

The finite time ruin probability in a risk model with capital injections

Ciyu Nie, David C M Dickson and Shuanming Li

Abstract

We consider a risk model with capital injections as described in Nie et al. (2011). We show that in the Sparre Andersen framework the density of the time to ruin for the model with capital injections can be expressed in terms of the density of the time to ruin in an ordinary Sparre Andersen risk process. In the special case of Erlang inter-claim times and exponential claims we show that there exists a readily computable formula for the density of the time to ruin. When the inter-claim time distribution is exponential, we obtain an explicit solution for the density of the time to ruin when the individual claim amount distribution is Erlang(2), and we explain techniques to find the moments of the time to ruin. In the final section we consider the related problem of the distribution of the duration of negative surplus in the classical risk model, and we obtain explicit solutions for the (defective) density of the total duration of negative surplus for two individual claim amount distributions.

1 Introduction

In this paper we consider the finite time ruin probability in the risk model considered in Nie et al (2011). In that paper we considered a surplus process into which capital was injected each time the surplus process, starting from level $u \geq k$, fell between 0 and k . The capital injections were assumed to be secured by a reinsurance arrangement, and each capital injection restored the surplus level to k . Ruin occurred if the surplus fell below 0. The focus of that study was on the probability of ruin in infinite time, and this measure was used to determine quantities of interest, such as the value of k that minimised the ultimate ruin probability for a given level of initial capital.

The scope of this study is more limited as we assume u and k are given and we consider the calculation of the ruin probability in finite time. We

derive general results for the density of the time to ruin in the framework of a Sparre Andersen risk model, i.e. with a general claim inter-arrival time distribution. In principle, these results lead to explicit solutions for the density of the time to ruin in cases where explicit solutions exist for the density of the time to ruin in standard Sparre Andersen surplus processes, but these expressions can be complicated. However, in the special case when claims are exponentially distributed we can exploit the fact that there is a general expression for the density of the time to ruin for a standard Sparre Andersen model, and we illustrate the use of this expression when claim inter-arrival times are Erlang(n) distributed. We make use of generalised binomial functions to deal with convolutions of certain functions, and express solutions in terms of generalised hypergeometric functions for computational efficiency.

We also consider the situation when claims occur as a Poisson process. We show how the density of the time to ruin can be found when the individual claim amount distribution is Erlang(2), and we show that if the joint density of the time to ruin and the deficit at ruin (in the classical risk model) satisfies a particular factorisation, then we can calculate moments of the time to ruin. In the final section we discuss the related problem of the distribution of the duration of negative surplus in the classical risk model, where these techniques can also be applied.

2 Model and notation

In the standard Sparre Andersen surplus process, the surplus at time t is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where u is the initial surplus, c is the rate of premium income per unit time, $\{N(t)\}_{t \geq 0}$ is a counting process for the number of claims and $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables representing individual claim amounts. We assume that X_1 has distribution function F , with $F(0) = 0$, density function f , and mean m_1 . We also assume that $cE[V_1] > m_1$ where $\{V_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, independent of $\{X_i\}_{i=1}^{\infty}$, representing the claim inter-arrival times.

We define T_u to be the time to ruin from initial surplus u , and define

$$\psi(u, t) = \Pr(T_u \leq t)$$

to be the finite time ruin probability, with (defective) density $w_u(t) = \frac{\partial}{\partial t} \psi(u, t)$.

Define Y_u to be the deficit at ruin from initial surplus u , and define

$$W(u, y, t) = \Pr(T_u \leq t, Y_u \leq y)$$

to be the probability of ruin by time t with a deficit of at most y at ruin, and let

$$w_u(y, t) = \frac{\partial^2}{\partial t \partial y} W(u, y, t)$$

denote the (defective) joint density of T_u and Y_u .

Further, let

$$\psi(u) = 1 - \bar{\psi}(u) = \lim_{t \rightarrow \infty} \psi(u, t)$$

be the ultimate ruin probability, and let

$$G(u, y) = \lim_{t \rightarrow \infty} W(u, y, t)$$

be the probability of ultimate ruin with a deficit of at most y at ruin.

We now introduce notation for the Sparre Andersen process with capital injections as described in the previous section. Let $T_{u,k}$ denote the time to ruin and let

$$\psi_k(u) = \Pr(T_{u,k} < \infty)$$

be the ultimate ruin probability. Nie et al (2011) give a general expression for $\psi_k(u)$. Next, let

$$W_{u,k}(t) = \Pr(T_{u,k} \leq t)$$

be the finite time ruin probability, with (defective) density $w_{u,k}(t) = \frac{\partial}{\partial t} W_{u,k}(t)$.

We use the notation M_X to denote the moment generating function of a random variable X , and we assume throughout that the required moment generating functions exist. We denote the Laplace transform of a function ϕ by $\tilde{\phi}(s) = \int_0^\infty e^{-sx} \phi(x) dx$.

3 General results

In this section we derive a general expression for the density of $T_{u,k}$, and, as in problems considered in Nie et al (2011), we consider first the situation when $u = k$, as this leads to solutions in the case $u > k$.

First, we observe that if ruin occurs on the n th ($n = 1, 2, 3, \dots$) occasion that the surplus falls below k , then there must have been $n - 1$ falls below k to a level between 0 and k , resulting in a capital injection to restore the surplus process to level k , and the n th fall below k results in a surplus below 0. Thus, the probability that ruin is caused by n falls in the surplus level below k is $G(0, k)^{n-1}(\psi(0) - G(0, k))$.

Second, we note that if ruin occurs on the n th occasion that the surplus falls below k , the time to ruin is the sum of $n - 1$ variables with distribution function

$$D_{0,1}(t) = \Pr(T_0 \leq t | T_0 < \infty \text{ and } Y_0 \leq k) = \frac{W(0, k, t)}{G(0, k)}$$

and of a further random variable with distribution function

$$D_{0,2}(t) = \Pr(T_0 \leq t | T_0 < \infty \text{ and } Y_0 > k) = \frac{\psi(0, t) - W(0, k, t)}{\psi(0) - G(0, k)}.$$

These observations lead to

$$W_{k,k}(t) = \sum_{n=1}^{\infty} G(0, k)^{n-1} (\psi(0) - G(0, k)) D_{0,1}^{(n-1)*} * D_{0,2}(t), \quad (3.1)$$

with $D_{0,1}^{0*} * D_{0,2}(t)$ defined to be the same as $D_{0,2}(t)$.

