The Time of Recovery and the Maximum Severity of Ruin in a Sparre Andersen Model

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Abstract

Phase-type distributions are one of the most general classes of distributions permitting a Markovian interpretation. Sparre Andersen risk models with phase-type claim inter-arrival times or phase-type claims can be analyzed using Markovian techniques and results can be expressed in compact matrix forms. Computations involved are readily programmable in practice.

This paper studies some quantities associated with the first passage time and the time of ruin in a Sparre Andersen risk model with phase-type inter-claim times. Li (2008) obtains a matrix expression for the Laplace transform of the first time that the surplus process reaches a given target from the initial surplus. Using this result, we analyze (i) the Laplace transform of the recovery time after ruin, (ii) the probability that the surplus attains a certain level before ruin and (iii) the distribution of the maximum severity of ruin. We also give a matrix expression for the expected discounted dividend payments prior to ruin for the Sparre Andersen model in the presence of a constant dividend barrier.

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1 Introduction

Much of the literature on ruin theory is concentrated on classical risk theory, in which the insurer starts with an initial surplus, and collects premiums continuously at a constant rate, while the aggregate claims process follows a compound Poisson process. Andersen (1957) let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then, random walks and queuing theory have provided a more general framework, which has led to explicit results in the case where the inter-claim times or the claim severities have distributions related to the Erlang or phase-type distributions.

The Sparre Andersen model with Erlang inter-claim times has been studied by Li and Garrido (2004) and Gerber and Shiu (2005). Willmot (1999) and Li and Garrido (2005) study the ruin probability and the Gerber-Shiu function in a Sparre Andersen model with inter-claim times being K_n distributed. The Sparre Andersen model with phase-type inter-claim times has been studied by Schmidli (2005), Albrecher and Boxma (2005), and Ren (2007). Asmussen (2000), Avram and Usabel (2004), Drekic et al. (2004) and Ng and Yang (2005) study the ruin probability and the distribution of the severity of ruin in risk models with phase-type claims. Willmot (2007) and Landriaut and Willmot (2008) study the Gerber-Shiu function in a Sparre Andersen model with general inter-claim times.

As mentioned in Bladt (2005), phase-type distributions constitute a class of distributions on positive real axis which seems to strike a balance between generality and tractability. Phase-type distributions have rational Laplace transforms and include combinations and mixtures of exponential and Erlang distributions as special cases. A phase-type distribution inherits the special structure from the Markov property of the underlying continuous-time Markov chain. Moreover, as stated in Ko and Ng (2007) that the class of phase-type distributions is one of the classes of distributions which are dense in the class of all positive distributions so that any distribution may be approximated arbitrarily closely by a suitable phase-type distribution. Sparre Andersen model with phase-type inter-claims can be analyzed using both renewal theory and Markovian techniques and this makes many derivations possible. Many results in the classical risk model can be extended to the Sparre Andersen model with phase-type inter-claim times and these results may be written in compact matrix forms.

We now consider a continuous-time Sparre Andersen model in which the surplus process is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \, , \quad t \geq 0 \, , \tag{1.1}$$

where $u \geq 0$ is the initial surplus, $c \geq 0$ is the premium income rate, while the random sum represents aggregate claims. The X_i's are i.i.d. random variables with common distribution function (d.f.) $P(x) = \mathbb{P}(X \leq x)$ $(P(0) = 0)$ and density function $p(x) = P'(x)$. X_i represents the i-th claim amount. Laplace transform (LT) of p is denoted as $\hat{p}(s) = \int_0^\infty e^{-sx}\, p(x)\, dx$. The renewal process $\{N(t)\,;\, t \geq 0\}$ denotes the number of claims up to time t and is independent of $\{X_i\}_{i\geq 1}$.

For $k = 1, 2, \ldots$, let V_k denote the time when the k-th claim occurs. Let $W_1 = V_1$ and $W_i = V_i - V_{i-1}$ for $i = 2, 3, \ldots$. We assume that $W_1, W_2, \ldots$ are independent and the inter-claim times $W_2, W_3, \ldots$ have a common distribution function $A(x) = \mathbb{P}(W \leq x)$ and density function $a(x) = A'(x)$. Denote by $\hat{a}(s) = \int_0^\infty e^{-sx}\, a(x)\, dx$ the LT of a. Further assume that $c\,\mathbb{E}[W] > \mathbb{E}[X]$, providing a positive safety loading factor.

In this paper, we assume that the distribution of the inter-claim times A is phase-type with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{B})$, where $\mathbf{B} = (b_{i,j})_{i,j=1}^n$ is an $n \times n$ matrix with $b_{i,i} < 0$, $b_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^n b_{i,j} \leq 0$ for any $i = 1, 2, \ldots, n$, $\vec{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$. Then

$$\begin{aligned} A(t) &= 1 - \vec{\boldsymbol{\alpha}} e^{t\mathbf{B}} \vec{\mathbf{e}}^\top, \qquad t \geq 0, \\ a(t) &= \vec{\boldsymbol{\alpha}} e^{t\mathbf{B}} \vec{\mathbf{b}}^\top, \qquad t \geq 0, \\ \hat{a}(s) &= \vec{\boldsymbol{\alpha}} (s\mathbf{I} - \mathbf{B})^{-1} \vec{\mathbf{b}}^\top, \\ \mathbb{E}[W] &= \int_0^\infty t\, a(t) dt = -\vec{\boldsymbol{\alpha}} \mathbf{B}^{-1} \vec{\mathbf{e}}^\top, \end{aligned}$$

where $\vec{\mathbf{e}}$ is a row vector of length n with each element being 1, $\mathbf{I}$ is the $n \times n$ identity matrix, and $\vec{\mathbf{b}}^\top = -\mathbf{B}\vec{\mathbf{e}}^\top$.

