THE CHAIN LADDER AND TWEEDIE DISTRIBUTED CLAIMS DATA

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Summary. The chain ladder algorithm is known to provide maximum likelihood (ML) parameter estimates for a model with multiplicative accident period and development period effects, provided that all observations are over-dispersed Poisson (ODP) distributed.

Mack (1991a) obtained the ML equations for the corresponding situation in which cells of the data triangle were gamma rather than ODP distributed.

These two choices of distribution correspond to the cases p=1 and p=2 when cell distributions are assumed to come from the Tweedie family. Section 3 places these results in a more general context by deriving the ML equations for parameter estimation in the case of a general member of the Tweedie family ($p \leq 0$ or $p \geq 1$).

The intermediate cases, with $1<p<2$, represent compound Poisson cell distributions, such as considered by Mack (1991a).

While ML estimates are **not** chain ladder for Tweedie distributions other than ODP, Section 3 indicates why they will be close to chain ladder under certain circumstances. Section 4 also demonstrates that the ML estimates for the general Tweedie case can be obtained by application of the chain ladder algorithm to transformed data. This is illustrated numerically.

Section 5 notes that the models underlying the chain ladder and separation methods are the same apart from an interchange of the roles of rows and diagonals of the data set. Consequently, each result on ML chain ladder estimation in Sections 3 and 4 has its counterpart for the separation method.

Keywords. Chain ladder, maximum likelihood, separation method, Tweedie distribution

1. Introduction

The chain ladder is a widely used algorithm for loss reserving. It is formulated in Mack (1993). From its heuristic beginnings, it was shown to give maximum likelihood (ML) estimates of model parameters (Hachemeister & Stanard, 1975; Mack, 1991a; Renshaw & Verrall, 1998) when:

- observations are independently Poisson distributed; and
- their means are modelled as the product of a row effect and a column effect.

This result was extended from the Poisson to the over-dispersed Poisson (ODP) distribution by England & Verrall (2002).

Mack (1991a) considered another model in which observations were gamma distributed, and gave a number of earlier references to the same model. ML parameter estimates were obtained which, while not identical to chain ladder estimates, have sometimes been found by subsequent authors (e.g. Wüthrich, 2003) to be numerically similar.

The ODP likelihood lies within the Tweedie family (Tweedie, 1984), a subset of the exponential dispersion family (Nelder & Wedderburn, 1972). Wüthrich (2003) made a numerical study of ML fitting in the case of Tweedie distributed observations. Again the results were similar to chain ladder estimation.

The purpose of the present very brief note is to consider ML estimation in this Tweedie case, to derive the earlier results as special cases of it, and to indicate the reasons for the numerical similarity of their results.

2. Preliminaries

Framework and notation

The data set will consist throughout of a triangle of insurance claims data. Let $i=1,2,\ldots,n$ denote period of origin, $j=1,2,\ldots,n$ denote development period, and $Y_{ij}\geq 0$ the observation in the (i,j) cell of the triangle. The triangle of data consists of the set $\{Y_{ij}: i=1,2,\ldots,n; j=1,2,\ldots,n-i+1\}$. It is assumed that $E[Y_{ij}]$ is finite for each (i,j).

Define cumulative row sums

$$S_{ij} = \sum_{k=1}^{j} Y_{ik} \tag{2.1}$$

Further, let $\sum^{R(i)} x_{ij}$ denote summation over the entire row i of the triangle of quantities x_{ij} indexed by i,j, i.e. over cells (i,j) with i fixed and $j=1,2,\ldots,n-i+1$. Similarly, let $\sum^{C(j)}$ denote summation over the entire column j of the triangle, and let $\sum^{D(k)}$ denote summation over the entire diagonal k.

