frac{1}{2}(N+1) \binom{2N+1}{N+1} \int_0^1 y^N (1-y)^N dy \\ + (N+1) \binom{2N+1}{N+1} \int_{\frac{1}{2}}^{t_N + \frac{1}{2}} y^N (1-y)^N dy. \tag{5.4}$$

If $t_N < 0$, we interpret the second integral as being minus the integral from $t_N + \frac{1}{2}$ to $\frac{1}{2}$. The first term here is half the binomial probability of obtaining at least $N+1$ draws when the probability of an up-move is 1, and is therefore equal to a half. We thus have

$$\Phi(2N+1, N+1, s_N) = \frac{1}{2} + (N+1) \binom{2N+1}{N+1} \int_{\frac{1}{2}}^{\frac{1}{2}+t_N} y^N (1-y)^N dy. \tag{5.5}$$

Shifting the variable of integration by 1/2, we are reduced to analyzing

$$\phi_N = (N+1) \binom{2N+1}{N+1} \int_0^{t_N} \left(y + \frac{1}{2}\right)^N \left(\frac{1}{2} - y\right)^N dy. \tag{5.6}$$

We can write this as

$$\phi_N = 2^{-2N}(N+1)\begin{pmatrix} 2N+1 \\ N+1 \end{pmatrix}\int_0^{t_N}\left(1-4y^2\right)^N dy. \tag{5.7}$$

This can be factorized as

$$2^{-2N}\begin{pmatrix} 2N+1 \\ N+1 \end{pmatrix}(N+1)N^{-\frac{1}{2}},$$

and

$$N^{\frac{1}{2}}\int_0^{t_N}\left(1-4y^2\right)^N dy.$$

It follows from Stirling's formula that the first of these is a smooth function of $1/N$, i.e. it has an asymptotic expansion in powers of $1/N$ as $N \to \infty$ with the first term a constant. It therefore remains to analyze the asymptotics of ψ_N defined by

$$\psi_N = \int_0^{t_N}\left(1-4y^2\right)^N dy. \tag{5.8}$$

Using the properties of s_N we can replace t_N with $N^{-1/2}t(\frac{1}{N})$ for some smooth function t. To complete the proof, we need to show that

$$f(x) = x^{-1/2}\int_0^{x^{1/2}t(x)}(1-4y^2)^{1/x}dy, \tag{5.9}$$

is smooth as a function of x on $[0,\epsilon]$ After a change of variables, $y' = 2y$, we can rewrite the integrand as

$$e^{\frac{\log(1-y^2)}{x}}.$$

Examining the analytic expansion of $\log(1-z)$, we have

$$\log(1-y^2) = -y^2h(y), \tag{5.10}$$

where h is analytic near 0, is even, and $h(0) = 1$. This implies that $h^{\frac{1}{2}}$ is also analytic and even near zero. Let

$$z = yh^{\frac{1}{2}}(y), \tag{5.11}$$

and then

$$y = zk(z), \tag{5.12}$$

for some even analytic function k. Changing variables to z, our integral becomes

$$x^{-\frac{1}{2}} \int_0^{x^{\frac{1}{2}}\tilde{t}(x)} e^{-z^2/x} l(z)dz,$$

with l analytic and even. The function $\tilde{t}$ will have the same asymptotic properties as t since our change of variables was smooth.

We now carry out one final change of variables. Let $z = x^{\frac{1}{2}}w$. We then have

$$\int_0^{\tilde{t}(x)} e^{-w^2} l(wx^{\frac{1}{2}})dw.$$

Since l is even and analytic, it is a smooth function of x. Expanding l by Taylor expansion to p terms, we have

$$\sum_{j=0}^{p-1} x^j \int_0^{\tilde{t}(x)} e^{-w^2} \frac{d^{2j}l}{dw^{2j}}(0) w^{2j} dw + \mathcal{O}(x^p).$$

The integral of a smooth function between smooth limits is smooth, and our main result is now clear.