It follows that W corresponds to the time to absorption in a terminating continuous-time Markov chain $\{J(t)\}_{t\geq 0}$ with $n+1$ states, one of which is absorbing. The state space of $\{J(t)\}_{t\geq 0}$ is $\{1, 2, 3, \ldots, n, 0\} = E \cup \{0\}$ and the initial distribution is $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0)$. The generator of $\{J(t)\}_{t\geq 0}$ is

$$\begin{pmatrix} \mathbf{B} & \vec{\mathbf{b}}^\top \\ \mathbf{0} & 0 \end{pmatrix}.$$

A detailed introduction to phase-type distributions and their properties can be found in Neuts (1981), Asmussen (1992), and references therein.

Note that W_1 may not follow the same distribution as the inter-claim times. In the case of an ordinary renewal risk process, W_1 follows the distribution A, that is to say, a claim has taken place at time 0 and u is the surplus after the claim is paid. For $i \in E$, if $J(0) = i$, then W_1 follows a distribution with density function

$\vec{\mathbf{e}}_i e^{t\mathbf{B}}\vec{\mathbf{b}}^\top$, where $\vec{\mathbf{e}}_i$ is a $1 \times n$ row vector with the i-th element being 1 and all other elements being 0. In the rest of the paper, we use $\mathbb{P}$ to denote the probability measure when the surplus process is an ordinary renewal risk process and use $\mathbb{P}_i$ to denote the probability measure given $J(0) = i$, i.e., the density function of W_1 is $\vec{\mathbf{e}}_i e^{t\mathbf{B}}\vec{\mathbf{b}}^\top$. $\mathbb{E}$ and $\mathbb{E}_i$ represent expectation operators under $\mathbb{P}$ and $\mathbb{P}_i$, respectively.

Let T denote the time of ruin,

$$T = \inf\{t \geq 0\,;\, U(t) < 0\}$$

($T = \infty$ if ruin does not occur). Define

$$\Psi(u) = \mathbb{P}(T < \infty \,\big|\, U(0) = u)\ , \qquad u \geq 0\ ,$$

as the infinite-horizon ruin probability in the corresponding ordinary renewal risk model. Further, define

$$\Psi_i(u) = \mathbb{P}_i(T < \infty \,\big|\, U(0) = u)\ , \qquad i \in E, u \geq 0\ ,$$

to be the ruin probability given that the initial state is i. Then

$$\Psi(u) = \vec{\boldsymbol{\alpha}}\vec{\boldsymbol{\Psi}}(u), \tag{1.2}$$

where $\vec{\boldsymbol{\Psi}}(u) = (\Psi_1(u), \Psi_1(u), \ldots, \Psi_n(u))^\top$. In Section 3, each $\Psi_i(u)$ will be further decomposed into n components according to the state at the time of recovery after ruin.

Li (2008) obtained a matrix exponential expression for the Laplace transform of the first time when the surplus attains a given level. The matrix $\mathbf{K}$ has the same eigenvalues as the matrix $\mathbf{Q}$ defined in Ren (2007) who shows that $\mathbf{Q}$ plays an important role in the discounted joint distribution of the surplus before ruin and the deficit at ruin. In this paper, we show how $\mathbf{K}$ can be used to extend some existing results from the classical risk model to a Sparre Andersen model with phase-type inter-claim times.

The rest of the paper is organized as follows. Section 2 reviews the result obtained in Li (2008): the Laplace transform of the first time that the surplus hits a certain level. Using this result, the Laplace transform of the time of recovery is obtained in Section 3. The probability that the surplus attains a certain level before ruin and the probability that ruin occurs without the surplus ever reaching a certain level are obtained in Section 4. Section 5 gives an expression for the distribution of the maximum severity of ruin. The expected discounted dividend payments prior to ruin for the model modified by adding a constant dividend barrier is studied in Section 6.

2 When Does the Surplus Attain a Certain Level?

For $b \geq u$, define

$$T_b = \min\{t \geq 0 : U(t) = b\} \tag{2.3}$$

to be the first time when the surplus reaches level b and define for $\delta \geq 0$

$$R(u;b) = \mathbb{E}\left[e^{-\delta T_b} \middle| U(0) = u\right]$$

to be the Laplace transform of T_b. Further define

$$R_{i,j}(u;b) = \mathbb{E}_i\left[e^{-\delta T_b} I(J(T_b) = j) \middle| U(0) = u\right], \qquad i,j = 1,2,\ldots,n,$$

to be the Laplace transform of T_b and the state when the process hits b is j, given that the initial state is i and the initial surplus is u. Then $R(u;b)$ may be computed by

$$R(u;b) = \vec{\boldsymbol{\alpha}}\mathbf{R}(u;b)\vec{\mathbf{e}}^\top, \tag{2.4}$$

where $\mathbf{R}(u;b) = \big(R_{i,j}(u;b)\big)_{i,j=1}^n$.