Next, we consider the case $u > k$. The situation is similar to the case $u = k$, but now either

- the time to ruin is the time to the first drop of the surplus process below k if that drop results in ruin, or
- the time to ruin is the sum of the time to the first drop below k , if that drop does not result in ruin, and the time to ruin from surplus level k .

Define

$$\begin{aligned} D_{u-k,1}(t) &= \Pr(T_{u-k} \leq t | T_{u-k} < \infty \text{ and } Y_{u-k} \leq k) \\ &= \frac{W(u-k, k, t)}{G(u-k, k)} \end{aligned}$$

to be the distribution of the time to the first drop of the surplus process below k given that this drop does not result in ruin, and define

$$\begin{aligned} D_{u-k,2}(t) &= \Pr(T_{u-k} \leq t | T_{u-k} < \infty \text{ and } Y_{u-k} > k) \\ &= \frac{\psi(u-k, t) - W(u-k, k, t)}{\psi(u-k) - G(u-k, k)} \end{aligned}$$

to be the distribution of the time to ruin given that ruin occurs on the first drop of the surplus process below level k . Then, as the process re-starts from k at the time of the first capital injection, we have

$$W_{u,k}(t) = (\psi(u-k) - G(u-k, k)) D_{u-k,2}(t) + G(u-k, k) D_{u-k,1} * W_{k,k}(t). \quad (3.2)$$

This is a general formula for $W_{u,k}(t)$. Formulae for the distribution functions $D_{u,1}$ and $D_{u,2}$ can be obtained for some claim amount distributions and claim inter-arrival time distributions – see, for example, Dickson (2008) or Landriault and Willmot (2009) in the case of the classical risk model, or Dickson and Li (2010) for the Erlang(2) risk model. In the case of exponentially distributed individual claims, the distribution functions $D_{u,1}$ and $D_{u,2}$ are easily found, facilitating calculation of $w_{u,k}(t)$, as shown in Section 4.1.

We can also derive moments of the time to ruin given that ruin occurs. Let $\tau_{u,1}$ and $\tau_{u,2}$ be random variables with distribution functions $D_{u,1}$ and $D_{u,2}$ respectively. Define $T_{k,k}^c$ to be the time to ruin given that ruin occurs from initial surplus k . Then from (3.1) and $\psi_k(k) = (\psi(0) - G(0, k))/(1 - G(0, k))$ (see Nie et al (2011)), we have

$$\begin{aligned} M_{T_{k,k}^c}(r) &= \sum_{n=1}^{\infty} G(0, k)^{n-1} (1 - G(0, k)) M_{\tau_{0,1}}(r)^{n-1} M_{\tau_{0,2}}(r) \\ &= \frac{(1 - G(0, k)) M_{\tau_{0,2}}(r)}{1 - G(0, k) M_{\tau_{0,1}}(r)}. \end{aligned}$$

Similarly, we can write (3.2) as

$$\begin{aligned} \Pr(T_{u,k}^c \leq t) &= \frac{\psi(u - k) - G(u - k, k)}{\psi_k(u)} \Pr(\tau_{u-k,2} \leq t) \\ &\quad + \frac{G(u - k, k) \psi_k(k)}{\psi_k(u)} \Pr(\tau_{u-k,1} + T_{k,k}^c \leq t) \end{aligned}$$

which gives

$$M_{T_{u,k}^c}(r) = \frac{\psi(u - k) - G(u - k, k)}{\psi_k(u)} M_{\tau_{u-k,2}}(r) + \frac{G(u - k, k) \psi_k(k)}{\psi_k(u)} M_{\tau_{u-k,1}}(r) M_{T_{k,k}^c}(r).$$

From this moment generating function we find

$$E[T_{u,k}^c] = \frac{\psi(u - k) - G(u - k, k)}{\psi_k(u)} E[\tau_{u-k,2}] + \frac{G(u - k, k) \psi_k(k)}{\psi_k(u)} (E[\tau_{u-k,1}] + E[T_{k,k}^c]) \quad (3.3)$$

and

$$\begin{aligned} E[(T_{u,k}^c)^2] &= \frac{\psi(u - k) - G(u - k, k)}{\psi_k(u)} E[\tau_{u-k,2}^2] \\ &\quad + \frac{G(u - k, k) \psi_k(k)}{\psi_k(u)} \left(E[\tau_{u-k,1}^2] + 2E[\tau_{u-k,1}]E[T_{k,k}^c] + E[(T_{k,k}^c)^2] \right). \end{aligned}$$

These give

$$E[T_{k,k}^c] = E[\tau_{0,2}] + \frac{G(0, k) E[\tau_{0,1}]}{1 - G(0, k)} \quad (3.4)$$

and

$$E \left[(T_{k,k}^c)^2 \right] = E[\tau_{0,2}^2] + \frac{G(0, k)}{1 - G(0, k)} \left(2E[\tau_{0,2}]E[\tau_{0,1}] + \frac{2G(0, k) E[\tau_{0,1}]^2}{1 - G(0, k)} + E[\tau_{0,1}^2] \right).$$

The calculation of moments of $\tau_{u,1}$ and $\tau_{u,2}$ depends on the density $w_u(y, t)$ existing in a suitable form. We discuss this further in Section 4.3.

4 Some explicit results

4.1 Exponential claims

In this section we consider the situation when $f(x) = \alpha e^{-\alpha x}$, $x > 0$. Initially we assume a general claim inter-arrival time distribution, before moving to a specific assumption in Section 4.1.1. It is well known that by the memoryless property of the exponential distribution

$$W(u, y, t) = \psi(u, t) (1 - e^{-\alpha y}),$$

with $G(u, y) = \psi(u) (1 - e^{-\alpha y})$. This leads to

$$D_{u,1}(t) = D_{u,2}(t) = \frac{\psi(u, t)}{\psi(u)}$$

for all $u \geq 0$. The density of $T_{k,k}$ can then be found from (3.1) as

$$w_{k,k}(t) = \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} w_0^{n*}(t)$$

and using this and (3.2), the density of $T_{u,k}$ is

$$w_{u,k}(t) = e^{-\alpha k} w_{u-k}(t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} w_0^{n*} * w_{u-k}(t). \quad (4.1)$$

From Dickson and Li (2010) we know that

$$w_u(t) = \sum_{j=1}^{\infty} w_0^{j*}(t) \frac{(\alpha u)^{j-1} e^{-\alpha u}}{\Gamma(j)}$$

which means that we can write (4.1) as

$$w_{u,k}(t) = e^{-\alpha u} \sum_{n=0}^{\infty} (1 - e^{-\alpha k})^n \sum_{j=1}^{\infty} w_0^{(n+j)*}(t) \frac{(\alpha(u-k))^{j-1}}{\Gamma(j)}. \quad (4.2)$$

Thus, if we can evaluate convolutions $w_0^{n*}(t)$ then we can find $w_{u,k}(t)$. We now show how this can be done for Erlang(n) claim inter-arrival times.