Chain ladder

The chain ladder model is formulated by Mack (1991b, 1993) as follows:

Assumption CL1: $E[S_{i,j+1} \mid S_{i1},S_{i2}\ldots,S_{ij}] = S_{ij}\, f_j$, $j=1,2,\ldots,n-1$, independently of i (2.2)

for some set of parameters f_j; and also

Assumption CL2: Rows of the data triangle are stochastically independent, i.e. Y_{ij} and Y_{kl} are independent for $i\neq k$.

It may be observed that (2.2) implies

$$E[S_{ij} \mid S_{i1}] = S_{i1}\, f_1\, f_2 \ldots f_{j-1} \tag{2.3}$$

or, equivalently,

$$E[Y_{ij}] = \alpha_i\beta_j \tag{2.4}$$

for parameters α_i, β_j, where $E[S_{ij}]$ denotes the unconditional mean of S_{ij}, and

$$f_j = \sum^{j+1}_{k=1} \beta_k \,/\, \sum^{j}_{k=1} \beta_k \qquad (2.5)$$

The chain ladder estimate of f_j is

$$F_j = \sum^{n-j}_{i=1} S_{i,j+1} \,/\, S_{ij} \qquad (2.6)$$

The F_j may be converted to estimates $\hat{\alpha}_i, \hat{\beta}_j$ of the α_i, β_j by means of the following relations:

$$\hat{\beta}_j = \beta_1\, [F_1 \ldots F_{j-2}\, (F_{j-1} - 1)] \qquad (2.7)$$

subject to some linear constraint on the β_j, such as

$$\sum^{n}_{k=1} \beta_k = 1 \qquad (2.8)$$

and

$$\hat{\alpha}_i = S_{i,n-i+1} / \sum^{R(i)} \hat{\beta}_j \qquad (2.9)$$

Exponential dispersion and Tweedie families of distributions

Exponential dispersion family

The following family of log likelihoods (or quasi-likelihoods) is called the **exponential dispersion family** (EDF) (Nelder & Wedderburn, 1972):

$$\ell(y;\theta,\lambda) = c(\lambda)[y\theta - b(\theta)] + a(y,\lambda) \qquad (2.10)$$

for some functions $a(.,.)$, $b(.)$ and $c(.)$ and parameters θ and λ.

It may be shown that, for Y subject to this log likelihood,

$$\mu = E[Y] = b'(\theta), \quad Var[Y] = b''(\theta)/c(\lambda) \qquad (2.11)$$

Tweedie family

A sub-family of the EDF is that defined by the relations:

$$c(\lambda) = \lambda \qquad (2.12)$$
$$Var[Y] = \mu^p/\lambda \text{ for some } p \leq 0 \text{ or } p \geq 1 \qquad (2.13)$$

This is the **Tweedie family** of exponential dispersion likelihoods (Tweedie, 1984). The restriction on the moment relations (2.11) implies that

$$b'(\theta) = [(1-p)(\theta+k)]^{1/(1-p)} \qquad (2.14)$$
$$b(\theta) = (2-p)^{-1}[(1-p)(\theta+k)]^{(2-p)/(1-p)} \qquad (2.15)$$

for some constant k. This parameterization is found, for example, in Jorgensen & Paes de Souza (1994) and Wüthrich (2003) with k=0.

It follows from (2.11), (2.14) and (2.15) that

$$\theta = \mu^{1-p}/(1-p) - k \tag{2.16}$$
$$b(\theta) = \mu^{2-p}/(2-p) \tag{2.17}$$

3. Maximum likelihood estimation for Tweedie chain ladder

Consider the model (2.4), together with the assumption that all Y_{ij} are stochastically independent. Note that this is not the same as the chain ladder model, as defined in Section 2, because the latter does not make the same independence assumption. Indeed, Assumption CL1 specifically postulates dependencies between observations from within the same row.