It follows from Li (2008) that

$$\mathbf{R}(u;b) = e^{-\mathbf{K}(b-u)}, \qquad u \leq b, \tag{2.5}$$

and

$$R(u;b) = \vec{\boldsymbol{\alpha}} e^{-\mathbf{K}(b-u)}\vec{\mathbf{e}}^\top, \qquad u \leq b, \tag{2.6}$$

where $\mathbf{K}$ is an $n \times n$ matrix which satisfies the following equation:

$$c\mathbf{K} = (\delta\,\mathbf{I} - \mathbf{B}) - \vec{\mathbf{b}}^\top\vec{\boldsymbol{\alpha}} \int_0^\infty p(x) e^{-\mathbf{K}x} dx. \tag{2.7}$$

To solve the matrix equation above, we let

$$\mathbf{L}_\delta(s) = \left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{B} + \frac{1}{c}\hat{p}(s)\vec{\mathbf{b}}^\top\vec{\boldsymbol{\alpha}}.$$

It follows from Ren (2007) that the solutions to the equation

$$\det[\mathbf{L}_\delta(s)] = 0 \tag{2.8}$$

and the solutions to Lundberg's fundamental equation

$$\hat{a}(\delta - c\,s)\hat{p}(s) = 1 \tag{2.9}$$

are identical. It can be proved by Rouché's theorem that equation (2.9) has exactly n solutions, denoted as $\rho_1, \rho_2, \ldots, \rho_n$, in the right half of the complex plane, In the following we assume that these n solutions are distinct.

Li (2008) shows that

$$\mathbf{K} = \mathbf{H}\boldsymbol{\Delta}\mathbf{H}^{-1}, \tag{2.10}$$

where $\boldsymbol{\Delta} = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$ and $\mathbf{H} = (\vec{\mathbf{h}}_1, \vec{\mathbf{h}}_2, \ldots, \vec{\mathbf{h}}_n)$, with column vector $\vec{\mathbf{h}}_i$ being an eigenvector of $\mathbf{L}_\delta(\rho_i)$ corresponding to the eigenvalue 0, i.e., $\mathbf{L}_\delta(\rho_i)\vec{\mathbf{h}}_i = \vec{\mathbf{0}}$, for $i = 1, 2, \ldots, n$. Then

$$R(u; b) = \vec{\boldsymbol{\alpha}}\mathbf{H}e^{-\boldsymbol{\Delta}(b-u)}\mathbf{H}^{-1}\vec{\mathbf{e}}^\top, \qquad u \le b, \tag{2.11}$$

When $\delta = 0$, $R_{i,j}(u; b)$ simplifies to the probability that the surplus first hits b at state j from the initial state i and the initial surplus u, i.e.,

$$R_{i,j}(u; b) = \mathbb{P}_i\left(J(T_b) = j \middle| U(0) = u\right), \qquad i, j = 1, 2, \ldots, n.$$

In this case, we denote $\mathbf{R}_0(u; b) = (R_{i,j}(u; b))_{i,j=1}^n$. It follows from (2.5) that

$$\mathbf{R}_0(u; b) = e^{-\mathbf{K}_0\,(b-u)}, \qquad u \le b, \tag{2.12}$$

where matrix $\mathbf{K}_0$ satisfies the following matrix equation:

$$c\mathbf{K}_0 = -\mathbf{B} - \vec{\mathbf{b}}^\top\vec{\boldsymbol{\alpha}}\int_0^\infty e^{-\mathbf{K}_0\,x}p(x)dx. \tag{2.13}$$

Example 1 (Erlang(n) inter-claim times)

When the distribution of inter-claim times is Erlang(n) with parameter λ, we have $\vec{\boldsymbol{\alpha}} = (1, 0, 0, \ldots, 0)$, $\vec{\mathbf{b}} = (0, 0, \ldots, 0, \lambda)$, and

$$\mathbf{B} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda \end{pmatrix}.$$

It is easy to show that $\mathbf{L}_\delta(s)$ has the following form:

$$\mathbf{L}_\delta(s) = \frac{1}{c}\begin{pmatrix} cs-(\lambda+\delta) & \lambda & 0 & \cdots & 0 & 0 \\ 0 & cs-(\lambda+\delta) & \lambda & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & cs-(\lambda+\delta) & \lambda \\ \lambda\hat{p}(s) & 0 & \cdots & \cdots & 0 & cs-(\lambda+\delta) \end{pmatrix},$$

and $\mathbf{H}$ is a Vandermonde matrix having the following form

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ \frac{\lambda+\delta-c\rho_1}{\lambda} & \frac{\lambda+\delta-c\rho_2}{\lambda} & \cdots & \cdots & \frac{\lambda+\delta-c\rho_n}{\lambda} \\ \left(\frac{\lambda+\delta-c\rho_1}{\lambda}\right)^2 & \left(\frac{\lambda+\delta-c\rho_2}{\lambda}\right)^2 & \cdots & \cdots & \left(\frac{\lambda+\delta-c\rho_n}{\lambda}\right)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\lambda+\delta-c\rho_1}{\lambda}\right)^{n-1} & \left(\frac{\lambda+\delta-c\rho_2}{\lambda}\right)^{n-1} & \cdots & \cdots & \left(\frac{\lambda+\delta-c\rho_n}{\lambda}\right)^{n-1} \end{pmatrix}.$$

Then

$$R(u;b) = \vec{\boldsymbol{\alpha}}\mathbf{H}e^{-\boldsymbol{\Delta}(b-u)}\mathbf{H}^{-1}\vec{\mathbf{e}}^\top = \sum_{i=1}^{n}\left[\prod_{j=1,j\neq i}^{n}\frac{\rho_j-\delta/c}{\rho_j-\rho_i}\right]e^{-\rho_i(b-u)}. \tag{2.14}$$

3 The Time of Recovery

In this section, we extend the definition of the stopping time T_b as given by (2.3). For a real number b, we now let T_b denote the time of the first upcrossing of the surplus process through the level b. For $b > u$, the definition is same as in (2.3). For $b < u$, the surplus has to drop below the level b before it can ever upcross through b. The stopping time T_0 is called the time of recovery; it is the first time the surplus reaches zero after ruin. The definition of the time of recovery for the classical risk model can be found in Gerber (1990), Egídio dos Reis (1993), Gerber and Shiu (1998) and the references therein. As presented in Gerber (1990), sometimes, the event ruin has a very small probability and the surplus process we are investigating is from one out of many existing lines of insurance business in the company. The company has enough funds available to support some negative surplus for some time, in the hope that the surplus will recovery in the future, allowing the company to keep this line of business alive. The purpose of this section is to study how long it takes for the surplus process to return to zero. See Picard (1994) for more interpretation of the time of recovery.

