Ruin probabilities for a risk model with two classes of risk processes

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Abstract

In this paper a risk model with two classes of business is considered, in which claim number processes are modeled by two independent Erlang(2) processes, aiming to calculate probabilities of ruin caused by a claim from a certain class. To do so, integro-differential equations for the ruin probabilities are derived and their Laplace transforms are then obtained. At the end of this paper, numerical results for the ruin probabilities are calculated for individual claim sizes with exponential and Gamma distributions.

Keywords: Erlang risk process; Integro-differential equations; Laplace transforms

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1 Introduction

During the last two decades, Erlang processes have been frequently employed by authors to model the number of claims received by an insurance company. Works in respect of Erlang(2) processes can be found in Dickson (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Sun and Yang (2004) and Tsai and Sun (2004). Meanwhile, Li and Garrido (2004a) and Li and Dickson (2006) considered risk models with Erlang(n) processes. Moreover, the generalized Erlang(n) processes are explored by Albrecher et al.(2005), Dickson and Drekic (2004), Gerber and Shiu (2003, 2005), and Li and Garrido (2004b).

Erlang processes also participated in modeling insurance business with multiple classes of claims. Yuen et al. (2002) derived a system of integro-differential equations for the survival probabilities for a risk model with two classes of business in which $\{N_1(t); t \geq 0\}$ is a Poisson process and $\{N_2(t); t \geq 0\}$ is an Erlang(2) process. Li and Garrido (2005) further explored the survival probabilities for a more general model where $N_1(t)$ is a Poisson process and $N_2(t)$ is a generalized Erlang(2) process. For the same model, Li and Lu (2005) considered the expected discounted penalty functions at ruin, given that ruin is caused by a claim of a certain class j.

This paper is devoted to studying a risk model which is constructed from two independent Erlang(2) risk processes. The generalized Erlang(2) processes are avoided in this paper because of the tediousness of mathematical derivations, but not theoretical difficulties. Definitions and notation related to the model are introduced in the next section. Ultimate probabilities of ruin due to a claim from a certain class are defined and remain the centre of interest thereafter. Similar ruin probabilities are defined in Li and Lu (2005). In Section 3, a system of integro-differential equations for the ruin probabilities is developed, and their Laplace transforms are derived in Section 4. Considering the large number of equations and expressions, key results are presented in a matrix form. At the end of this paper, numerical examples of the ruin probabilities are provided with claim sizes following exponential and Gamma distributions.

2 The Model

We consider a continuous time risk model with two classes of business in an insurance company that has two streams of individual claims, denoted by $X_i, i = 1, 2, \ldots$ and $Y_j, j = 1, 2, \ldots$, respectively. We assume that individual claim amounts X_i's are independent and identically distributed (i.i.d.) and have a common distribution function (d.f.) F_X, a probability density function (p.d.f.) f_X and a survival function (s.f.) $\bar{F}_X := 1 - F_X$. Similarly, claim amounts Y_j's are i.i.d. following the common d.f. F_Y, p.d.f. f_Y and s.f. $\bar{F}_Y := 1 - F_Y$. Let $\mathbb{E}[X_1] = \mu_X$, and $\mathbb{E}[Y_1] = \mu_Y$. In addition, let $\{X_i\}_{i \geq 1}$ and $\{Y_j\}_{j \geq 1}$ be mutually independent from each other.

Renewal processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are employed, respectively, to denote the number of claims that occur up to time t in the first and second class. Both $N_1(t)$ and $N_2(t)$ have i.i.d. inter-arrival times. Let $V_{1i} = L_{i1}^{(1)} + L_{i2}^{(1)}$ be the ith inter-arrival time of $N_1(t)$. Assume that $\{L_{i1}^{(1)}\}_{i\geq 1}$ and $\{L_{i2}^{(1)}\}_{i\geq 1}$ are independent of each other and are both i.i.d. exponentially distributed with parameter $\lambda_1 > 0$. Then $\{V_{1i}\}_{i\geq 1}$ follows an Erlang(2) distribution. Similarly, for $N_2(t)$, the inter-arrival times are sums of two independent r.v.'s, i.e., $V_{2i} = L_{i1}^{(2)} + L_{i2}^{(2)}$, where $\{L_{i1}^{(2)}\}_{i\geq 1}, \{L_{i2}^{(2)}\}_{i\geq 1}$ are i.i.d. exponential r.v.'s with parameter $\lambda_2 > 0$. These two claim number processes are presumed to be independent of each other and are independent from all the claim size random variables as well. Then we are able to define the following surplus process consisting of these two classes of business,

$$S(t) = u + ct - \sum_{i=1}^{N_1(t)} X_i - \sum_{j=1}^{N_2(t)} Y_j, \qquad t > 0, \tag{2.1}$$

where $S(t)$ is the amount of surplus of the company at time t and $S(0) = u$. As usual, $u \geq 0$ is the initial surplus, and $c > 0$ is the rate of premium received by the company. The positive safety loading condition for (2.1) is

$$c > \frac{\lambda_1}{2}\mu_X + \frac{\lambda_2}{2}\mu_Y,$$

and the safety loading factor θ satisfies

$$\frac{1}{1+\theta} = \frac{1}{2c}(\lambda_1\mu_X + \lambda_2\mu_Y).$$

Notice that due to the independence between the two classes of business, θ is the overall safety loading factor for the combined business, which is not necessarily the safety loading factor for each class.

For the risk model (2.1), we define $T := \inf\{t > 0 : S(t) < 0\}(\infty \text{ otherwise})$ to be the time of ruin, and $\psi(u) := \mathbb{P}\{T < \infty | S(0) = u\}$ to be the ultimate ruin probability. Then $\phi(u) := 1 - \psi(u)$ is the ultimate survival probability. Further, we let J be the cause-of-ruin random variable. If the ruin is caused by a claim from class j, $j = 1, 2$, then $J = j$. Thus the ruin probability $\psi(u)$ can be decomposed as $\psi(u) = \psi_1(u) + \psi_2(u)$, where

$$\psi_j(u) := \mathbb{P}\{T < \infty, J = j | S(0) = u\}, u \geq 0, j = 1, 2,$$

is the ruin probability due to a claim from class j.