Let Y denote the entire set $\{Y_{ij}\}$ of observations, and let $\ell(Y)$ denote the log likelihood of Y. Suppose that each Y_{ij} has a Tweedie distribution defined by (2.12) and the following generalization of (2.13):

$$\mathrm{Var}[Y_{ij}] = \mu_{ij}^{p}/\lambda w_{ij} \tag{3.1}$$

i.e. λ is replaced by λ/w_{ij} in (2.12). In common parlance w_{ij} is the weight associated with Y_{ij}. This model will be called the **Tweedie chain ladder model**.

With the replacement just $\lambda \leftarrow \lambda/w_{ij}$ given, and substitution of (2.16) and (2.17) into (2.10),

$$\ell(Y) = \sum \{ \lambda w_{ij} [y_{ij} [\mu_{ij}^{1-p}/(1-p) - k] - \mu_{ij}^{2-p}/(2-p)] + a(y_{ij},\lambda)\} \tag{3.2}$$

where the summation runs over all observations in the data set Y.

The ML equations with respect to the α_i are:

$$\partial L/\partial\alpha_i = \sum^{R(i)} \lambda w_{ij} [y_{ij}\, \mu_{ij}^{-p} - \mu_{ij}^{1-p}]\, \beta_j = 0,\ i=1,\dots,n \tag{3.3}$$

where use has been made of (2.4). This may be equivalently represented as follows:

Lemma 3.1. The ML equations with respect to the α_i for the Tweedie chain ladder model are:

$$\sum^{R(i)} w_{ij}\, \mu_{ij}^{1-p} [y_{ij} - \mu_{ij}] = 0,\ i=1,\dots,n \tag{3.4}$$

Similarly, the ML equations with respect to the β_j are:

$$\sum^{C(j)} w_{ij}\, \mu_{ij}^{1-p}\, [y_{ij} - \mu_{ij}] = 0,\ j=1,\ldots,n \qquad (3.5)$$

Corollary 3.2. The case of ODP Y_{ij} is represented by p=1, w_{ij}=1. The ML equations are then

$$\sum^{R(i)} [y_{ij} - \mu_{ij}] = 0,\ i=1,\ldots,n \qquad (3.6)$$

$$\sum^{C(j)} [y_{ij} - \mu_{ij}] = 0,\ j=1,\ldots,n \qquad (3.7)$$

These imply the chain ladder estimation of the α_i, β_j set out in (2.6)-(2.9).

Proof. See Hachemeister & Stanard (1975), Mack (1991a) or Renshaw & Verrall (1998). □

Corollary 3.3. The case of gamma Y_{ij} is represented by p=2. The ML equations are then

$$\sum^{R(i)} w_{ij}\, [y_{ij} / \mu_{ij} - 1] = 0,\ i=1,\ldots,n \qquad (3.8)$$

$$\sum^{C(j)} w_{ij}\, [y_{ij} / \mu_{ij} - 1] = 0,\ j=1,\ldots,n \qquad (3.9)$$

Substitution of $\alpha_i\beta_j$ for μ_{ij}, followed by minor rearrangement, gives

$$\alpha_i = w_{i\cdot}^{-1} \sum^{R(i)} w_{ij}\, y_{ij} / \beta_j,\ i=1,\ldots,n \qquad (3.10)$$

$$\beta_j = w_{\cdot j}^{-1} \sum^{C(j)} w_{ij}\, y_{ij} / \alpha_i,\ j=1,\ldots,n \qquad (3.11)$$

where

$$w_{i\cdot} = \sum^{R(i)} w_{ij} \qquad (3.12)$$

$$w_{\cdot j} = \sum^{C(j)} w_{ij} \qquad (3.13)$$

These are essentially the results obtained by Mack (1991a) for gamma distributed cells.

Remark 3.4. Mack's assumption of a gamma distribution is, in fact, an approximation to a compound Poisson distribution in each cell of the triangle in which each cell has a gamma severity distribution with the same shape parameter. Mack notes that the shape parameter would need to take a smallish value in order to attribute a n0n-negligible probability to Y_{ij} in the vicinity of zero.