Now define for $\delta \geq 0$

$$\psi(u) = \mathbb{E}\left[e^{-\delta T_0}I(T<\infty)\middle|U(0)=u\right], \qquad u \geq 0,$$

to be the Laplace transform of T_0 with argument δ. Further define

$$\psi_{i,j}(u) = \mathbb{E}_i\left[e^{-\delta T_0}I(J(T_0)=j, T<\infty)|U(0)=u\right], \quad i,j\in E, u\geq 0,$$

to be the Laplace transform of the time of recovery if the process upcrosses 0 at state j after ruin given that the surplus starts from an initial state i and initial surplus u. Then

$$\psi(u) = \vec{\boldsymbol{\alpha}}\boldsymbol{\psi}(u)\vec{\mathbf{e}}^\top, \tag{3.1}$$

where $\boldsymbol{\psi}(u) = (\psi_{i,j}(u))_{i,j=1}^n$.

In particular, when $\delta = 0$, $\psi(u)$ simplifies to ruin probability $\Psi(u)$ and $\psi_{i,j}(u)$ simplifies to the probability of ruin with the state of recovery being j given that the surplus starts from initial state i and initial surplus u. We use $\Psi_{i,j}(u)$ to denote $\psi_{i,j}(u)$ when $\delta = 0$, i.e.,

$$\Psi_{i,j}(u) = \mathbb{P}_i \left(J(T_0) = j, T < \infty | U(0) = u \right), \quad i, j \in E, u \geq 0.$$

Then $\Psi_i(u)$ can be decomposed as

$$\Psi_i(u) = \sum_{j=1}^{n} \Psi_{i,j}(u), \quad i = 1, 2, \ldots, n.$$

In matrix notation,

$$\vec{\boldsymbol{\Psi}}(u) = \boldsymbol{\Psi}(u)\vec{\mathbf{e}}^\top, \tag{3.2}$$

$$\Psi(u) = \vec{\boldsymbol{\alpha}}\boldsymbol{\Psi}(u)\vec{\mathbf{e}}^\top, \tag{3.3}$$

where $\boldsymbol{\Psi}(u) = (\Psi_{i,j}(u))_{i,j=1}^n$.

Theorem 1

$$\boldsymbol{\psi}(u) = \mathbb{E}\left[e^{-\delta T + \mathbf{K}\, U(T)} I(T < \infty) | U(0) = u \right]. \tag{3.4}$$

Proof: Define

$$\psi_{\cdot,j} = \mathbb{E}\left[e^{-\delta T_0} I(T < \infty, J(T_0) = j) \big| U(0) = u \right], \qquad u \geq 0, j \in E.$$

Then $\psi_{\cdot,j} = \vec{\boldsymbol{\alpha}}\boldsymbol{\psi}(u)\vec{\mathbf{e}}_j^\top$. It follows from (2.5) that

$$\mathbb{E}\left[e^{-\delta(T_0 - T)} I(J(T_0) = j) | T < \infty, U(T) \right] = \vec{\boldsymbol{\alpha}} e^{\mathbf{K}\, U(T)} \vec{\mathbf{e}}_j^\top. \tag{3.5}$$

Then by using the law of iterated expectation and (3.5), we have

$$\begin{aligned} \psi_{\cdot,j}(u) &= \mathbb{E}\left[e^{-\delta T_0} I(T < \infty, J(T_0) = j) | U(0) = u \right] \\ &= \mathbb{E}\left[e^{-\delta T} e^{-\delta(T_0 - T)} I(T < \infty, J(T_0) = j) | U(0) = u \right] \\ &= \mathbb{E}\left[e^{-\delta T} \vec{\boldsymbol{\alpha}} e^{\mathbf{K}\, U(T)} \vec{\mathbf{e}}_j^\top I(T < \infty) | U(0) = u \right] \\ &= \vec{\boldsymbol{\alpha}} \mathbb{E}\left[e^{-\delta T + \mathbf{K}\, U(T)} I(T < \infty) | U(0) = u \right] \vec{\mathbf{e}}_j^\top. \end{aligned} \tag{3.6}$$

Since $\psi_{\cdot,j} = \vec{\boldsymbol{\alpha}}\boldsymbol{\psi}(u)\vec{\mathbf{e}}_j^\top$ holds true for any $\vec{\boldsymbol{\alpha}}$ and for any $j \in E$, then we conclude that $\boldsymbol{\psi}(u)$ has an expression in (3.4). $\square$

Remarks:

1. For the case $n = 1$, the risk model reduces to the classical compound Poisson model, and accordingly, formula (3.6) becomes

$$\psi(u) = \mathbb{E}[e^{-\delta T+\rho U(T)} I(T < \infty)|U(0) = u], \tag{3.7}$$

where ρ satisfies the following equation

$$c\,\rho - (\lambda + \delta) + \lambda \int_0^\infty p(x)e^{-\rho x}dx = 0.$$

Eq. (3.7) is in agreement with Eq. (6.4) in Gerber and Shiu (1998).