3 Integro-differential equations for ruin probabilities

In this section we will derive a system of integro-differential equations for the ultimate ruin probabilities for the surplus process (2.1) defined in section 2, where both

$\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are Erlang(2) processes. Note that as remarked by Zhu and Yang (2009), the differentiability of the ultimate ruin probabilities is not guaranteed. A counter example can be constructed by just letting the claim size distributions to be discontinuous (see page 168 of Rolski et al. (1998)). Therefore, to make sure that the ruin probabilities are differentiable, we need to assume that all claim size distributions are absolutely continuous within the rest of this paper.

Li and Garrido (2005) commented that because of the Erlang(2) distributed claim inter-arrival times from the second class in their model, the ruin probability is no longer time-homogeneous. So for the probability of ultimate ruin, they assumed that a claim from the second class occurs exactly at time 0. It results in the consideration of a type of bivariate ruin probabilities. For the same reason, the ultimate ruin probabilities, $\psi(u)$ and $\psi_j(u), j = 1, 2$, defined above are not time-homogeneous either. We assume that $\psi(u)$ is the ultimate ruin probability for two new lines of business that both commence exactly at time 0. Then we define a ruin probability, denoted by $\psi(u, \tau_1, \tau_2)$, to be a multivariate function of the current surplus u, the length of time τ_1, elapsed since the time of a claim from the first class of business, and the length of time τ_2, elapsed since the time of a claim from the second class. Not surprisingly, we say $\psi(u, \tau_1, \tau_2) = \psi_1(u, \tau_1, \tau_2) + \psi_2(u, \tau_1, \tau_2)$, where ψ_j is the probability of ruin if the claim that causes ruin is from class j, $j = 1, 2$. Naturally we are interested in the ruin probabilities at the time of the realization of $L_{11}^{(1)}$ and $L_{11}^{(2)}$, which are distinguished by the following four situations due to the lack of memory of $L_{11}^{(1)}$ and $L_{11}^{(2)}$. For $j = 1, 2$:

- when $L_{11}^{(1)} > \tau_1$ and $L_{11}^{(2)} > \tau_2$, $\gamma_{j0}(u) := \psi_j(u, \tau_1, \tau_2) = \psi_j(u, 0, 0) = \psi_j(u)$;
- when $L_{11}^{(1)} < \tau_1$ and $L_{11}^{(2)} > \tau_2$, $\gamma_{j1}(u) := \psi_j(u, \tau_1, \tau_2) = \psi_j(u, L_{11}^{(1)}, 0)$;
- when $L_{11}^{(1)} > \tau_1$ and $L_{11}^{(2)} < \tau_2$, $\gamma_{j2}(u) := \psi_j(u, \tau_1, \tau_2) = \psi_j(u, 0, L_{11}^{(2)})$;
- when $L_{11}^{(1)} < \tau_1$ and $L_{11}^{(2)} < \tau_2$, $\gamma_{j3}(u) := \psi_j(u, \tau_1, \tau_2) = \psi_j(u, L_{11}^{(1)}, L_{11}^{(2)})$.

Then $\psi(u) = \gamma_{10}(u) + \gamma_{20}(u)$. Using the total probability formula we have

$$\begin{aligned}
\psi_j(u, \tau_1, \tau_2) &= \gamma_{j0}(u)\mathbb{P}\{L_{11}^{(1)} > \tau_1, L_{11}^{(2)} > \tau_2\} + \gamma_{j1}(u)\mathbb{P}\{L_{11}^{(1)} < \tau_1, L_{11}^{(2)} > \tau_2\} \\
&\quad + \gamma_{j2}(u)\mathbb{P}\{L_{11}^{(1)} > \tau_1, L_{11}^{(2)} < \tau_2\} + \gamma_{j3}(u)\mathbb{P}\{L_{11}^{(1)} < \tau_1, L_{11}^{(2)} < \tau_2\} \\
&= e^{-\lambda_1 \tau_1} e^{-\lambda_2 \tau_2} \gamma_{j0}(u) + \left(1 - e^{-\lambda_1 \tau_1}\right) e^{-\lambda_2 \tau_2} \gamma_{j1}(u) \\
&\quad + e^{-\lambda_1 \tau_1} \left(1 - e^{-\lambda_2 \tau_2}\right) \gamma_{j2}(u) + \left(1 - e^{-\lambda_1 \tau_1}\right)\left(1 - e^{-\lambda_2 \tau_2}\right) \gamma_{j3}(u), \quad j = 1, 2.
\end{aligned}$$

In the following we will derive integro-differential equations for the ultimate ruin probabilities $\gamma_{ji}(u), j = 1, 2, i = 0, 1, 2, 3$. We let $\mathbf{\Gamma}_j(u) = (\gamma_{j0}(u), \gamma_{j1}(u), \gamma_{j2}(u), \gamma_{j3}(u))^T$ to be a 4×1 vector, and $\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{\Gamma}_j(u) = \left(\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{j0}(u), \frac{\mathrm{d}}{\mathrm{d}u}\gamma_{j1}(u), \frac{\mathrm{d}}{\mathrm{d}u}\gamma_{j2}(u), \frac{\mathrm{d}}{\mathrm{d}u}\gamma_{j3}(u)\right)^T$, $j = 1, 2$. Assuming an integral of a matrix consists of integrals of elements in the integrand matrix, then we have the following result.