It may be noted that, as shown by Jorgensen and Paes de Souza (1994), the compound Poisson itself may be accommodated within the Tweedie family (with $1 \leq p < 2$) and so this element of approximation eliminated.

Remark 3.5. The ML equations (3.6) and (3.7) also show that the chain ladder estimates are marginal sum estimates in the ODP case (see Mack, 1991a; Schmidt & Wünsche, 1998). In the general Tweedie case (equations

(3.4) and (3.5)), while **not** equivalent to the chain ladder, they are **weighted** marginal sum estimates.

This provides an indication of the reason why past investigations have shown chain ladder estimates to be close to ML estimates in various Tweedie cases. For example, this was a finding of Wüthrich (2003).

To elaborate on this, write the general weighted marginal sum equation corresponding to (3.4) in the form

$$\sum^{R(i)} \omega_{ij} [y_{ij} - \hat{\mu}_{ij}] = 0 \tag{3.14}$$

where the ω_{ij} are general weights and the term $\hat{\mu}_{ij}$ recognizes that the solution of the equations provides only an estimate of μ_{ij}. A parallel to the following argument about (3.4) may be given in relation to (3.5).

Now re-write the left side of (3.14) as

$$\sum^{R(i)} \omega_{ij} [\varepsilon_{ij} + \eta_{ij}] \tag{3.15}$$

where $\varepsilon_{ij} = y_{ij} - \mu_{ij}$ and $\eta_{ij} = \mu_{ij} - \hat{\mu}_{ij}$, both of which are random variables with zero means (assuming a correctly specified model).

Now consider the substitution of the solutions $\hat{\mu}_{ij}$ of (3.14) in the unweighted form of the same system of equations:

$$\begin{aligned} \omega_i \sum^{R(i)} [y_{ij} - \hat{\mu}_{ij}] &= \omega_i \sum^{R(i)} [\varepsilon_{ij} + \eta_{ij}] \\ &= \sum^{R(i)} \omega_{ij} [\varepsilon_{ij} + \eta_{ij}] + \sum^{R(i)} (\omega_i - \omega_{ij}) [\varepsilon_{ij} + \eta_{ij}] \\ &= \sum^{R(i)} (\omega_i - \omega_{ij}) [\varepsilon_{ij} + \eta_{ij}] \qquad \text{[by (3.14)]} \end{aligned} \tag{3.16}$$

where $\omega_i = \sum^{R(i)} \omega_{ij} / (n-i+1)$.

The right side of (3.16) has a mean of zero and a variance of $\sum^{R(i)} (\omega_i - \omega_{ij}) \sigma_{ij}^2$ where $\sigma_{ij}^2 = \text{Var}[\varepsilon_{ij} + \eta_{ij}] = \text{Var}[y_{ij} - \hat{\mu}_{ij}]$. Hence the value of (3.16) will be small if either or both of the following conditions hold:

- Weights vary little across a row;
- The variances of observations around values fitted by (3.14) are small.

In this case, the solutions to (3.4) will also be approximate solutions to the unweighted form:

$$\sum^{R(i)} [y_{ij} - \hat{\mu}_{ij}] = 0$$

which is the chain ladder solution.

In summary, under the right conditions the chain ladder will approximate the solution to the weighted marginal sum estimates given by (3.4) and (3.5).

An example of this approximation is provided by Wüthrich (2003), who made a numerical study of ML fitting of the Tweedie chain ladder model in which the parameters α_i, β_j, λ and p were all treated as free and the weights w_{ij} as known. In the example, the w_{ij} varied comparatively little with i and j, and p was estimated to be 1.17.

Hence the weights $\omega_{ij} = w_{ij}\,\mu_{ij}^{p-1}$ show not too much variation over the triangle and the ML estimates of the Tweedie chain ladder are expected to approximate those of the standard chain ladder, as was indeed found by Wüthrich.