4.1.1 Erlang(n) claim inter-arrival times

Still assuming $f(x) = \alpha e^{-\alpha x}$, we now consider the situation when the distribution of claim inter-arrival times is Erlang(n, β) with density function

$$e_{n,\beta}(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{\Gamma(n)}$$

for $t > 0$. From Borovkov and Dickson (2008) we know that

$$w_0(t) = n\beta^n \sum_{m=0}^{\infty} \frac{(\alpha c \beta^n)^m t^{(n+1)m+n-1} e^{-(\beta+\alpha c)t}}{m!(n(m+1))!}$$

and this function has Laplace transform

$$\begin{aligned} \tilde{w}_0(s) &= n\beta^n \sum_{m=0}^{\infty} \frac{(\alpha c \beta^n)^m ((n+1)m+n-1)!}{m!(n(m+1))! (\beta + \alpha c + s)^{(n+1)m+n}} \\ &= \left(\frac{\beta}{\beta + \alpha c + s} \right)^n \sum_{m=0}^{\infty} \binom{(n+1)m+n}{m} \frac{n Z^m}{(n+1)m+n} \end{aligned}$$

where $Z = \alpha c \beta^n / (\beta + \alpha c + s)^{n+1}$. We see that $\tilde{w}_0(s)$ can be written in terms of the generalised binomial function defined as

$$\mathcal{B}_t(z) = \sum_{k=0}^{\infty} \binom{tk+1}{k} \frac{1}{tk+1} z^k,$$

with the property that

$$\mathcal{B}_t(z)^r = \sum_{k=0}^{\infty} \binom{tk+r}{k} \frac{r}{tk+r} z^k.$$

See Graham et al (1994) for details. Thus

$$\tilde{w}_0(s) = \left(\frac{\beta}{\beta + \alpha c + s} \right)^n \mathcal{B}_{n+1}(Z)^n.$$

Using properties of Laplace transforms and generalised binomial functions, this gives

$$\begin{aligned}\tilde{w}_0^{r*}(s) &= \tilde{w}_0(s)^r = \left(\frac{\beta}{\beta + \alpha c + s}\right)^{nr} \mathcal{B}_{n+1}(Z)^{nr} \\ &= \beta^{nr} \sum_{m=0}^{\infty} \binom{(n+1)m + nr}{m} \frac{nr}{(n+1)m + nr} \frac{(\alpha c \beta^n)^m}{(\beta + \alpha c + s)^{m(n+1)+nr}},\end{aligned}$$

and inversion yields

$$w_0^{r*}(t) = \beta^{nr} t^{nr-1} n r e^{-(\beta+\alpha c)t} \sum_{m=0}^{\infty} \frac{(\alpha c \beta^n t^{n+1})^m}{m!(n(r+m))!}.$$

Finally, for ease of computation with software, we can apply techniques described in Graham et al (1994) to write $w_0^{r*}(t)$ in terms of generalised hypergeometric functions as

$$w_0^{r*}(t) = \frac{\beta^{nr} t^{nr-1} e^{-(\beta+\alpha c)t}}{\Gamma(nr)} {}_0F_n \left(r + \frac{1}{n}, r + \frac{2}{n}, \dots, r + \frac{n}{n}; \frac{\alpha c \beta^n t^{n+1}}{n^n} \right) \quad (4.3)$$

where

$${}_pF_q(B_1, B_2, \dots, B_p, C_1, C_2, \dots, C_q; Z) = \sum_{m=0}^{\infty} \frac{(B_1)_m (B_2)_m \dots (B_p)_m}{(C_1)_m (C_2)_m \dots (C_q)_m} \frac{Z^m}{m!}$$

is the generalised hypergeometric function, with $(a)_m = \Gamma(a+m)/\Gamma(a)$.

Combining (4.2) and (4.3) we have a readily computable formula for $w_{u,k}(t)$. Of course, infinite sums must be truncated at a suitable point. Figure 4.1 gives the density $w_{10,k}(t)/\psi_k(10)$ for $k = 1, 2, 3$ under the classical risk model ($n = \beta = 1$) with $\alpha = 1$ and $c = 1.2$. The highest peak is the case $k = 1$ whilst the lowest is the case $k = 3$. The shape of each density is not surprising; these densities have the same sort of shape as in the classical risk model (the case $k = 0$).

4.2 Classical risk model with Erlang(2) claims

In this section we assume that the claim arrival process is a Poisson process with parameter λ and we consider the situation when the claim size distribution is Erlang(2, α). From Dickson (2008) we know that $w_u(y, t)$ is of the form

$$w_u(y, t) = l_1(u, t) \alpha^2 y e^{-\alpha y} + l_2(u, t) \alpha e^{-\alpha y}.$$

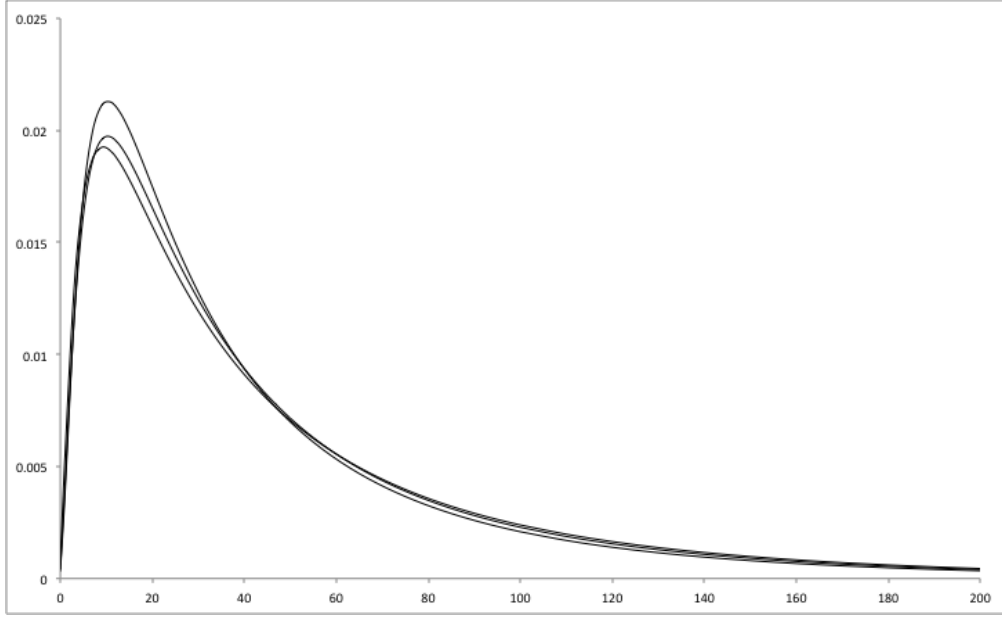


Figure 4.1: $w_{10,k}(t)/\psi_{10}(k)$ for $k = 1, 2, 3$

In particular,

$$l_1(0, t) = \lambda \sum_{n=0}^{\infty} \frac{(\lambda \alpha^2 c^2)^n t^{3n} e^{-(\lambda + \alpha c)t}}{n!(2n + 1)!}$$