Theorem 1 *The ruin probability vector* $\mathbf{\Gamma}_j(u), j = 1, 2$ *for risk model (2.1) satisfies the following integro-differential equation:*

$$c\frac{d}{du}\mathbf{\Gamma}_j(u) = \mathbf{A}\mathbf{\Gamma}_j(u) - \int_0^u [\mathbf{g}_1(x) + \mathbf{g}_2(x)]\mathbf{\Gamma}_j(u-x)dx - \bar{\mathbf{G}}_j(u)\mathbf{1}, \tag{3.1}$$

where

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} \lambda & -\lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda & 0 & -\lambda_2 \\ 0 & 0 & \lambda & -\lambda_1 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$\mathbf{g}_1(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 f_X(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 f_X(x) & 0 \end{pmatrix},$$

$$\mathbf{g}_2(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_2 f_Y(x) & 0 & 0 & 0 \\ 0 & \lambda_2 f_Y(x) & 0 & 0 \end{pmatrix},$$

$\lambda = \lambda_1 + \lambda_2$, $\mathbf{G}_j(u) = \int_0^u \mathbf{g}_j(x)dx$, $\bar{\mathbf{G}}_j(u) = \int_u^\infty \mathbf{g}_j(x)dx$, *and* $\mathbf{1} = (1, 1, 1, 1)^T$.

Proof. Let W be the minimum of $L_{11}^{(1)}$ and $L_{11}^{(2)}$. Note that similar random variables were defined by Yuen et al. (2002) and Li and Garrido (2005) to study the ultimate ruin probabilities for their models. We can work out the following probabilities

$$\begin{aligned} &\mathbb{P}\{W = L_{11}^{(1)}\} = \mathbb{P}\{L_{11}^{(1)} < L_{11}^{(2)}\} = \frac{\lambda_1}{\lambda}, \\ &\mathbb{P}\{W = L_{11}^{(2)}\} = \mathbb{P}\{L_{11}^{(1)} > L_{11}^{(2)}\} = \frac{\lambda_2}{\lambda}, \\ &\mathbb{P}\{W > t | W = L_{11}^{(1)}\} = \mathbb{P}\{W > t | W = L_{11}^{(2)}\} = e^{-\lambda t}. \end{aligned}$$

Obviously, the two conditional distributions are exponential with parameter λ. We first consider $J = 1$. Using these probabilities and conditioning on the values of W in the surplus process $S(t)$, we can write the following equation:

$$\begin{aligned} \gamma_{10}(u) &= \int_0^\infty \mathbb{P}\{W = t, W = L_{11}^{(1)}\}\gamma_{11}(u+ct)dt \\ &\quad + \int_0^\infty \mathbb{P}\{W = t, W = L_{11}^{(2)}\}\gamma_{12}(u+ct)dt \\ &= \lambda_1 \int_0^\infty e^{-\lambda t}\gamma_{11}(u+ct)dt + \lambda_2 \int_0^\infty e^{-\lambda t}\gamma_{12}(u+ct)dt. \end{aligned} \tag{3.2}$$

Let $W_1 = \min(L_{12}^{(1)}, L_{11}^{(2)})$. By similar arguments, we have that

$$\begin{aligned}\gamma_{11}(u) &= \int_0^\infty \mathbb{P}\{W_1 = t, W_1 = L_{12}^{(1)}\}\Big[\int_0^{u+ct} \gamma_{10}(u+ct-x) f_X(x)dx \\ &\quad + \bar{F}_X(u+ct)\Big]dt + \int_0^\infty \mathbb{P}\{W_1 = t, W_1 = L_{11}^{(2)}\}\gamma_{13}(u+ct)dt \\ &= \lambda_1 \int_0^\infty e^{-\lambda t}\Big[\int_0^{u+ct} \gamma_{10}(u+ct-x) f_X(x)dx + \bar{F}_X(u+ct)\Big]dt \\ &\quad +\lambda_2 \int_0^\infty e^{-\lambda t}\gamma_{13}(u+ct)dt. \qquad (3.3)\end{aligned}$$

Parallel to (3.2) and (3.3), one can write the following equations for $\gamma_{12}(u)$ and $\gamma_{13}(u)$:

$$\begin{aligned}\gamma_{12}(u) &= \lambda_2 \int_0^\infty e^{-\lambda t} \int_0^{u+ct} \gamma_{10}(u+ct-y) f_Y(y)dydt \\ &\quad +\lambda_1 \int_0^\infty e^{-\lambda t}\gamma_{13}(u+ct)dt, \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\gamma_{13}(u) &= \lambda_1 \int_0^\infty e^{-\lambda t}\Big[\int_0^{u+ct} \gamma_{12}(u+ct-x) f_X(x)dx + \bar{F}_X(u+ct)\Big]dt \\ &\quad +\lambda_2 \int_0^\infty e^{-\lambda t} \int_0^{u+ct} \gamma_{11}(u+ct-y) f_Y(y)dydt. \qquad (3.5)\end{aligned}$$

Notice that in (3.4) and (3.5), a claim from the second class, say Y_1, can't cause ruin when we evaluate the probabilities for $J = 1$. Letting $s = u + ct$, equations (3.2) - (3.5) can be rewritten as

$$\begin{aligned}c\gamma_{10}(u) &= \lambda_1 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\gamma_{11}(s)ds + \lambda_2 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\gamma_{12}(s)ds, \\ c\gamma_{11}(u) &= \lambda_1 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\Big[\int_0^s \gamma_{10}(s-x) f_X(x)dx + \bar{F}_X(s)\Big]ds \\ &\quad +\lambda_2 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\gamma_{13}(s)ds, \\ c\gamma_{12}(u) &= \lambda_2 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\int_0^s \gamma_{10}(s-y) f_Y(y)dyds \\ &\quad +\lambda_1 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\gamma_{13}(s)ds,\end{aligned}$$

and

$$\begin{aligned}c\gamma_{13}(u) &= \lambda_1 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\Big[\int_0^s \gamma_{12}(s-x) f_X(x)dx + \bar{F}_X(s)\Big]ds \\ &\quad +\lambda_2 \int_0^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\}\int_0^s \gamma_{11}(s-y) f_Y(y)dyds.\end{aligned}$$