4. Maximum likelihood estimation for general Tweedie

Parameters of the general Tweedie chain ladder model may be estimated by the use of GLM software. However, an interesting special case arises under the sole constraint that the weights w_{ij} also have the multiplicative structure:

$$w_{ij} = u_i\, v_j \tag{4.1}$$

Note that this includes the unweighted case $w_{ij} = 1$.

The ML equations for estimation of the α_i, β_j were derived as (3.4) and (3.5). Rewrite these with the substitutions:

$$Z_{ij} = w_{ij}\,\mu_{ij}^{1-p}\, Y_{ij} \tag{4.2}$$
$$\nu_{ij} = w_{ij}\,\mu_{ij}^{2-p} = u_i v_j\,(\alpha_i\beta_j)^{2-p} = a_i b_j \tag{4.3}$$

where

$$a_i = u_i\,\alpha_i^{2-p} \tag{4.4}$$
$$b_j = v_j\,\beta_j^{2-p} \tag{4.5}$$

This yields

$$\sum^{R(i)} [z_{ij} - \nu_{ij}] = 0,\ i=1,\ldots,n \tag{4.6}$$

$$\sum^{C(j)} [z_{ij} - \nu_{ij}] = 0,\ i=1,\ldots,n \tag{4.7}$$

Note that these are the same equations as (3.6) and (3.7) in Corollary 3.2. The lemma therefore implies the following result.

Lemma 4.1. Consider the Tweedie chain ladder model with general (admissible) p and subject to (3.1) with constraint (4.1). ML estimates of a_i, b_j (and hence of α_i, β_j, by (4.4) and (4.5)) are obtained by application of the chain ladder algorithm (2.6)-(2.9) to the data triangle $Z=\{Z_{ij}\}$. □

In the application of this result $\mu_{ij} = \alpha_i\beta_j$ must be known in order to formulate the "data" Z_{ij}, whereas α_i, β_j are estimands of the theorem. However, a solution can be obtained by an iterative procedure.

Let a superscript (r) denote the r-th iteration of the estimate to which it is attached, e.g. $\mu^{(r)}_{ij}$. Define

$$Z^{(r)}_{ij} = w_{ij} [\mu^{(r)}_{ij}]^{1-p} Y_{ij} \quad (4.8)$$
$$\nu^{(r)}_{ij} = w_{ij} [\mu^{(r)}_{ij}]^{2-p} = u_i v_j (\alpha^{(r)}_i \beta^{(r)}_j)^{2-p} = a^{(r)}_i b^{(r)}_j \quad (4.9)$$

Then define $a^{(r+1)}_i$, $b^{(r+1)}_j$ as the estimates obtained in place of a_i, b_j when the chain ladder algorithm is applied to the data triangle $\{Z^{(r)}_{ij}\}$ in place of Z. By this iterative means, obtain the sequence of estimates $\{a^{(r)}_i, b^{(r)}_j, r=0,1,\ldots\}$, initiated at r=0 by some simple choice, such as setting $a^{(r)}_i$, $b^{(r)}_j$ equal to the estimates of α_i, β_j given by the conventional chain ladder.

If this sequence converges, then the limit is taken as an estimate of the a_i, b_j.

This procedure has been applied to the data set in the Appendix with p=2, and convergence of the estimate loss reserve to an accuracy of 0.05% in the estimated loss reserve obtained in 5 iterations. Convergence becomes slower as p increases. For p=2.4, 24 iterations were required to achieve an accuracy of 0.1%.

5. The "separation method"

Taylor (1977) introduced the procedure that subsequently became known as the "separation method". This produces parameter estimates for a model of the form

$$E[Y_{ij}] = \alpha_{i+j-1}\beta_j \quad (5.1)$$

which is the parallel of (2.4), but with the α parameter applying to diagonal i+j-1 rather than row i.