and

$$l_2(0, t) = 2\lambda \alpha c \sum_{n=0}^{\infty} \frac{(\lambda \alpha^2 c^2)^n t^{3n+1} e^{-(\lambda + \alpha c)t}}{n!(2n + 2)!}$$

so that

$$\begin{aligned} \tilde{l}_1(0, s) &= \lambda \sum_{n=0}^{\infty} \frac{(\lambda \alpha^2 c^2)^n \Gamma(3n + 1)}{n!(2n + 1)! (\lambda + \alpha c + s)^{3n+1}} \\ &= \frac{\lambda}{\lambda + \alpha c + s} \mathcal{B}_3(\tilde{Z}) \end{aligned}$$

where

$$\tilde{Z} = \lambda \alpha^2 c^2 / (\lambda + \alpha c + s)^3, \quad (4.4)$$

and

$$\tilde{l}_2(0, s) = \frac{\lambda \alpha c}{(\lambda + \alpha c + s)^2} \mathcal{B}_3(\tilde{Z})^2.$$

Now consider $w_{k,k}(t)$. We have

$$\begin{aligned} d_{0,1}(t) &= \frac{\frac{\partial}{\partial t}W(0, k, t)}{G(0, k)} = \frac{l_1(0, t) (1 - e^{-\alpha k} - \alpha k e^{-\alpha k}) + l_2(0, t) (1 - e^{-\alpha k})}{G(0, k)} \\ &= \frac{a_k l_1(0, t) + b_k l_2(0, t)}{G(0, k)} \end{aligned}$$

where $a_k = 1 - e^{-\alpha k} - \alpha k e^{-\alpha k}$ and $b_k = 1 - e^{-\alpha k}$, and

$$\begin{aligned} d_{0,2}(t) &= \frac{w_0(t) - \frac{\partial}{\partial t}W(0, k, t)}{\psi(0) - G(0, k)} = \frac{l_1(0, t)(e^{-\alpha k}(1 + \alpha k)) + l_2(0, t)e^{-\alpha k}}{\psi(0) - G(0, k)} \\ &= \frac{(1 - a_k) l_1(0, t) + (1 - b_k) l_2(0, t)}{\psi(0) - G(0, k)}. \end{aligned}$$

It then follows from (3.1) that

$$\tilde{w}_{k,k}(s) = \sum_{n=1}^{\infty} \left(a_k \tilde{l}_1(0, s) + b_k \tilde{l}_2(0, s) \right)^{n-1} \left((1 - a_k) \tilde{l}_1(0, s) + (1 - b_k) \tilde{l}_2(0, s) \right).$$

From this we obtain

$$\begin{aligned} \tilde{w}_{k,k}(s) &= (1 - a_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \tilde{l}_1(0, s)^{j+1} \tilde{l}_2(0, s)^{n-1-j} \\ &\quad + (1 - b_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \tilde{l}_1(0, s)^j \tilde{l}_2(0, s)^{n-j} \quad (4.5) \\ &= (1 - a_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \frac{\lambda^n (\alpha c)^{n-1-j}}{(\lambda + \alpha c + s)^{2n-1-j}} \mathcal{B}_3(\tilde{Z})^{2n-1-j} \\ &\quad + (1 - b_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \frac{\lambda^n (\alpha c)^{n-j}}{(\lambda + \alpha c + s)^{2n-j}} \mathcal{B}_3(\tilde{Z})^{2n-j} \\ &= (1 - a_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \lambda^n (\alpha c)^{n-1-j} \\ &\quad \times \sum_{i=0}^{\infty} \binom{3i + 2n - 1 - j}{i} \frac{2n - 1 - j}{3i + 2n - 1 - j} \frac{(\lambda \alpha^2 c^2)^i}{(\lambda + \alpha c + s)^{3i + 2n - 1 - j}} \\ &\quad + (1 - b_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \lambda^n (\alpha c)^{n-j} \end{aligned}$$

$$\times \sum_{i=0}^{\infty} \binom{3i+2n-j}{i} \frac{2n-j}{3i+2n-j} \frac{(\lambda\alpha^2c^2)^i}{(\lambda+\alpha c+s)^{3i+2n-j}}.$$

The inverse of

$$\sum_{i=0}^{\infty} \binom{3i+2n-1-j}{i} \frac{2n-1-j}{3i+2n-1-j} \frac{(\lambda\alpha^2c^2)^i}{(\lambda+\alpha c+s)^{3i+2n-1-j}}$$

is

$$\begin{aligned} & \sum_{i=0}^{\infty} \binom{3i+2n-1-j}{i} \frac{2n-1-j}{3i+2n-1-j} \frac{(\lambda\alpha^2c^2)^i e^{-(\lambda+\alpha c)t} t^{3i+2n-2-j}}{\Gamma(3i+2n-1-j)} \\ &= e^{-(\lambda+\alpha c)t} t^{2n-2-j} \sum_{i=0}^{\infty} \frac{(2n-1-j)(\lambda\alpha^2c^2t^3)^i}{i!(2i+2n-1-j)!} \\ &= \frac{e^{-(\lambda+\alpha c)t} t^{2n-2-j}}{(2n-2-j)!} {}_0F_2 \left(n - \frac{j}{2}, n - \frac{j}{2} + \frac{1}{2}; \frac{\lambda\alpha^2c^2t^3}{4} \right). \end{aligned}$$

Similarly, the inverse of

$$\sum_{i=0}^{\infty} \binom{3i+2n-j}{i} \frac{2n-j}{3i+2n-j} \frac{(\lambda\alpha^2c^2)^i}{(\lambda+\alpha c+s)^{3i+2n-j}}$$

is

$$\frac{e^{-(\lambda+\alpha c)t} t^{2n-1-j}}{(2n-1-j)!} {}_0F_2 \left(n - \frac{j}{2} + \frac{1}{2}, n - \frac{j}{2} + 1; \frac{\lambda\alpha^2c^2t^3}{4} \right).$$