Differentiating the above equations with respect to u yields the following system of integro-differential equations:

$$\begin{aligned}
c\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{10}(u) &= \lambda\gamma_{10}(u)-\lambda_1\gamma_{11}(u)-\lambda_2\gamma_{12}(u),\\
c\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{11}(u) &= -\lambda_1\int_0^u\gamma_{10}(u-x)f_X(x)dx-\lambda_1\bar{F}_X(u)+\lambda\gamma_{11}(u)-\lambda_2\gamma_{13}(u),\\
c\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{12}(u) &= -\lambda_2\int_0^u\gamma_{10}(u-y)f_Y(y)dy+\lambda\gamma_{12}(u)-\lambda_1\gamma_{13}(u),\\
c\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{13}(u) &= -\lambda_2\int_0^u\gamma_{11}(u-y)f_Y(y)dy-\lambda_1\int_0^u\gamma_{12}(u-x)f_X(x)dx\\
&\quad -\lambda_1\bar{F}_X(u)+\lambda\gamma_{13}(u),
\end{aligned}$$

or in a matrix form,

$$c\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{\Gamma}_1(u)=\mathbf{A}\mathbf{\Gamma}_1(u)-\int_0^u[\mathbf{g}_1(x)+\mathbf{g}_2(x)]\mathbf{\Gamma}_1(u-x)dx-\bar{\mathbf{G}}_1(u)\mathbf{1}. \qquad (3.6)$$

The integro-differential equation for $\mathbf{\Gamma}_2(u)$ can be derived similarly. □

To end this section, we derive a relation between the initial values $\gamma_{ji}(0), i=0,1,2,3$, which has a similar form to equation (11) in Li and Garrido (2005). After integrating both sides of equation (3.1) from 0 to u, we obtain

$$\begin{aligned}
c\Big[\mathbf{\Gamma}_j(u)-\mathbf{\Gamma}_j(0)\Big] &= \int_0^u\mathbf{A}\mathbf{\Gamma}_j(s)ds-\int_0^u\int_0^s[\mathbf{g}_1(x)+\mathbf{g}_2(x)]\mathbf{\Gamma}_j(s-x)dxds\\
&\quad -\int_0^u\int_s^\infty\mathbf{g}_j(x)\mathbf{1}dxds\\
&= \int_0^u\mathbf{A}\mathbf{\Gamma}_j(s)ds-\int_0^u\left[\mathbf{G}_1(s)+\mathbf{G}_2(s)\right]\mathbf{\Gamma}_j(u-s)ds-\int_0^\infty x\mathbf{g}_j(x)\mathbf{1}dx\\
&= \int_0^u\mathbf{C}\mathbf{\Gamma}_j(s)ds+\int_0^u[\bar{\mathbf{G}}_1(s)+\bar{\mathbf{G}}_2(s)]\mathbf{\Gamma}_j(u-s)ds-\int_0^\infty x\mathbf{g}_j(x)\mathbf{1}dx,
\end{aligned}$$

where

$$\mathbf{C}=\begin{pmatrix}\lambda & -\lambda_1 & -\lambda_2 & 0\\ -\lambda_1 & \lambda & 0 & -\lambda_2\\ -\lambda_2 & 0 & \lambda & -\lambda_1\\ 0 & -\lambda_2 & -\lambda_1 & \lambda\end{pmatrix}.$$

It is straightforward to show

$$\int_0^\infty x\mathbf{g}_1(x)\mathbf{1}dx=\begin{pmatrix}0\\ \lambda_1\mu_X\\ 0\\ \lambda_1\mu_X\end{pmatrix}, \qquad \int_0^\infty x\mathbf{g}_2(x)\mathbf{1}dx=\begin{pmatrix}0\\ 0\\ \lambda_2\mu_Y\\ \lambda_2\mu_Y\end{pmatrix}.$$

By the Monotone Convergence Theorem and the fact that $\mathbf{\Gamma}_j(\infty) = \mathbf{0} = (0,0,0,0)^T, j = 1,2$, from the above equation, as $u \to \infty$, we have

$$\mathbf{\Gamma}_j(0) = -\frac{1}{c}\int_0^\infty \mathbf{C}\mathbf{\Gamma}_j(s)ds + \frac{1}{c}\int_0^\infty x\mathbf{g}_j(x)\mathbf{1}dx, \qquad j = 1,2.$$

Since one can easily verify that $\frac{1}{4}\mathbf{1}^T\mathbf{C} = \mathbf{0}^T$, we obtain the following results for $\mathbf{\Gamma}_1(0)$ and $\mathbf{\Gamma}_2(0)$:

$$\begin{aligned}
\frac{1}{4}\mathbf{1}^T\mathbf{\Gamma}_1(0) &= \frac{1}{4c}\mathbf{1}^T\int_0^\infty x\mathbf{g}_1(x)\mathbf{1}dx = \frac{1}{2c}\lambda_1\mu_X, \\
\frac{1}{4}\mathbf{1}^T\mathbf{\Gamma}_2(0) &= \frac{1}{4c}\mathbf{1}^T\int_0^\infty x\mathbf{g}_2(x)\mathbf{1}dx = \frac{1}{2c}\lambda_2\mu_Y, \\
\frac{1}{4}\mathbf{1}^T[\mathbf{\Gamma}_1(0) + \mathbf{\Gamma}_2(0)] &= \frac{1}{1+\theta},
\end{aligned} \tag{3.7}$$

where θ is the safety loading factor defined in section 2. Results in (3.7) are used in the derivation of Laplace transforms of the ruin probabilities $\gamma_{ji}(u), i = 0,1,2,3, j = 1,2$, in section 4.

4 Laplace transforms of the ruin probabilities

Having obtained the integro-differential equations for the ultimate ruin probabilities $\gamma_{ji}(u)(i = 0,1,2,3, j = 1,2)$ of model (2.1), in the following we will derive the Laplace transforms for the ruin probabilities and consider the inversion of these Laplace transforms for some particular claim size distributions.

Firstly, we define the following Laplace transforms:

$$\begin{aligned}
\hat{\gamma}_{ji}(s) &= \int_0^\infty e^{-su}\gamma_{ji}(u)du, \qquad j = 1,2, \quad i = 0,1,2,3, \\
\hat{f}_X(s) &= \int_0^\infty e^{-sx}f_X(x)dx, \qquad \hat{f}_Y(s) = \int_0^\infty e^{-sy}f_Y(y)dy,
\end{aligned}$$

and $\hat{\mathbf{\Gamma}}_j(s) = (\hat{\gamma}_{j0}(s), \hat{\gamma}_{j1}(s), \hat{\gamma}_{j2}(s), \hat{\gamma}_{j3}(s))^T, j = 1,2$.