2. When $\delta = 0$, $\boldsymbol{\psi}(u)$ simplifies to $\boldsymbol{\Psi}(u)$ and Eq. (3.4) simplifies to

$$\boldsymbol{\Psi}(u) = \mathbb{E}\left[e^{\mathbf{K}_0 U(T)} I(T < \infty)|U(0) = u\right]. \tag{3.8}$$

Further, if $n = 1$, $\mathbf{K}_0$ simplifies to 0, then $\boldsymbol{\Psi}(u)$ simplifies to ruin probability $\Psi(u)$.

Now we can rewrite

$$\begin{aligned} \boldsymbol{\psi}(u) &= \mathbb{E}\left[e^{-\delta T+\boldsymbol{\Delta}\, U(T)} I(T < \infty)|U(0) = u\right] \\ &= \mathbf{H}\,\mathrm{diag}(\theta_1(u), \theta_2(u), \ldots, \theta_n(u))\mathbf{H}^{-1}, \end{aligned} \tag{3.9}$$

where

$$\theta_i(u) = \mathbb{E}[e^{-\delta T}\omega_i\left(U(T-), |U(T)|\right) I(T < \infty)|U(0) = u]$$

is a Gerber-Shiu function with penalty function $\omega_i(x, y) = e^{-\rho_i y}$, for $i = 1, 2, \ldots, n$.

The evaluation of the Gerber-Shiu function, first introduced in Gerber and Shiu (1998), is now one of the main research problems in ruin theory. See Ren (2007) for a detailed literature review.

Example 1 continued

We now revisit Example 1 in Section 2. It follows from (2.14) and (3.9) that

$$\psi(u) = \sum_{i=1}^n \left[\prod_{j=1, j\neq i}^n \frac{\rho_j - \delta/c}{\rho_j - \rho_i}\right] \theta_i(u), \qquad u \geq 0. \tag{3.10}$$

Further if claim amounts are exponentially distributed, i.e., $p(x) = \beta e^{-\beta x}$, then

$$\theta_i(u) = \frac{\beta - \gamma}{\beta + \rho_i} e^{-\gamma\, u},$$

where the adjustment coefficient $-\gamma < 0$ is the solution of $\hat{a}(\delta - cs)\hat{p}(s) = 1$. See Gerber and Shiu (2005) or Willmot (2007) for a derivation of this result.

4 The Probability That the Surplus Attains a Certain Level Before Ruin

For $b > u \geq 0$, define

$$\xi(u;b) = \mathbb{P}\left(\sup_{0\leq t\leq T} U(t) < b, T < \infty \middle| U(0) = u\right)$$

to be the probability that ruin occurs without the surplus reaching b prior to ruin in the ordinary renewal risk model. Alternatively, $\xi(u;b)$ can be viewed as the distribution of the maximum surplus before ruin. Further, define

$$\xi_{i,j}(u;b) = \mathbb{P}_i\left(\sup_{0\leq t\leq T} U(t) < b, T < \infty, J(T_0) = j \middle| U(0) = u\right), \qquad i \in E,$$

to be the probability that ruin occurs from initial surplus u and the state at the time of recovery is j without the surplus process reaching level b prior to ruin given that the process starts from initial state i. Then

$$\xi(u;b) = \vec{\boldsymbol{\alpha}}\boldsymbol{\xi}(u;b)\vec{\mathbf{e}}^{\top}, \qquad 0 \leq u \leq b, \tag{4.11}$$

where $\boldsymbol{\xi}(u;b) = \big(\xi_{i,j}(u;b)\big)_{i,j=1}^{n}$.

For $0 \leq u \leq b$ and $i, j \in E$, define $\chi_{i,j}(u;b)$ to be the probability that the surplus process attains level b at state j from initial state i and initial surplus u without first falling below zero. Clearly, $\chi_{i,j}(b;b) = I(i=j)$ for $i,j \in E$. Then

$$\chi(u;b) = \vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}^{\top}, \quad 0 \leq u \leq b, \tag{4.12}$$

is the probability that the surplus process attains level b without first falling below zero in the corresponding ordinary renewal risk model, where $\boldsymbol{\chi}(u,b) = \big(\chi_{i,j}(u;b)\big)_{i,j=1}^{n}$ with $\boldsymbol{\chi}(b;b) = \mathbf{I}$.

By considering whether or not the surplus reaches $b\,(> u)$ before ruin, we have

$$\Psi_{i,j}(u) = \xi_{i,j}(u;b) + \sum_{k=1}^{n} \chi_{i,k}(u;b)\Psi_{k,j}(b)\,, \qquad i,j \in E\,,$$

or in matrix notation,

$$\boldsymbol{\Psi}(u) = \boldsymbol{\xi}(u;b) + \boldsymbol{\chi}(u;b)\boldsymbol{\Psi}(b). \tag{4.13}$$

We will show in what follows that $\boldsymbol{\chi}(u;b)$ can be expressed in terms of the ruin probability matrix $\boldsymbol{\Psi}(u)$ and therefore the distribution of the maximum surplus before ruin given in (4.13) can also be expressed in terms of the ruin probability matrix.