6. Richardson extrapolation

Whilst our main interest in this paper is theoretical, we present some numerical work for illustration. As observed in [10], one advantage of developing an asymptotic expansion for binomial prices is that it allows the application of Richardson extrapolation to remove error terms. We have shown that for the Black–Scholes price, BS, of a call option or digital call option, we have that our approximation after n terms satisfies

$$\phi_n = \mathrm{BS} + \frac{A}{n} + \mathcal{O}(n^{-2}),$$

and after $2n+1$ terms (since only defined for n odd,)

$$\phi_{2n+1} = \mathrm{BS} + \frac{A}{2n+1} + \mathcal{O}(n^{-2}).$$

We therefore have

$$\left(1 - \frac{n}{2n+1}\right)^{-1} \left(\phi_{2n+1} - \frac{n}{2n+1}\phi_n\right) = \mathrm{BS} + \mathcal{O}(n^{-2}). \tag{6.1}$$

Clearly, one could achieve higher order convergence via further applications of Richardson extrapolation since we have established a complete

steps	LR error times n^2	adjusted error times n	adjusted extrap. error times n^2	CRR error times n	CRR extrap. error times n
3	-0.273	2.025	-0.721	0.999	-1.637
5	-0.312	2.013	-0.706	0.532	-2.573
7	-0.334	2.005	-0.671	0.063	-3.501
9	-0.347	2.000	-0.643	-0.405	-2.542
11	-0.356	1.997	-0.622	-0.872	-0.244
13	-0.363	1.994	-0.605	-1.338	1.821
15	-0.368	1.992	-0.592	-1.804	3.691
17	-0.372	1.990	-0.582	-2.179	5.220
19	-0.375	1.989	-0.574	-1.742	5.027
21	-0.378	1.988	-0.567	-1.351	4.793
23	-0.380	1.987	-0.561	-0.999	4.525
25	-0.382	1.986	-0.556	-0.681	4.227
201	-0.403	1.978	-0.500	0.571	-4.054

TABLE 1. Errors for the three methods after n steps.

asymptotic expansion. For further discussion of Richardson extrapolation, see for example [8].

We present numbers for a case studied in [10]. We have $S = 100, K = 105, r = 0.05, \sigma = 0.2$, and $T = 1$. The Black–Scholes price is 8.0214. We present results for the Leisen–Reimer tree, the new adjusted method and the CRR tree. The Leisen–Reimer tree is known to have order of convergence n^{-2} from numerical work but no proof has been given. We therefore multiply its error by n^2. The Leisen–Reimer tree is method (C) from their paper [9], and we have adapted our implementation from that in [11].

For the other two trees, since we know the error is of order n^{-1}, we multiply the error by n for the unextrapolated case. In the extrapolated case, we multiply by n^2 for the adjusted tree, but not for the CRR tree where the lead term is still present. We see that the Leisen–Reimer tree does much better than the adjusted tree without extrapolation, but its (unextrapolated) value is similar to that of the adjusted tree after extrapolation. Clearly, extrapolation could be applied to both to get higher order errors.

Note that both the Leisen–Reimer tree and the extrapolated adjusted tree have errors that are more than a 1000 times smaller than that for the CRR tree when 201 steps are used. Note also that the computational cost for the adjusted tree is the same as that for CRR.

Ultimately, the interest in binomial trees is for the pricing of American options. Whilst we do not address the issue here, we note that in that case the Leisen–Reimer is known numerically to have lead term of order -1, see [9] or [11]. It will then be the form of extrapolation in equation (6.1) that will be applied even to the Leisen–Reimer tree despite its higher order convergence for European options. We leave the study of American options to future work.

7. Conclusion

We have shown that for a suitably adjusted implementation of the binomial tree, not only does the lead term have non-oscillatory asymptotics but that a complete asymptotic expansion in powers of n^{-1} exists for the prices of European digital options as well as for vanilla call and put options. This is the first time that such a result has been demonstrated for a binomial tree. This allows the application of Richardson extrapolation, and the achievement of high accuracy with small numbers of steps.

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