Using standard properties of Laplace transforms, we obtain from the integro-differential equation (3.1) that

$$\begin{aligned}
c\left[s\hat{\mathbf{\Gamma}}_j(s) - \mathbf{\Gamma}_j(0)\right] &= \mathbf{A}\hat{\mathbf{\Gamma}}_j(s) - [\hat{\mathbf{g}}_1(s) + \hat{\mathbf{g}}_2(s)]\,\hat{\mathbf{\Gamma}}_j(s) \\
&\quad -\hat{\hat{\mathbf{G}}}_j(s)\mathbf{1}, \qquad j = 1,2,
\end{aligned} \tag{4.1}$$

where

$$\hat{\mathbf{g}}_1(s) + \hat{\mathbf{g}}_2(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_1\hat{f}_X(s) & 0 & 0 & 0 \\ \lambda_2\hat{f}_Y(s) & 0 & 0 & 0 \\ 0 & \lambda_2\hat{f}_Y(s) & \lambda_1\hat{f}_X(s) & 0 \end{pmatrix},$$

$$\hat{\hat{\mathbf{G}}}_1(s)\mathbf{1} = \frac{\lambda_1}{s}\begin{pmatrix} 0 \\ 1-\hat{f}_X(s) \\ 0 \\ 1-\hat{f}_X(s) \end{pmatrix},$$

$$\hat{\hat{\mathbf{G}}}_2(s)\mathbf{1} = \frac{\lambda_2}{s}\begin{pmatrix} 0 \\ 0 \\ 1-\hat{f}_Y(s) \\ 1-\hat{f}_Y(s) \end{pmatrix}.$$

As a result, equation (4.1) is rewritten as,

$$\left[s\mathbf{I} - \frac{1}{c}\left(\mathbf{A} - \hat{\mathbf{g}}_1(s) - \hat{\mathbf{g}}_2(s)\right)\right]\hat{\mathbf{\Gamma}}_j(s) = \mathbf{\Gamma}_j(0) - \frac{1}{c}\hat{\hat{\mathbf{G}}}_j(s)\mathbf{1}, \quad j = 1, 2, \tag{4.2}$$

where $\mathbf{I} = \text{diag}(1,\, 1,\, 1,\, 1)$. Let $\mathbf{D}(s) = cs\mathbf{I} - \mathbf{A} + \hat{\mathbf{g}}_1(s) + \hat{\mathbf{g}}_2(s)$, which has the form

$$\begin{pmatrix} cs-\lambda & \lambda_1 & \lambda_2 & 0 \\ \lambda_1\hat{f}_X(s) & cs-\lambda & 0 & \lambda_2 \\ \lambda_2\hat{f}_Y(s) & 0 & cs-\lambda & \lambda_1 \\ 0 & \lambda_2\hat{f}_Y(s) & \lambda_1\hat{f}_X(s) & cs-\lambda \end{pmatrix}.$$

If the inverse of $\mathbf{D}(s)$ exists, i.e., $\det(\mathbf{D}(s)) \neq 0$, then equation (4.2) is solvable. So one will be interested to know the solutions of equation $\det(\mathbf{D}(s)) = 0$, where the determinant of $\mathbf{D}(s)$ is:

$$\det(\mathbf{D}(s)) = \left[(cs-\lambda)^2 - \lambda_1^2\hat{f}_X(s) - \lambda_2^2\hat{f}_Y(s)\right]^2 - 4\lambda_1^2\lambda_2^2\hat{f}_X(s)\hat{f}_Y(s).$$

It is shown in the following theorem that the equation $\det(\mathbf{D}(s)) = 0$ has only three positive real roots.

Theorem 2 *The equation*

$$\left[(cs-\lambda)^2 - \lambda_1^2\hat{f}_X(s) - \lambda_2^2\hat{f}_Y(s)\right]^2 - 4\lambda_1^2\lambda_2^2\hat{f}_X(s)\hat{f}_Y(s) = 0 \tag{4.3}$$

has exactly three positive real solutions, say, ρ_1, ρ_2 and ρ_3.

Proof. For our convenience, we rewrite equation (4.3) as $\alpha^2(s) = \beta(s)$, where $\alpha(s) = (cs-\lambda)^2 - \lambda_1^2\hat{f}_X(s) - \lambda_2^2\hat{f}_Y(s)$, $\beta(s) = 4\lambda_1^2\lambda_2^2\hat{f}_X(s)\hat{f}_Y(s)$.

The first step is to show equation $\alpha(s) = 0$ has only two positive real roots. We need to prove that the equation

$$\alpha'(s) = 2c^2s - 2c\lambda - \lambda_1^2\hat{f}'_X(s) - \lambda_2^2\hat{f}'_Y(s) = 0 \tag{4.4}$$

has only one positive real root, say, s_0. This is because the function $2c^2s$ is a strictly increasing function taking values between 0 and ∞. The function $2c\lambda + \lambda_1^2\hat{f}'_X(s) +$

$\lambda_2^2 \hat{f}'_Y(s)$ is also strictly increasing from $2c\lambda - \lambda_1^2\mu_X - \lambda_2^2\mu_Y (> 0)$ to $2c\lambda$. Therefore equation (4.4) has only one positive real root. In addition, $\alpha'(0) = \lambda_1^2\mu_X + \lambda_2^2\mu_Y - 2c\lambda < 0$ and $\alpha'(\infty) = \infty$, so one can see that $\alpha'(s) < 0$ for $s \in [0, s_0)$ and $\alpha'(s) > 0$ for $s \in (s_0, \infty)$. It means that $\alpha(s)$ decreases for $s \in [0, s_0)$ and is increasing for $s \in (s_0, \infty)$. Moreover, the facts that $\alpha(0) = \lambda^2 - \lambda_1^2 - \lambda_2^2 = 2\lambda_1\lambda_2 > 0$ and $\alpha(\frac{\lambda}{c}) = -\lambda_1^2 \hat{f}_X(\frac{\lambda}{c}) - \lambda_2^2 \hat{f}_Y(\frac{\lambda}{c}) < 0$ show that equation $\alpha(s) = 0$ has only two positive real roots, say, s_1 and s_2, satisfying $s_1 < s_0 \leq \frac{\lambda}{c} < s_2$, as $\alpha'(\frac{\lambda}{c}) \geq 0$.