For $a < u < b$, let T_a and T_b be two stopping times as defined in Section 3. For $i, j = 1, 2, \ldots, n$, we define the following two functions,

$$A_{i,j}(a, b|u) = \mathbb{E}_i[e^{-\delta T_a} I(T_a < T_b, J(T_a) = j)|U(0) = u]$$

and

$$B_{i,j}(a, b|u) = \mathbb{E}_i[e^{-\delta T_b} I(T_a > T_b, J(T_b) = j)|U(0) = u].$$

$A_{i,j}(a, b|u)$ is the Laplace transform of the time for the surplus process to upcross level a at state j before the process reaches level b from initial surplus u and initial state i, while $B_{i,j}(a, b|u)$ is the Laplace transform of the time to reach level b at state j for the first time provided that the process has not dropped below a from initial surplus u and initial state i. In particular, when $a = 0$, $A_{i,j}(0, b|u)$ is the Laplace transform of the time of recovery at state j before the process ever reaches level b from initial surplus u and initial state i, and $B_{i,j}(0, b|u)$ is the Laplace transform of the time for the surplus process to reach level b at state j before ruin has occured from initial surplus u and initial state i. Let $\mathbf{A}(a, b|u) = (A_{i,j}(a, b|u))_{i,j=1}^n$ and $\mathbf{B}(a, b|u) = (B_{i,j}(a, b|u))_{i,j=1}^n$.

Use the same methodology in Section 6 of Gerber and Shiu (1998), we have the following results:

$$\begin{aligned}
\mathbf{A}(a, b|u) &= \mathbf{A}(a + c, b + c|u + c), \\
\mathbf{B}(a, b|u) &= \mathbf{B}(a + c, b + c|u + c), \qquad \forall c \in \mathbb{R}, \\
\mathbf{A}(a, \infty|u) &= \lim_{b\to\infty} \mathbf{A}(a, b|u) = \lim_{b\to\infty} \mathbf{A}(0, b - a|u - a) = \boldsymbol{\psi}(u - a), \quad (4.14) \\
\mathbf{B}(-\infty, b|u) &= \mathbf{R}(u; b). \quad (4.15)
\end{aligned}$$

Furthermore, for $a \le u < b < b'$, we have

$$\begin{aligned}
A_{i,j}(a, b'|u) &= A_{i,j}(a, b|u) + \sum_{k=1}^{n} B_{i,k}(a, b|u) A_{k,j}(a, b'|b), \\
B_{i,j}(-\infty, b|u) &= B_{i,j}(a, b|u) + \sum_{k=1}^{n} A_{i,k}(a, b|u) B_{k,j}(-\infty, b|a),
\end{aligned}$$

which can be rewritten in the matrix form

$$\begin{aligned}
\mathbf{A}(a, b'|u) &= \mathbf{A}(a, b|u) + \mathbf{B}(a, b|u)\mathbf{A}(a, b'|b), \quad (4.16) \\
\mathbf{B}(-\infty, b|u) &= \mathbf{B}(a, b|u) + \mathbf{A}(a, b|u)\mathbf{B}(-\infty, b|a). \quad (4.17)
\end{aligned}$$

Substituting (4.14) and (4.15) in (4.16) and (4.17) and setting $a = 0$ and $b' = \infty$, we have

$$\begin{aligned} \boldsymbol{\psi}(u) &= \mathbf{A}(0, b|u) + \mathbf{B}(0, b|u)\boldsymbol{\psi}(b), & (4.18)\\ \mathbf{R}(u; b) &= \mathbf{A}(0, b|u)\mathbf{R}(0; b) + \mathbf{B}(0, b|u). & (4.19) \end{aligned}$$

By combining (4.18) and (4.19), one obtains

$$\begin{aligned} \mathbf{A}(0, b|u) &= [\boldsymbol{\psi}(u) - \mathbf{R}(u; b)\boldsymbol{\psi}(b)]\,[\mathbf{I} - \mathbf{R}(0; b)\boldsymbol{\psi}(b)]^{-1} \\ &= \left[\boldsymbol{\psi}(u) - e^{-\mathbf{K}(b-u)}\boldsymbol{\psi}(b)\right]\left[\mathbf{I} - e^{-\mathbf{K}b}\boldsymbol{\psi}(b)\right]^{-1}, & (4.20)\\ \mathbf{B}(0, b|u) &= [\mathbf{R}(u; b) - \boldsymbol{\psi}(u)\mathbf{R}(0; b)]\,[\mathbf{I} - \boldsymbol{\psi}(b)\mathbf{R}(0; b)]^{-1} \\ &= [e^{\mathbf{K}u} - \boldsymbol{\psi}(u)][e^{\mathbf{K}b} - \boldsymbol{\psi}(b)]^{-1}, \quad 0 \le u \le b. & (4.21) \end{aligned}$$

It follows from (3.9) that $\mathbf{B}(0, b|u)$ can be rewritten as

$$\mathbf{B}(0, b|u) = \mathbf{H}\,\mathrm{diag}\left(\frac{e^{\rho_1 u} - \theta_1(u)}{e^{\rho_1 b} - \theta_1(b)}, \ldots, \frac{e^{\rho_n u} - \theta_n(u)}{e^{\rho_n b} - \theta_n(b)}\right)\mathbf{H}^{-1}.$$

Remarks:

1. When $n = 1$, matrix $\mathbf{K}$ simplifies to ρ, $\boldsymbol{\psi}(u)$ simplifies to the Laplace transform of the time of recovery in the classical risk model, and Eqs. (4.20) and (4.21) simplify to

   $$A(0, b|u) = \frac{e^{\rho b}\psi(u) - e^{\rho u}\psi(b)}{e^{\rho b} - \psi(b)}, \quad B(0, b|u) = \frac{e^{\rho u} - \psi(u)}{e^{\rho b} - \psi(b)}.$$

   These two formulas are (6.24) and (6.25) in Gerber and Shiu (1998).