The heuristic equations given by Taylor for parameter estimation were:

$$\sum^{D(k)} [y_{ij} - \mu_{ij}] = 0, k=1,\ldots,n \quad (5.2)$$

$$\sum^{C(j)} [y_{ij} - \mu_{ij}] = 0, j=1,\ldots,n \quad (5.3)$$

It is evident that these equations yield marginal sum estimates. Taylor (1977) gives the explicit algorithm for generating estimates of the α_{i+j-1}, β_j. This will be referred to as **separation method estimation**, and is as follows:

$$\alpha_k = \sum^{D(k)} Y_{ij} / [1 - \sum^{n}_{j=n-k} \beta_j] \quad (5.4)$$

$$\beta_j = \sum^{C(j)} Y_{ij} / \sum^{n}_{k=j} \alpha_k \quad (5.5)$$

these equations being applied alternately for k=n, j=n, k=n-1, etc.

The model resulting from replacement of (2.4) by (5.1) in the Tweedie chain ladder model will be referred to as the **Tweedie separation model**. It is the same as the Tweedie chain ladder model except for the interchange of rows and diagonals, and so a result parallel to each of those of Sections 3 and 4 is obtainable.

Lemma 5.1. The ML equations with respect to the α_k, β_j for the Tweedie separation model are:

$$\sum^{D(k)} w_{ij} \mu_{ij}^{1-p} [y_{ij} - \mu_{ij}] = 0, i=1,\ldots,n \quad (5.6)$$

$$\sum^{C(j)} w_{ij} \mu_{ij}^{1-p} [y_{ij} - \mu_{ij}] = 0, j=1,\ldots,n \quad (5.7)$$

Corollary 5.2. The case of ODP Y_{ij} is represented by p=1, w_{ij}=1. The ML equations are then

$$\sum^{D(k)} [y_{ij} - \mu_{ij}] = 0, i=1,\ldots,n \quad (5.8)$$

$$\sum^{C(j)} [y_{ij} - \mu_{ij}] = 0, j=1,\ldots,n \quad (5.9)$$

These imply the separation method estimation of the α_k, β_j set out in (5.4) and (5.5).

Remark 5.3. This result was known for the simple Poisson case since Verbeek (1972), actually earlier than the corresponding result for the chain ladder (Corollary 3.2).

Corollary 5.4. The case of gamma Y_{ij} is represented by p=2. The ML equations are then

$$\sum^{D(k)} w_{ij} [y_{ij} / \mu_{ij} - 1] = 0, i=1,\ldots,n \quad (5.10)$$

$$\sum^{C(j)} w_{ij} [y_{ij} / \mu_{ij} - 1] = 0, j=1,\ldots,n \quad (5.11)$$

Remark 5.5. In the case of the general Tweedie separation model, the separation method algorithm (5.4) and (5.5) will approximate the ML solution (5.6) and (5.7) if either or both of the following conditions hold:

- Weights vary little over the triangle;
- The variances of observations around values fitted by (5.6) and (5.7) are small.

Lemma 5.6. Consider the Tweedie separation model with general (admissible) p and subject to (3.1) with constraint

$$w_{i+j-1,j} = u_{i+j-1} v_j \quad (5.12)$$

Define by (4.2), and also define

$$v_{i+j-1,j} = w_{i+j-1,j} \mu_{i+j-1,j}^{2-p} = u_{i+j-1} v_j (\alpha_{i+j-1}\beta_j)^{2-p} = a_{i+j-1} b_j \quad (5.13)$$

where

$$a_k = u_k \alpha_k^{2-p} \tag{5.14}$$
$$b_j = v_j \beta_j^{2-p} \tag{5.15}$$

ML estimates of a_k, b_j (and hence of α_k, β_j) are obtained by application of the separation method algorithm (5.4) and (5.5) to the data triangle $Z=\{Z_{ij}\}$.

6. Acknowledgement

Thanks are due to Hugh Miller, who provided the numerical detail reported in Section 4.