Next we will examine the equation $\alpha^2(s) = \beta(s)$. Clearly, the function $\beta(s)$ is a non-negative decreasing function for all $s \in (0, \infty)$, $\alpha^2(s)$ is non-negative for all s, and the two positive real numbers s_1 and s_2 satisfy $\alpha^2(s) = 0$. It is not hard to find that the function $\alpha^2(s)$ is decreasing over intervals $[0, s_1)$ and $[s_0, s_2)$, and is increasing over $[s_1, s_0)$ and $[s_2, \infty)$. We then show the following facts which will lead to three positive roots for the equation $\alpha^2(s) = \beta(s)$. The first fact is $\alpha^2(0) = 4\lambda_1^2\lambda_2^2 = \beta(0)$. The second one is $\frac{d}{ds}[\alpha^2(s)]|_{s=0} < \beta'(0)$, which is because

$$
\begin{aligned}
& \frac{d}{ds}[\alpha^2(s)]|_{s=0} - \beta'(0) \\
= \; & 4\lambda_1\lambda_2(\lambda_1^2\mu_X + \lambda_2^2\mu_Y - 2c\lambda) + 4\lambda_1^2\lambda_2^2(\mu_X + \mu_Y) \\
= \; & 4\lambda_1\lambda_2\Big[(\lambda_1 + \lambda_2)(\lambda_1\mu_X + \lambda_2\mu_Y) - 2c(\lambda_1 + \lambda_2)\Big] \\
< \; & 4\lambda_1\lambda_2\Big[(\lambda_1 + \lambda_2)(\lambda_1\mu_X + \lambda_2\mu_Y) - (\lambda_1\mu_X + \lambda_2\mu_Y)(\lambda_1 + \lambda_2)\Big] = 0.
\end{aligned}
$$

Thirdly, $\alpha^2(\frac{\lambda}{c}) > \beta(\frac{\lambda}{c})$. The proof is straightforward. These three facts, as a whole, show that the equation $\alpha^2(s) = \beta(s)$ has only three positive real roots, say, ρ_1, ρ_2 and ρ_3, satisfying $0 < s_1 < \rho_1 < \frac{\lambda}{c} < \rho_2 < s_2 < \rho_3 < \infty$ (see Figure 1 below). □

[Figure 1: An example figure for $\alpha^2(s)$ and $\beta(s)$]

From Theorem 2 we know when $s \neq \rho_i, i = 1, 2, 3$, the inverse of $\mathbf{D}(s)$ exists satisfying $\mathbf{D}^{-1}(s) = [\det(\mathbf{D}(s))]^{-1}\mathbf{D}^*(s)$, where $\mathbf{D}^*(s)$ has the form:

$$
\begin{pmatrix}
(cs-\lambda)\alpha(s) & -\lambda_1\xi_1(s) & -\lambda_2\xi_2(s) & 2\lambda_1\lambda_2(cs-\lambda) \\
-\lambda_1\hat{f}_X(s)\xi_1(s) & (cs-\lambda)\alpha(s) & 2\lambda_1\lambda_2(cs-\lambda)\hat{f}_X(s) & -\lambda_2\xi_2(s) \\
-\lambda_2\hat{f}_Y(s)\xi_2(s) & 2\lambda_1\lambda_2(cs-\lambda)\hat{f}_Y(s) & (cs-\lambda)\alpha(s) & -\lambda_1\xi_1(s) \\
2\lambda_1\lambda_2(cs-\lambda)\hat{f}_X(s)\hat{f}_Y(s) & -\lambda_2\hat{f}_Y(s)\xi_2(s) & -\lambda_1\hat{f}_X(s)\xi_1(s) & (cs-\lambda)\alpha(s)
\end{pmatrix}
$$

in which $\xi_1(s) = \alpha(s) + 2\lambda_2^2\hat{f}_Y(s)$, and $\xi_2(s) = \alpha(s) + 2\lambda_1^2\hat{f}_X(s)$. Therefore, solving equation (4.2) yields

$$\hat{\mathbf{\Gamma}}_j(s) = c\mathbf{D}^{-1}(s)\mathbf{\Gamma}_j(0) - \mathbf{D}^{-1}(s)\hat{\hat{\mathbf{G}}}_j(s)\mathbf{1}, \qquad j = 1, 2. \tag{4.5}$$

For $j = 1, 2$, $\mathbf{\Gamma}_j(0)$ is determined as follows. Since $\hat{\gamma}_{ji}(s), i = 0, 1, 2, 3$ are all finite for all $s > 0$, Theorem 2 implies that at $s = \rho_k$ $(k = 1, 2, 3)$, $c\mathbf{D}^*(s)\mathbf{\Gamma}_j(0) - \mathbf{D}^*(s)\hat{\hat{\mathbf{G}}}_j(s)\mathbf{1} =$

$\mathbf{0}$. Further, since the rank of matrix $\mathbf{D}^*(\rho_k)(k = 1, 2, 3)$ equals 1, then using the second row vector of $\mathbf{D}^*(\rho_k)$, denoted by $\mathbf{d}_2^*(\rho_k)$, we can write the following equations

$$c\mathbf{d}_2^*(\rho_k)\mathbf{\Gamma}_j(0) - \mathbf{d}_2^*(\rho_k)\hat{\hat{\mathbf{G}}}_j(\rho_k)\mathbf{1} = 0, \qquad k = 1, 2, 3. \tag{4.6}$$