2. When $\delta = 0$ we can see $\chi_{i,j}(u; b) = B_{i,j}(0, b|u)$ for $i, j = 1, 2, \ldots, n$ and we have from (4.21) that

   $$\boldsymbol{\chi}(u; b) = [e^{\mathbf{K}_0 u} - \boldsymbol{\Psi}(u)][e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}(b)]^{-1}, \quad 0 \le u \le b. \tag{4.22}$$

It follows from (4.13) that

$$\begin{aligned} \boldsymbol{\xi}(u; b) &= \boldsymbol{\Psi}(u) - \boldsymbol{\chi}(u; b)\boldsymbol{\Psi}(b) \\ &= \boldsymbol{\Psi}(u) - [e^{\mathbf{K}_0 u} - \boldsymbol{\Psi}(u)][e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}(b)]^{-1}\boldsymbol{\Psi}(b). & (4.23) \end{aligned}$$

Finally, Eq. (4.11) gives

$$\xi(u; b) = \vec{\boldsymbol{\alpha}}\big\{\boldsymbol{\Psi}(u) - [e^{\mathbf{K}_0 u} - \boldsymbol{\Psi}(u)][e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}(b)]^{-1}\boldsymbol{\Psi}(b)\big\}\vec{\mathbf{e}}^{\top}. \tag{4.24}$$

In particular, when $n = 1$, the model simplifies to the classical risk model, $\boldsymbol{\Psi}(u)$ simplifies to $\Psi(u)$ and $\mathbf{K}_0$ simplifies to 0, then Eq. (4.24) gives

$$\xi(u; b) = \frac{\Psi(u) - \Psi(b)}{1 - \Psi(b)}, \qquad 0 \le u \le b.$$

This formula can be found in Dickson and Gray (1984).

5 The Distribution of the Maximum Severity of Ruin

In this section, we allow the surplus process to continue if ruin occurs, and consider the insurer's maximum severity of ruin which describes the worst situation the company would experience before reaching the time of recovery. Since we assume that the positive loading condition holds, it is certain that the surplus process will attain this level after ruin. For the classical risk model, Picard (1994) gives an explicit expression in terms of the ruin probability for the distribution of the maximum severity of ruin. Li and Dickson (2006) study the distribution of the maximum severity of ruin for the Sparre Andersen risk model with Erlang claim inter-arrival times. Li and Lu (2008) study the distribution of the maximum severity of ruin in a Markov-modulated risk model.

In Section 3, T_0 is defined to be the time of recovery after ruin, i.e.,

$$T_0 = \inf\{t : t > T, U(t) \geq 0\},$$

and define

$$M_u = \sup\{|U(t)|, T \leq t \leq T_0\}$$

to be the maximum severity of ruin. Let

$$F(z;u) = \mathbb{P}(M_u \leq z \big| \tau = 0, T < \infty), \quad z \geq 0,$$

denote the distribution function of the maximum severity of ruin given that ruin occurs.

If follows from the reasoning in Dickson (2005, p. 164) that

$$F(z;u) = \int_0^z \frac{g(y|u)}{\Psi(u)} \chi(z-y;z) dy = \frac{\vec{\boldsymbol{\alpha}} \int_0^z g(y|u) \boldsymbol{\chi}(z-y;z) dy \, \vec{\mathbf{e}}^\top}{\Psi(u)},$$

where $\Psi(u) = \vec{\boldsymbol{\alpha}} \boldsymbol{\Psi}(u) \vec{\mathbf{e}}^\top$ and $g(y|u) = \partial G(y|u)/\partial y$ with

$$G(y|u) = \mathbb{P}\big(T < \infty, |U(T)| \leq y \big| U(0) = u\big), \qquad u, y \geq 0,$$

being the probability that ruin occurs and the deficit at ruin is at most y given that the initial surplus is u.

It follows from (4.22) that

$$F(z;u) = \frac{\vec{\boldsymbol{\alpha}} \int_0^z g(y|u) \left[e^{\mathbf{K}_0 (z-y)} - \boldsymbol{\Psi}(z-y)\right] dy \left[e^{\mathbf{K}_0 z} - \boldsymbol{\Psi}(z)\right]^{-1} \vec{\mathbf{e}}^\top}{\Psi(u)}. \tag{5.25}$$

To evaluate the integral above, we consider a surplus process starting from initial surplus $u+z$. Conditioning on the amount by which the surplus drops below level z for the first time, we can show that

$$\mathbf{\Psi}(u+z) = \mathbb{E}\left[e^{\mathbf{K}_0 U(T)} I(T<\infty) | U(0) = u+z\right]$$

has the following expression:

$$\mathbf{\Psi}(u+z) = \int_0^z g(y|u)\mathbf{\Psi}(z-y)dy + \int_z^\infty g(y|u)e^{\mathbf{K}_0(z-y)}dy\,. \tag{5.26}$$

Then

$$\begin{aligned}\int_0^z g(y|u)\left[e^{\mathbf{K}_0\,(z-y)} - \mathbf{\Psi}(z-y)\right]dy &= \int_0^\infty g(y|u)e^{\mathbf{K}_0(z-y)}dy - \mathbf{\Psi}(u+z) \\ &= e^{\mathbf{K}_0\,z}\int_0^\infty g(y|u)e^{-\mathbf{K}_0\,y}dy - \mathbf{\Psi}(u+z).\end{aligned}$$