These results give

$$\begin{aligned} w_{k,k}(t) &= (1-a_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \lambda^n (\alpha c)^{n-1-j} \\ &\quad \times \frac{e^{-(\lambda+\alpha c)t} t^{2n-2-j}}{(2n-2-j)!} {}_0F_2 \left(n - \frac{j}{2}, n - \frac{j}{2} + \frac{1}{2}; \frac{\lambda\alpha^2c^2t^3}{4} \right) \\ &\quad + (1-b_k) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \lambda^n (\alpha c)^{n-j} \\ &\quad \times \frac{e^{-(\lambda+\alpha c)t} t^{2n-1-j}}{(2n-1-j)!} {}_0F_2 \left(n - \frac{j}{2} + \frac{1}{2}, n - \frac{j}{2} + 1; \frac{\lambda\alpha^2c^2t^3}{4} \right), \end{aligned}$$

a formula which is easy to compute with appropriate truncation of the infinite sums. The derivation of this result has a further application, as shown in Section 5.

We now look at the case where $u \geq k$. We have

$$W(u, k, t) = \int_0^k \int_0^t w_u(y, \tau) d\tau dy = a_k \int_0^t l_1(u, \tau) d\tau + b_k \int_0^t l_2(u, \tau) d\tau$$

and hence

$$\psi(u, t) = \int_0^t (l_1(u, \tau) + l_2(u, \tau)) d\tau.$$

Then from equation (3.2) we have

$$W_{u,k}(t) = \psi(u - k, t) - W(u - k, k, t) + W_{k,k} * W(u - k, k, t),$$

and hence, by differentiating and taking the Laplace transform, we have

$$\begin{aligned} \tilde{w}_{u,k}(s) &= (1 - a_k)\tilde{l}_1(u - k, s) + (1 - b_k)\tilde{l}_2(u - k, s) \\ &\quad + [a_k\tilde{l}_1(u - k, s) + b_k\tilde{l}_2(u - k, s)]\tilde{w}_{k,k}(s). \end{aligned} \quad (4.6)$$

Dickson (2008) gives formulae for the functions $l_1(u, t)$ and $l_2(u, t)$ and shows that their Laplace transforms are

$$\begin{aligned} \tilde{l}_1(u, s) &= \int_0^\infty e^{-st} l_1(u, t) dt \\ &= \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \binom{p}{q} \tilde{l}_1(0, s)^{q+1} \tilde{l}_2(0, s)^{p-q} e_{p+q+1, \alpha}(u), \end{aligned} \quad (4.7)$$

$$\begin{aligned} \tilde{l}_2(u, s) &= \int_0^\infty e^{-st} l_2(u, t) dt \\ &= \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \binom{p}{q} \tilde{l}_1(0, s)^{q+1} \tilde{l}_2(0, s)^{p-q} e_{p+q+2, \alpha}(u) \\ &\quad + \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \binom{p}{q} \tilde{l}_1(0, s)^q \tilde{l}_2(0, s)^{p-q+1} e_{p+q+1, \alpha}(u). \end{aligned} \quad (4.8)$$

Substituting these results and equation (4.5) for $\tilde{w}_{k,k}(\delta)$ into equation (4.6), and by inverting the Laplace transform using the property of the generalised binomial function and the same method used in deriving $w_{k,k}(t)$, we obtain our expression for $w_{u,k}(t)$ as

$$\begin{aligned}
& w_{u,k}(t) \\
= & (1 - a_k) l_1(u - k, t) + (1 - b_k) l_2(u - k, t) \\
& + \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^p \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1 - a_k) \left(\frac{\lambda}{\alpha c}\right)^{n+p+1} \\
& [a_k e_{p+q+1, \alpha}(u - k) + b_k e_{p+q+2, \alpha}(u - k)] \\
& \times e_{2n+2p-i-q, \alpha c}(t) {}_0F_2 \left(\frac{2n+2p-i-q+1}{2}, \frac{2n+2p-i-q+2}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right) \\
& + \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^p \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-1-i} \left(\frac{\lambda}{\alpha c}\right)^{n+p+1} \\
& [(a_k + b_k - 2a_k b_k) e_{p+q+1, \alpha}(u - k) + b_k (1 - b_k) e_{p+q+2}(u - k)] \\
& \times e_{2n+2p-i-q+1, \alpha c}(t) {}_0F_2 \left(\frac{2n+2p-i-q+2}{2}, \frac{2n+2p-i-q+3}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right) \\
& + (1 - b_k) \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^p \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-i} \left(\frac{\lambda}{\alpha c}\right)^{n+p+1} e_{p+q+1, \alpha}(u - k) \\
& \times e_{2n+2p-i-q+2, \alpha c}(t) {}_0F_2 \left(\frac{2n+2p-i-q+3}{2}, \frac{2n+2p-i-q+4}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right).
\end{aligned}$$

This formula can easily be implemented by appropriate truncation of the infinite sums.

4.3 Moments

We now illustrate how $E[T_{u,k}^c]$ can be found. The techniques hinge on being able to decompose the joint density $w_u(y, t)$ as

$$w_u(y, t) = \sum_{i=1}^n h_i(u, t) p_i(y)$$

for some functions $\{h_i(u, t)\}_{i=1}^n$ and densities $\{p_i(y)\}_{i=1}^n$. The key point is that we do not need to know the functions $\{h_i(u, t)\}_{i=1}^n$. Our techniques readily extend to higher moments, and so we omit details. We work through the case $u = k$ in detail, then illustrate the key points for the case $u > k$.