These three equations together with (3.7) could be written in a matrix form as follows:

$$\mathbf{E}\mathbf{\Gamma}_j(0) = \mathbf{e}_j, \tag{4.7}$$

where

$$\begin{aligned}
\mathbf{E} &= \left(\frac{1}{4}\mathbf{1}, \mathbf{d}_2^*(\rho_1)^T, \mathbf{d}_2^*(\rho_2)^T, \mathbf{d}_2^*(\rho_3)^T\right)^T, \\
\mathbf{e}_1 &= \left(\frac{1}{2c}\lambda_1\mu_X, \mathbf{d}_2^*(\rho_1)\hat{\hat{\mathbf{G}}}_1(\rho_1)\mathbf{1}, \mathbf{d}_2^*(\rho_2)\hat{\hat{\mathbf{G}}}_1(\rho_2)\mathbf{1}, \mathbf{d}_2^*(\rho_3)\hat{\hat{\mathbf{G}}}_1(\rho_3)\mathbf{1}\right)^T, \\
\mathbf{e}_2 &= \left(\frac{1}{2c}\lambda_2\mu_Y, \mathbf{d}_2^*(\rho_1)\hat{\hat{\mathbf{G}}}_2(\rho_1)\mathbf{1}, \mathbf{d}_2^*(\rho_2)\hat{\hat{\mathbf{G}}}_2(\rho_2)\mathbf{1}, \mathbf{d}_2^*(\rho_3)\hat{\hat{\mathbf{G}}}_2(\rho_3)\mathbf{1}\right)^T.
\end{aligned}$$

Solving (4.7) gives

$$\mathbf{\Gamma}_j(0) = \mathbf{E}^{-1}\mathbf{e}_j, \qquad j = 1, 2, \tag{4.8}$$

where $\mathbf{E}^{-1}$ is the inverse of $\mathbf{E}$. Using results (4.5) and (4.8), the ultimate ruin probabilities can be obtained by inverting their Laplace transforms. In particular, $\psi_1(u) = \gamma_{10}(u)$, $\psi_2(u) = \gamma_{20}(u)$, and $\psi(u) = \psi_1(u) + \psi_2(u)$.

5 Numerical Examples

In this section, we examine an insurance business having two classes of individual claims, which follow exponential and Gamma distributions, respectively. Given the exact distribution information, using the derived formulae in Section 4 we are able to obtain numerical expressions for the ruin probabilities of interest. The computation involved is conducted by the software *Mathematica*.

Example 1: In this example, we assume both X_1 and Y_1 follow different exponential distributions, i.e., $f_X(x) = \exp\{-x\}, f_Y(y) = 0.5\exp\{-0.5y\}$. So $\hat{f}_X(s) = (1 + s)^{-1}$, $\hat{f}_Y(s) = (1 + 2s)^{-1}$, $\mu_X = 1$, and $\mu_Y = 2$. Let $\lambda_1 = 1, \lambda_2 = 0.5, c = 1.1$, then we have $\lambda = 1.5$ and $\theta = 0.1$. From the definition of $\alpha(s)$, we obtain

$$\alpha(s) = (1.1s - 1.5)^2 - \frac{1}{1+s} - \frac{1}{4(1+2s)},$$

and equation (4.3) is

$$\left[(1.1s - 1.5)^2 - \frac{1}{1+s} - \frac{1}{4(1+2s)}\right]^2 - \frac{1}{(1+s)(1+2s)} = 0.$$

It has three positive real roots $\rho_1 = 0.98191, \rho_2 = 1.70019$, and $\rho_3 = 2.08155$, giving

$$\alpha(\rho_1) = -0.41260, \quad \alpha(\rho_2) = -0.29011, \quad \alpha(\rho_3) = 0.25070.$$

Substituting the numbers into (4.7) and solving the equations yields:

$$\begin{aligned}
\mathbf{\Gamma}_1(0) &= (0.40390, 0.61302, 0.29788, 0.50339)^T, \\
\mathbf{\Gamma}_2(0) &= (0.45373, 0.29946, 0.61872, 0.44627)^T.
\end{aligned}$$

Substituting them into (4.5) gives

$$\hat{\mathbf{\Gamma}}_1(s) = \eta^{-1}(s) \begin{pmatrix} 0.40388(0.5+s)(0.57974+s)(0.878+s) \\ 0.61302(0.41155+s)(0.5+s)(0.90495+s) \\ 0.29788(0.51426+s)(0.69374+1.62218s+s^2) \\ 0.50339(0.44702+s)(0.52292+1.4198s+s^2) \end{pmatrix},$$
$$\hat{\mathbf{\Gamma}}_2(s) = \eta^{-1}(s) \begin{pmatrix} 0.45373(0.48798+s)(0.96494+s)(1+s) \\ 0.29946(0.49096+s)(0.93994+s)(1.56446+s) \\ 0.61872(0.47841+s)(0.80396+s)(1+s) \\ 0.44627(0.50264+s)(0.78571+s)(1.36585+s) \end{pmatrix},$$

where $\eta(s) = (0.07898+s)(0.48942+s)(0.76272+1.7407s+s^2)$. Inverting these Laplace transforms yields $\mathbf{\Gamma}_1(u)$ and $\mathbf{\Gamma}_2(u)$ which give

$$\begin{aligned}
\psi_1(u) &= [0.14382\cos(0.07218u) + 0.03877\sin(0.07218u)]e^{-0.87035u} \\
&\quad -0.00243e^{-0.48942u} + 0.26251e^{-0.07898u}, \\
\psi_2(u) &= [-0.13309\cos(0.07218u) - 0.044\sin(0.07218u)]e^{-0.87035u} \\
&\quad +0.00258e^{-0.48942u} + 0.58424e^{-0.07898u}, \\
\psi(u) &= \psi_1(u) + \psi_2(u) \\
&= \Big[0.01073\cos(0.07218u) - 0.00523\sin(0.07218u)\Big]e^{-0.87035u} \\
&\quad +0.00015e^{-0.48942u} + 0.84675e^{-0.07898u},
\end{aligned}$$

with $\psi_1(0) = 0.40390, \psi_2(0) = 0.45373$, and $\psi(0) = 0.85763$.