Setting $z=0$ in (5.26) gives

$$\mathbf{\Psi}(u) = \int_0^\infty g(y|u)e^{-\mathbf{K}_0\,y}dy.$$

Finally, we have

$$F(z;u) = \frac{\vec{\boldsymbol{\alpha}}\left[e^{\mathbf{K}_0\,z}\mathbf{\Psi}(u) - \mathbf{\Psi}(u+z)\right]\left[e^{\mathbf{K}_0\,z} - \mathbf{\Psi}(z)\right]^{-1}\vec{\mathbf{e}}^\top}{\vec{\boldsymbol{\alpha}}\mathbf{\Psi}(u)\vec{\mathbf{e}}^\top}\,. \tag{5.27}$$

Remarks:

1. When $n=1$, $\mathbf{K}_0$ simplifies to zero and $\mathbf{\Psi}(u)$ simplifies to the ruin probability $\Psi(u)$ for the classical risk model. Then (5.27) simplifies to

$$F(z;u) = \frac{\Psi(u) - \Psi(u+z)}{[1-\Psi(z)]\Psi(u)},$$

which was first given in Picard (1990).

2. When the distribution of inter-claim times is Erlang(n) with parameter λ,

$$F(z;u) = \frac{1}{\Psi(u)}\sum_{i=1}^n \left[\prod_{j=1,j\neq i}^n \frac{\rho_j}{\rho_j - \rho_i}\right]\frac{e^{\rho_i z}\theta_i(u) - \theta_i(u+z)}{e^{\rho_i z} - \theta_i(z)}\,,$$

where $\rho_1 = 0, \rho_2, \ldots, \rho_n$ are solutions of the Lundberg's equation (2.9) but with $\delta = 0$.

6 The Expected Discounted Dividend Payments

Cheung (2007) studies the moments of the discounted dividend payments prior to ruin in a Sparre Andersen model with phase-type claim inter-arrival times in the presence of a constant dividend barrier. The purpose of this section is to obtain an alternative expression for the expectation of the discounted dividend payments prior to ruin using the two functions obtained in Section 4.

Now we consider the surplus process (1.1) modified by the payment of dividends. When the surplus exceeds a constant barrier $b\,(\geq u)$, dividends are paid continuously so the surplus stays at the level b until a new claim occurs. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy and define $\bar{T} = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin. Let $\delta > 0$ be the force of interest for valuation and define

$$D_{u,b} = \int_0^{\bar{T}} e^{-\delta\, t} dD(t), \qquad 0 \leq u \leq b,$$

to be the present value of all dividends until the time of ruin $\bar{T}$ given that the initial surplus is u, where $D(t)$ is the aggregate dividends paid by time t. Define

$$V(u; b) = \mathbb{E}\big[D_{u,\,b}\big|U_b(0) = u\big], \qquad 0 \leq u \leq b,$$

to be the expected present value of the dividend payments before ruin and define

$$V_i(u; b) = \mathbb{E}_i\big[D_{u,\,b}\big|U_b(0) = u\big], \qquad 0 \leq u \leq b,\, i \in E,$$

to be the expected present value of the dividend payment before ruin given the initial state is i and the initial surplus is u. Then

$$V(u; b) = \vec{\boldsymbol{\alpha}}\vec{\mathbf{V}}(u; b), \quad 0 \leq u \leq b,$$

where $\vec{\mathbf{V}}(u; b) = (V_1(u; b), V_2(u; b), \ldots, V_n(u; b))^\top$.

Since no dividend are payable unless the surplus reaches the level b before ruin occurs, we have, for $0 \leq u \leq b$,

$$V_i(u; b) = \sum_{j=1}^{n} B_{i,j}(0, b|u) V_j(b; b), \quad i \in E.$$

In matrix form,

$$\vec{\mathbf{V}}(u; b) = \mathbf{B}(0, b|u)\vec{\mathbf{V}}(b; b), \qquad 0 \leq u \leq b.$$

It follows from Cheung (2007) that $\frac{\partial \vec{\mathbf{V}}(u;b)}{\partial u}\Big|_{u=b} = \vec{\mathbf{e}}^\top$, then

$$\begin{aligned}
\vec{\mathbf{V}}(b;b) &= \left[\frac{\partial \mathbf{B}(0,b|u)}{\partial u}\right]^{-1}\Big|_{u=b} \vec{\mathbf{e}}^\top = \left[e^{\mathbf{K}b} - \boldsymbol{\psi}(b)\right]\left[\mathbf{K}e^{\mathbf{K}b} - \boldsymbol{\psi}'(b)\right]^{-1}\vec{\mathbf{e}}^\top, \\
\vec{\mathbf{V}}(u;b) &= \left[e^{\mathbf{K}u} - \boldsymbol{\psi}(u)\right]\left[\mathbf{K}e^{\mathbf{K}b} - \boldsymbol{\psi}'(b)\right]^{-1}\vec{\mathbf{e}}^\top, \qquad 0 \le u \le b,
\end{aligned}$$

and

$$V(u;b) = \vec{\boldsymbol{\alpha}}\left[e^{\mathbf{K}u} - \boldsymbol{\psi}(u)\right]\left[\mathbf{K}e^{\mathbf{K}b} - \boldsymbol{\psi}'(b)\right]^{-1}\vec{\mathbf{e}}^\top. \tag{6.28}$$

In particular, when $n = 1$, (6.28) simplifies to

$$V(u;b) = \frac{e^{\rho u} - \psi(u)}{\rho e^{\rho b} - \psi'(b)}, \qquad 0 \le u \le b,$$

which is formula (7.5) in Gerber and Shiu (1998).

Acknowledgment

The author would like to thank Prof. David Dickson and two anonymous referees for their valuable comments and suggestions to improve the paper.

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