We consider the situation when

$$f(x) = p\alpha e^{-\alpha x} + q\beta e^{-\beta x} \quad (4.9)$$

where $p + q = 1$, $0 < p < 1$ and $\alpha < \beta$. To find $E[T_{k,k}^c]$ we require $E[\tau_{0,1}]$ and $E[\tau_{0,2}]$, given by

$$E[\tau_{0,1}] = \frac{1}{G(0, k)} \int_0^\infty t \frac{\partial}{\partial t} W(0, k, t) dt$$

and

$$E[\tau_{0,2}] = \frac{1}{\psi(0) - G(0, k)} \int_0^\infty t \frac{\partial}{\partial t} (\psi(0, t) - W(0, k, t)) dt.$$

From Dickson (2008) we know that

$$w_0(y, t) = \eta(0, t)\alpha e^{-\alpha y} + \kappa(0, t)\beta e^{-\beta y}$$

where

$$\tilde{\eta}(0, \delta) = \int_0^\infty e^{-\delta t} \eta(0, t) dt = \frac{\lambda}{c} \frac{p}{\rho + \alpha} \quad \text{and} \quad \tilde{\kappa}(0, \delta) = \frac{\lambda}{c} \frac{q}{\rho + \beta},$$

where $\rho(= \rho(\delta))$ is the unique positive solution of Lundberg's fundamental equation for the classical risk model, so that

$$\lambda + \delta - c\rho = \lambda \tilde{f}(\rho).$$

See Gerber and Shiu (1998). Thus,

$$\frac{\partial}{\partial t} W(0, k, t) = \eta(0, t) (1 - e^{-\alpha k}) + \kappa(0, t) (1 - e^{-\beta k})$$

and

$$\frac{\partial}{\partial t} (\psi(0, t) - W(0, k, t)) = \eta(0, t) e^{-\alpha k} + \kappa(0, t) e^{-\beta k}.$$

Hence

$$E[\tau_{0,1}] = \frac{1}{G(0, k)} \int_0^\infty t (\eta(0, t) (1 - e^{-\alpha k}) + \kappa(0, t) (1 - e^{-\beta k})) dt.$$

Now

$$\int_0^\infty t \eta(0, t) dt = -\frac{d}{d\delta} \tilde{\eta}(0, \delta) \Big|_{\delta=0} = \frac{\lambda}{c} \frac{p}{(\rho + \alpha)^2} \frac{d}{d\delta} \rho \Big|_{\delta=0} = \frac{\lambda p}{c \alpha^2 (c - \lambda m_1)}$$

since $\rho = 0$ when $\delta = 0$ and $\frac{d}{d\delta} \rho \Big|_{\delta=0} = 1/(c - \lambda m_1)$. Similarly,

$$\int_0^\infty t \kappa(0, t) dt = \frac{\lambda q}{c \beta^2 (c - \lambda m_1)}.$$

Hence

$$E[\tau_{0,1}] = \frac{1}{G(0, k)} \left(\frac{\lambda p (1 - e^{-\alpha k})}{c \alpha^2 (c - \lambda m_1)} + \frac{\lambda q (1 - e^{-\beta k})}{c \beta^2 (c - \lambda m_1)} \right)$$

and it is straightforward to show that

$$G(0, k) = \frac{\lambda p}{\alpha c} (1 - e^{-\alpha k}) + \frac{\lambda q}{\beta c} (1 - e^{-\beta k}).$$

Similarly,

$$E[\tau_{0,2}] = \frac{1}{\psi(0) - G(0, k)} \left(\frac{\lambda p e^{-\alpha k}}{c \alpha^2 (c - \lambda m_1)} + \frac{\lambda q e^{-\beta k}}{c \beta^2 (c - \lambda m_1)} \right),$$

giving everything required to find $E[T_{k,k}^c]$ from formula (3.4).

We now consider $E[T_{u,k}^c]$. From formula (3.3) we see that we require $E[\tau_{u-k,1}]$ and $E[\tau_{u-k,2}]$. We again know from Dickson (2008) that $w_u(y, t)$ is of the form

$$w_u(y, t) = \eta(u, t) \alpha e^{-\alpha y} + \kappa(u, t) \beta e^{-\beta y}. \quad (4.10)$$

The key quantities that we need to evaluate to obtain $E[\tau_{u-k,1}]$ and $E[\tau_{u-k,2}]$ are

$$\int_0^\infty t \eta(u - k, t) dt \quad \text{and} \quad \int_0^\infty t \kappa(u - k, t) dt.$$

We start by deriving formulae for these, then indicate an alternative approach.

A rather complex formula for $\int_0^t \eta(u, s) ds$ is given in Dickson (2008), from which we could obtain $\eta(u, t)$. However, to obtain $\int_0^\infty t \eta(u - k, t) dt$ it is considerably easier to note from Dickson (2008) that

$$\begin{aligned} & \int_0^\infty e^{-\delta t} \eta(u - k, t) dt \\ &= \frac{1}{\alpha} \sum_{n=0}^\infty \sum_{j=0}^n \binom{n}{j} \tilde{\eta}(0, \delta)^{j+1} \tilde{\kappa}(0, \delta)^{n-j} \\ & \quad \times e^{-\beta(u-k)} \left(\frac{\alpha}{\beta} \right)^j \frac{\alpha(\beta(u-k))^n}{n!} {}_1F_1(j+1, n+1, (\beta-\alpha)(u-k)). \end{aligned} \quad (4.11)$$

Consider

$$\frac{d}{d\delta} \tilde{\eta}(0, \delta)^{j+1} \tilde{\kappa}(0, \delta)^{n-j} = (j+1) \tilde{\eta}(0, \delta)^j \left(\frac{d}{d\delta} \tilde{\eta}(0, \delta) \right) \tilde{\kappa}(0, \delta)^{n-j}$$

$$+\tilde{\eta}(0, \delta)^{j+1}(n-j)\tilde{\kappa}(0, \delta)^{n-j-1}\left(\frac{d}{d\delta}\tilde{\kappa}(0, \delta)\right)$$

giving

$$\begin{aligned} -\frac{d}{d\delta}\tilde{\eta}(0, \delta)^{j+1}\tilde{\kappa}(0, \delta)^{n-j}\Big|_{\delta=0} &= (j+1)\left(\frac{\lambda p}{\alpha c}\right)^j\left(\frac{\lambda p}{c\alpha^2}\frac{1}{c-\lambda m_1}\right)\left(\frac{\lambda q}{c\beta}\right)^{n-j} \\ &\quad +\left(\frac{\lambda p}{\alpha c}\right)^{j+1}(n-j)\left(\frac{\lambda q}{c\beta}\right)^{n-j-1}\left(\frac{\lambda q}{c\beta^2}\frac{1}{c-\lambda m_1}\right) \\ &= \frac{1}{c-\lambda m_1}\left(\frac{\lambda p}{\alpha c}\right)^{j+1}\left(\frac{\lambda q}{c\beta}\right)^{n-j}\left(\frac{j+1}{\alpha}+\frac{n-j}{\beta}\right). \end{aligned}$$