Figure 2 shows the total probability of ruin for different values of u, as well as their decomposition into the ruin probabilities due to claims from class one and those from class two. One can see from the graph that $\psi_1(u)$ is a strictly decreasing function of u. It is reducing sharply when u is small (between 0 and 2) and turns to be flatter when u increases. On the contrary, $\psi_2(u)$ is a strictly increasing function when u is small (between 0 and approximately 2) and starts to decrease when u increases. Moreover, $\psi_2(u)$ is always greater than $\psi_1(u)$, which means the second class of business is riskier than the first one within the context of the combined business. Although individual claims received by the company are expected to be less frequent from class two, the higher expected individual claim amount indicates that claims from class two will cause ruin more likely than those from class one.

[Figure 2: Decomposition of the ruin probability in Example 1]

Example 2: We consider the following claim size distributions: $f_X(x) = 4x\exp\{-2x\}$ and $f_Y(y) = y\exp\{-y\}$, i.e., $X_1 \sim$ Gamma(2,2) and $Y_1 \sim$ Gamma(2,1). So $\hat{f}_X(s) = 4(2+s)^{-2}$, $\hat{f}_Y(s) = (1+s)^{-2}$, $\mu_X = 1$, and $\mu_Y = 2$. Using $\lambda_1 = 1, \lambda_2 = 0.5$, and $c = 1.1$, the function $\alpha(s)$ has the form

$$\alpha(s) = (1.1s - 1.5)^2 - \frac{4}{(2+s)^2} - \frac{1}{4(1+s)^2},$$

and equation (4.3) becomes

$$\left[(1.1s - 1.5)^2 - \frac{4}{(2+s)^2} - \frac{1}{4(1+s)^2}\right]^2 - \frac{4}{(1+s)^2(2+s)^2} = 0.$$

It has three positive real roots $\rho_1 = 0.983376$, $\rho_2 = 1.68756$, and $\rho_3 = 1.974$, which give

$$\alpha(\rho_1) = -0.338, \quad \alpha(\rho_2) = -0.201805, \quad \alpha(\rho_3) = 0.169224.$$

Substituting the numbers into (4.7) and solving the equations yield:

$$\begin{aligned}
\mathbf{\Gamma}_1(0) &= (0.389174, 0.633569, 0.279393, 0.516046)^T, \\
\mathbf{\Gamma}_2(0) &= (0.455215, 0.280531, 0.641064, 0.441372)^T.
\end{aligned}$$

Substituting them into (4.5) gives

$$\hat{\mathbf{\Gamma}}_1(s) = \eta^{-1}(s)\begin{pmatrix} 0.39(1+s)^2(1.48+s)(2.43+s)(2.74+s)(1.19+2.11s+s^2) \\ 0.63(0.62+s)(1+s)^2(2.42+s)(2.98+s)(1.87+2.73s+s^2) \\ 0.28(1.56+s)(2.45+s)(2.79+s)(1.05+2.04s+s^2)(1.54+2.04s+s^2) \\ 0.52(0.73+s)(1.17+s)(1.60+s)(2.46+s)(3.08+s)(1.04+1.76s+s^2) \end{pmatrix},$$

$$\hat{\mathbf{\Gamma}}_2(s) = \eta^{-1}(s)\begin{pmatrix} 0.46(0.87+s)(1.11+s)(2+s)^2(2.40+s)(2.25+2.97s+s^2) \\ 0.28(0.90+s)(1.08+s)(2.41+s)(2.19+2.94s+s^2)(6.77+4.51s+s^2) \\ 0.64(0.78+s)(1.66+s)(2+s)^2(2.46+s)(1.36+2.31s+s^2) \\ 0.44(1.23+s)(1.63+s)(2.46+s)(0.87+1.83s+s^2)(6.02+4.37s+s^2) \end{pmatrix},$$

where $\eta(s) = (0.12+s)(0.88+s)(1.10+s)(1.52+s)(2.44+s)(2.51+s)(1.85+2.62s+s^2)$. Inverting these Laplace transforms gives:

$$\begin{aligned}
\psi_1(u) &= -0.06603e^{-2.50908u} + 0.01657e^{-2.44128u} + 0.02344e^{-1.51647u} \\
&\quad + \left[0.15593\cos(0.36491u) - 0.07338\sin(0.36491u)\right]e^{-1.30987u} \\
&\quad + 0.00906e^{-1.10128u} - 0.01187e^{-0.88265u} + 0.26207e^{-0.1199u},
\end{aligned}$$

$$
\begin{aligned}
\psi_2(u) &= 0.05648e^{-2.50908u} - 0.01772e^{-2.44128u} - 0.01961e^{-1.51647u} \\
&\quad -\Big[0.14889\cos(0.36491u) - 0.05041\sin(0.36491u)\Big]e^{-1.30987u} \\
&\quad -0.00971e^{-1.10128u} + 0.01281e^{-0.88265u} + 0.58185e^{-0.1199u}, \\
\psi(u) &= -0.00955e^{-2.50908u} - 0.00115e^{-2.44128u} + 0.00383e^{-1.51647u} \\
&\quad +\Big[0.00704\cos(0.36491u) - 0.02297\sin(0.36491u)\Big]e^{-1.30987u} \\
&\quad -0.00065e^{-1.10128u} + 0.00093e^{-0.88265u} + 0.84393e^{-0.1199u},
\end{aligned}
$$

with $\psi_1(0) = 0.38917, \psi_2(0) = 0.45522$, and $\psi(0) = 0.84439$.

Similarly, Figure 3 shows the total probability of ruin, $\psi(u)$, for different values of u, as well as $\psi_1(u)$ and $\psi_2(u)$. The ruin probability $\psi_1(u)$ is still a strictly decreasing function of u, and it decreases rapidly when u is small (between 0 and 2). It decreases slower when u increases. On the contrary, when u is small (between 0 and slightly < 2) $\psi_2(u)$ is a strictly increasing function and starts to decrease when u increases. Again, $\psi_2(u)$ is always greater than $\psi_1(u)$, which is expected by the higher expected individual claim amount for the second class of business.

[Figure 3: Decomposition of the ruin probability in Example 2]

Remark. The author has also worked on a more general risk model that is formed by two generalized Erlang(2) processes. Due to the similar derivations, more tedious forms of results, and the fact that no new techniques are required, the author decides not to include the model within this paper.

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