Appendix
Data for numerical example

The following data triangle is extracted from Appendix B.3.3 to Taylor (2000).

Accident	Claim payments ($) in development year												
year	1	2	3	4	5	6	7	8	9	10	11	12	13
1983	1,897,289	5,200,926	6,766,124	5,390,019	1,495,905	2,031,888	2,493,553	506,813	128,100	75,943	308,205	8,899	18,813
1984	2,087,985	4,308,216	5,872,530	6,782,784	4,915,169	2,051,073	1,864,319	562,354	356,830	833,297	4,844	561,572	
1985	1,490,677	4,476,085	4,992,179	8,358,920	4,697,517	3,502,695	850,298	2,684,057	727,265	3,400	397,917		
1986	1,483,176	3,293,114	6,436,956	6,102,689	5,747,793	4,045,070	2,522,463	1,125,877	1,431,484	862,797			
1987	1,392,209	4,130,422	4,838,069	6,746,366	5,949,455	3,748,639	2,854,290	1,001,874	738,291				
1988	1,350,347	2,687,237	4,483,829	5,607,406	4,630,570	3,082,570	1,760,536	2,190,282					
1989	1,777,107	4,026,788	4,038,537	5,375,214	5,109,038	3,723,188	3,122,941						
1990	1,861,113	2,828,223	2,935,704	5,537,553	6,515,910	6,300,323							
1991	2,236,165	3,848,454	4,554,935	6,457,862	5,572,385								
1992	2,271,180	3,459,346	3,599,932	5,309,764									
1993	2,822,819	4,834,966	7,362,328										
1994	2,464,971	4,669,219											
1995	2,725,355												

References

England P D & Verrall R J (2002). Stochastic claims reserving in general insurance. **British Actuarial Journal**, 8(iii), 443-518.

Hachemeister C A & Stanard J N (1975). IBNR claims count estimation with static lag functions. Paper presented to the **XIIth Astin Colloquium**, Portimão, Portugal.

Jorgensen B & Paes de Souza M C (1994). Fitting Tweedie’s compound Poisson model to insurance claims data. **Scandinavian Actuarial Journal**, 69-93.

Mack T (1991a). A simple parametric model for rating automobile insurance or estimating IBNR claims reserves. **Astin Bulletin**, 21(1), 93-109.

Mack T (1991b). Which stochastic model is underlying the chain ladder method? **Insurance: mathematics & economics**, 15(2/3), 133-138.

Mack T (1993). Distribution-free calculation of the standard error of chain ladder reserve estimates. **Astin Bulletin**, 23(2), 213-225.

Nelder J.A. and Wedderburn R.W.M. (1972). Generalized linear models. **Journal of the Royal Statistical Society, Series A**, 135, 370-384.

Renshaw A E & Verrall R J (1998). A stochastic model underlying the chain-ladder technique. **British Actuarial Journal**, 4(iv), 903-923.

Schmidt K D & Wünsche A (1998). Chain ladder, marginal sum and maximum likelihood estimation. **Blätter der Versicherungsmathematiker**, 23, 267-277.

Taylor G. (1977). Separation and other effects from the distribution of non-life insurance claim delays. **Astin Bulletin,** 9, 217-230.

Taylor G. (2000). **Loss reserving: an actuarial perspective.** Kluwer Academic Publishers. London, New York, Dordrecht.

Tweedie M C K (1984). An index which distinguishes between some important exponential families. Appears in **Statistics: applications and new directions. Proceedings of the Indian Statistical Golden Jubilee International Conference** (eds. Ghosh J K & Roy J), Indian Statistical Institute, 579-604.

Verbeek H G (1972). An approach to the analysis of of claims experience in Motor Liability excess of loss reinsurance. **Astin Bulletin,** 6, 195-202.

Wüthrich M V (2003). Claims reserving using Tweedie's compound Poisson model. **Astin Bulletin**, 33(3), 331-346.