This gives

$$\begin{aligned} &\int_0^\infty t\eta(u-k, t)dt \\ &= \frac{1}{\alpha(c-\lambda m_1)}\sum_{n=0}^\infty\sum_{j=0}^n\binom{n}{j}\left(\frac{\lambda p}{\alpha c}\right)^{j+1}\left(\frac{\lambda q}{c\beta}\right)^{n-j}\left(\frac{j+1}{\alpha}+\frac{n-j}{\beta}\right) \\ &\quad \times e^{-\beta(u-k)}\left(\frac{\alpha}{\beta}\right)^j\frac{\alpha(\beta(u-k))^n}{n!}{}_1F_1(j+1, n+1, (\beta-\alpha)(u-k)). \end{aligned}$$

A similar argument yields

$$\begin{aligned} &\int_0^\infty t\kappa(u-k, t)dt \\ &= \frac{1}{\beta(c-\lambda m_1)}\sum_{n=0}^\infty\sum_{j=0}^n\binom{n}{j}\left(\frac{\lambda q}{c\beta}\right)^{j+1}\left(\frac{\lambda p}{\alpha c}\right)^{n-j}\left(\frac{j+1}{\beta}+\frac{n-j}{\alpha}\right) \\ &\quad \times e^{-\beta(u-k)}\left(\frac{\beta}{\alpha}\right)^j\frac{\beta(\alpha(u-k))^n}{n!}{}_1F_1(n-j, n+1, (\beta-\alpha)(u-k)) \end{aligned}$$

and we are thus able to compute $E[\tau_{u-k,1}]$ and $E[\tau_{u-k,2}]$. It is clear that if we differentiate (4.11) a second time we can obtain $\int_0^\infty t^2\eta(u-k, t)dt$. This is a straightforward task that involves no new ideas.

A second approach uses results given in Lin and Willmot (2000). They provide formulae for both $E[T_u I(T_u < \infty)]$ and $E[T_u |U(T_u)| I(T_u < \infty)]$, and their methodology can be applied to find quantities such as

$E[T_u^2 | U(T_u) | I(T_u < \infty)]$. It follows from formula (4.10) that

$$E[T_u I(T_u < \infty)] = \int_0^\infty t \eta(u, t) dt + \int_0^\infty t \kappa(u, t) dt$$

and

$$E[T_u | U(T_u) | I(T_u < \infty)] = \frac{1}{\alpha} \int_0^\infty t \eta(u, t) dt + \frac{1}{\beta} \int_0^\infty t \kappa(u, t) dt.$$

These two identities can then be used to find $\int_0^\infty t \eta(u, t) dt$ and $\int_0^\infty t \kappa(u, t) dt$.

Again, the decomposition of $w_u(y, t)$ is the key. If the individual claim amount distribution was a mixture of three exponentials, say

$$f(x) = \sum_{i=1}^3 w_i \alpha_i \exp\{-\alpha_i x\},$$

then we would have

$$w_u(y, t) = \sum_{i=1}^3 h_i(u, t) \alpha_i \exp\{-\alpha_i x\},$$

and in this case we would need to apply formulae given by Lin and Willmot (2000) to find $E[T_u | U(T_u)|^2 I(T_u < \infty)]$ in order to identify $\int_0^\infty t h_i(u, t) dt$ for $i = 1, 2, 3$.

Figure 4.2 shows $E[T_{u,k}^c]$ for $k = 0, 1, 2$ and 3 when f is given by (4.9) with $p = 1/3$, $\alpha = 1/2$ and $\beta = 2$, with $\lambda = 1$ and $c = 1.2$. Some features of this plot are not particularly surprising – as u increases for a given value of k , $E[T_{u,k}^c]$ increases and as k increases, $E[T_{u,k}^c]$ increases for a given value of u . An interesting feature of the plot is that the lines are almost parallel from around $u = 5$.

5 The duration of negative surplus

The distribution of the total duration of negative surplus in the classical risk model was studied by Dickson and Egídio dos Reis (1996). In that paper a recursion formula is given for the distribution function of the total duration of negative surplus, and expressions are given for the density of the duration of individual periods of negative surplus when the individual claim amount distribution is exponential and Erlang(2). We now revisit each of these examples and show that there is an explicit solution for the density of the total duration of negative surplus.

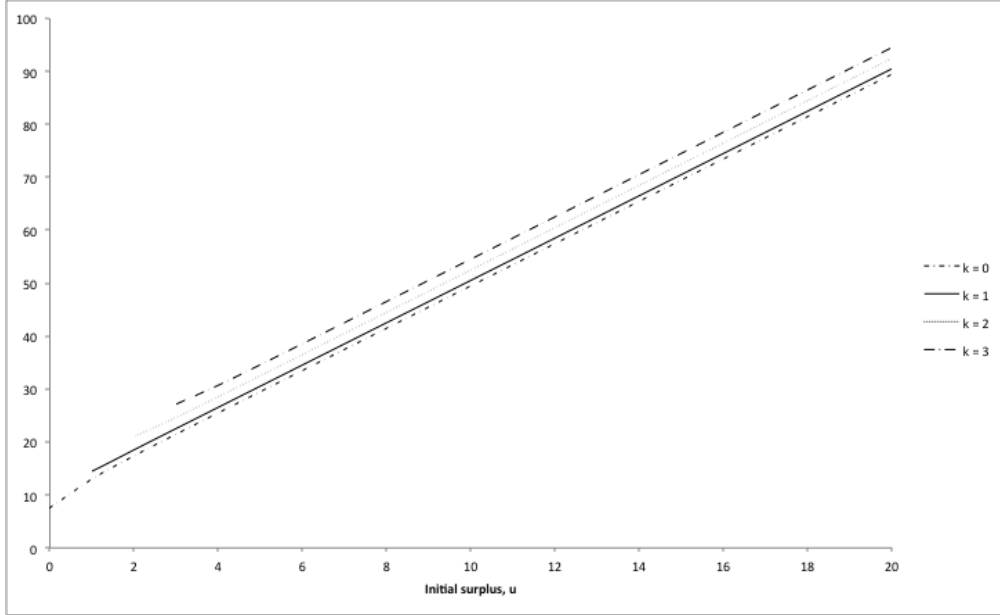


Figure 4.2: $E[T_{u,k}^c]$ for $k = 0, 1, 2$ and 3

We follow the notation of Dickson and Egídio dos Reis (1996): a denotes the density function of the first period of negative surplus given that ruin occurs, d denotes the density function of any subsequent period of negative surplus given that the surplus falls below 0 after the last recovery, and k denotes the (defective) density function of the total duration of negative surplus, which we denote by TT . (Note that $\Pr(TT = 0) = \bar{\psi}(u)$.) As in Section 4.2, claims occur as a Poisson process with parameter λ .

5.1 Exponential claims

When $f(x) = \alpha e^{-\alpha x}$, $x > 0$, we have $a(t) = d(t) = w_0(t)/\psi(0)$. Thus, from formula (3.1) of Dickson and Egídio dos Reis (1996) we have

$$\begin{aligned}
 k(t) &= \psi(u) \bar{\psi}(0) \sum_{n=0}^{\infty} \psi(0)^n a^{(n+1)*}(t) \\
 &= \frac{\psi(u) \bar{\psi}(0)}{\psi(0)} \sum_{n=1}^{\infty} w_0^{n*}(t) \\
 &= \frac{\psi(u) \bar{\psi}(0)}{\psi(0)} \sum_{n=1}^{\infty} \frac{\lambda^n t^{n-1} e^{-(\lambda+\alpha c)t}}{\Gamma(n)} {}_0F_1(n+1; \alpha c \lambda t^2), \quad (5.1)
 \end{aligned}$$

where the final line follows from formula (4.3). It is straightforward to implement this formula numerically by truncating the infinite sum. We can see from formula (5.1) that the conditional distribution of TT given that $TT > 0$ is independent of u , which is a consequence of the distribution of the deficit at ruin, given that ruin occurs, being independent of u .

5.2 Erlang(2) claims

We now consider the case when $f(x) = \alpha^2 x e^{-\alpha x}$, $x > 0$. Dickson and Egídio dos Reis (1996) give a formula for $a(t)$ (which contains a typographical error – the powers of t should be $3n$) from which we obtain

$$\begin{aligned}\tilde{a}(s) &= (1 - \gamma(u)) \frac{\alpha c}{\lambda + \alpha c + s} \mathcal{B}_3(\tilde{Z}) + \gamma(u) \frac{(\alpha c)^2}{(\lambda + \alpha c + s)^2} \mathcal{B}_3(\tilde{Z})^2 \\ &= \frac{\alpha c}{\lambda} \left((1 - \gamma(u)) \tilde{l}_1(0, s) + \gamma(u) \tilde{l}_2(0, s) \right)\end{aligned}$$

where \tilde{Z} is given by (4.4) and the function $\gamma(u)$ is given in Dickson and Egídio dos Reis (1996), and is such that $\gamma(0) = \frac{1}{2}$, meaning that

$$\tilde{d}(s) = \frac{\alpha c}{2\lambda} \left(\tilde{l}_1(0, s) + \tilde{l}_2(0, s) \right).$$

We remark that $d(t) = w_0(t)/\psi(0)$, and so $\tilde{d}(s) = \left(\tilde{l}_1(0, s) + \tilde{l}_2(0, s) \right) / \psi(0)$, with $\psi(0) = 2\lambda/(\alpha c)$. It is straightforward to show (e.g. from formula (3.2) of Dickson and Egídio dos Reis (1996) that

$$\tilde{k}(s) = \psi(u) \bar{\psi}(0) \sum_{n=0}^{\infty} \psi(0)^n \tilde{a}(s) \tilde{d}(s)^n$$

and so we get

$$\begin{aligned}\tilde{k}(s) &= \psi(u) \bar{\psi}(0) \sum_{n=0}^{\infty} \frac{\alpha c}{\lambda} \left((1 - \gamma(u)) \tilde{l}_1(0, s) + \gamma(u) \tilde{l}_2(0, s) \right) \left(\tilde{l}_1(0, s) + \tilde{l}_2(0, s) \right)^n \\ &= \psi(u) \bar{\psi}(0) \frac{\alpha c}{\lambda} (1 - \gamma(u)) \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \tilde{l}_1(0, s)^{i+1} \tilde{l}_2(0, s)^{n-i} \\ &\quad + \psi(u) \bar{\psi}(0) \frac{\alpha c}{\lambda} \gamma(u) \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \tilde{l}_1(0, s)^i \tilde{l}_2(0, s)^{n-i+1}.\end{aligned}$$

From our inversion of $\tilde{w}_{k,k}(s)$ in Section 4.2 we see that this yields

$$\begin{aligned}
k(t) = & \psi(u) \bar{\psi}(0) \frac{\alpha c}{\lambda} (1 - \gamma(u)) \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \frac{\lambda^{n+1} (\alpha c)^{n-i} t^{2n-i} e^{-(\lambda+\alpha c)t}}{(2n-i)!} \\
& \times {}_0F_2 \left(n+1 - \frac{i}{2}, n + \frac{3}{2} - \frac{i}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right) \\
& + \psi(u) \bar{\psi}(0) \frac{\alpha c}{\lambda} \gamma(u) \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \frac{\lambda^{n+1} (\alpha c)^{n+1-i} t^{2n+1-i} e^{-(\lambda+\alpha c)t}}{(2n+1-i)!} \\
& \times {}_0F_2 \left(n + \frac{3}{2} - \frac{i}{2}, n + 2 - \frac{i}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right).
\end{aligned}$$

Again it is straightforward to implement this formula numerically by truncating the infinite sums.

References

- [1] Borovkov, K.A. and Dickson, D.C.M. (2008) *On the ruin time distribution for a Sparre Andersen process with exponential claim sizes*. Insurance: Mathematics & Economics 42, 1104–1108.
- [2] Dickson, D.C.M. (2008) *Some explicit solutions for the joint density of the time of ruin and the deficit at ruin*. ASTIN Bulletin 38, 259–276.
- [3] Dickson, D.C.M. and Egidio dos Reis, A.D. (1996) *On the distribution of the duration of negative surplus*. Scandinavian Actuarial Journal 1996, 148–164.
- [4] Dickson, D.C.M. and Li, S. (2010) *Finite time ruin problems for the Erlang(2) risk model*. Insurance: Mathematics & Economics 46, 12–18.
- [5] Gerber, H.U. and Shiu, E.S.W. (1998). *On the time value of ruin*. North American Actuarial Journal 2, 1, 48–78.
- [6] Graham, R.L., Knuth, D.E. and Patashnik, O. (1994) Concrete Mathematics, 2nd edition. Addison-Wesley, Upper Saddle River, NJ.
- [7] Landriault, D. and Willmot, G.E. (2009) *On the joint distributions of the time to ruin, the surplus prior to ruin, and the deficit at ruin in the classical risk model*. North American Actuarial Journal 13, 2, 252–279.

- [8] Lin, X.S. and Willmot, G.E. (2000) *The moments of the time of ruin, the surplus before ruin, and the deficit at ruin*. Insurance: Mathematics & Economics 27, 19–44.
- [9] Nie, C., Dickson, D.C.M. and Li, S. (2011) *Minimizing the ruin probability through capital injections*. Annals of Actuarial Science 5, 2, 195–209.