Asymptotic Stability of a Brock-Mirman Economy with Unbounded Shock

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Abstract

New results in the asymptotic theory of Markov processes are applied to analysis of the long-run behavior exhibited by optimal growth models with unbounded productivity shock. The techniques developed here are geometrically intuitive, and are shown to imply global stability for a popular model specification. In the process, we present a simple new proof of a recent result pertaining to the stability of discrete dynamical systems on metric space.

1 Introduction

The evolution of stochastic growth models is characterized by a sequence of probability distributions through time, where each successive distribution governs the likelihood of outcomes for the state variables in that period. The objective of this paper is to establish strong convergence to a unique limit for the distributions generated by a popular parameterization of the standard optimal accumulation model, namely that with logarithmic utility, Cobb-Douglas production and lognormal shock.

For this unit elastic and lognormal model it is readily observable from the law of motion that weak (i.e. pointwise) convergence to a unique invariant distribution occurs for a log-linearized version of the system.[1] Of course, global stability of the linearized model does not automatically imply the same for the nonlinear model. To establish this implication we also require

[1] See, for example, Stokey, Lucas and Prescott [19], Chapter 2, Section 2. We are assuming here that returns to capital are diminishing.

metric convergence of the linearized model and verification of a topological conjugacy relation between the linear and nonlinear systems. This is a valid approach, but not the one pursued in this paper. Instead we establish strong (metric) convergence for the original nonlinear process.

The first detailed analysis of the asymptotic properties of a stochastic growth model with concave production is Brock and Mirman [4]. Using probabilistic methods, the authors show uniform convergence of distribution functions over time to a unique invariant distribution. Their analysis is based on the analytically convenient assumption that the probability measure corresponding to the production shock vanishes off a compact set.

Subsequent authors have maintained the compactness assumption. Stokey, Lucas and Prescott [19] find similar convergence results using an application of Helly's theorem (Chapter 13, Section 3). Hopenhayn and Prescott [8] obtain existence and strong convergence by applying the fixed point theorem of Tarski. Amir [1] extends the analysis of the latter to allow for some forms of increasing returns to fixed capital.

Although convenient, the compactness assumption excludes many common shock specifications.[2] In particular, none of the above studies can be applied to the model considered in this paper, where the shock is supported on the positive reals. Here an entirely different approach to global stability is investigated, using the auxiliary notions of strongly contractive operators and Lagrange stability.

Section 2 introduces an important class of systems called *semidynamical systems*. Section 3 shows that semidynamical systems which are both strongly contractive and Lagrange stable exhibit global stability. Section 4 shows how Markov processes can be regarded as a special class of semidynamical systems. Section 5 provides specialized techniques to determine whether these semidynamical systems generated by Markov processes satisfy strong contractiveness and Lagrange stability. Section 6 applies the techniques of Section 5 to stability analysis for the Markov processes that characterize unit elastic growth models.

The treatment of Markov processes employed in this paper is inherently operator-theoretic. This approach originated in the 1930s with Krylov and Bogolioubov [12]. Important contributions were made by Kakutani and Yoshida [11] and Hopf [9]. Operator-based methods have proved useful for investigating such macrostructure of Markov models as stability and ergodic properties.[3] Previous applications of the particular techniques used here

[2]Most of the "named" continuous distributions have everywhere positive density on an unbounded domain.

[3]For an early and extensive exposition of operator-theoretic techniques in economics see Futia [7].

include models of particle energy in an ideal gas, and of cell size in a proliferating cell population (Lasota, [13]).

In addition to a new application of recent mathematical results to the study of stochastic growth models, the paper includes a new proof of an important sufficient condition for the convergence of contractive dynamical systems, as well as a new proof of a classical result on Markov chains originally due to A. A. Markov.

2 Mathematical Techniques

This section outlines the mathematical techniques used in the paper. With regard to these methods, the primary reference is Lasota [13].

2.1 Discrete Dynamical Systems

Consider a dynamic economic model represented by a first order difference equation such as

$$x_{t+1} = Tx_t, \quad x \in U, \quad T\colon U \to U. \tag{1}$$

The model is fully determined by the properties of the set U and the map T. A realization or *trajectory* for the model is a sequence $(T^n x)$ in U generated by iterating the map T on initial condition x. A (globally stable) equilibrium—if it exists—is a point $\bar{x} \in U$ such that $T\bar{x} = \bar{x}$ and that every trajectory converges to it under continued iteration of T.

Formally, a *semidynamical system* is a pair (U, T), where U is a metric space (the state space) with distance d and T is a continuous mapping of U into itself.[4] Let (U, T) be a semidynamical system. A *fixed point* of T on U is a point $\bar{x} \in U$ such that $T\bar{x} = \bar{x}$. Fixed points are said to be *stationary* or *invariant* under T. Similar terminology also applies to sets. In particular, if $TA \subset A$ then A is said to be *invariant* under T. If $\bar{x}$ is a fixed point of T on U then by the *stable set* $S_T(\bar{x})$ of $\bar{x}$ we refer to the subset of U which is convergent to $\bar{x}$ under iteration of T. $\bar{x}$ is said to be *stable*, or an *attractor*, whenever there exists a set G open in U such that $\bar{x} \in G$ and $G \subset S_T(\bar{x})$. Finally, (U, T) is said to be *asymptotically stable* if there exists a unique fixed point $\bar{x}$ and $S_T(\bar{x}) = U$.

A considerable literature exists on sufficient and necessary conditions for asymptotic stability of semidynamical systems. The conditions are in the

[4] The system is called *dynamical* if, in addition, the mapping T is invertible with continuous inverse (i.e. is a homeomorphism).

form of metric, topological and linear restrictions on the state space and the map. Motivated by our interest in Markov processes, we focus in this paper on semidynamical systems which have two properties called strong contractiveness and Lagrange stability. It has recently been shown in the discrete context that every system with these two properties is asymptotically stable (Lasota [13]). Here a new proof of the same result is provided, based on the properties of strongly contractive operators on a compactum.

2.2 Stochastically Perturbed Dynamical Systems

Suppose that the system (1) is perturbed by a serially uncorrelated noise ε, the distribution of ε being governed by density $\varphi\colon U \to \mathbb{R}_+$:

$$x_{t+1} = T(x_t, \varepsilon_t), \quad x \in U, \quad \varepsilon \sim \varphi. \tag{2}$$

Let

$$p\colon U^2 \ni (x_t, x_{t+1}) \mapsto p(x_t, x_{t+1}) \in \mathbb{R}_+ \tag{3}$$

be the conditional density of x_{t+1} given x_t, which is calculated from our knowledge of the functions T and φ. Note that if ψ_t is a density which governs the location of the current state then

$$\psi_{t+1}\colon U \ni x_{t+1} \mapsto \int_U \psi_t(x_t) p(x_t, x_{t+1}) dx_t \in \mathbb{R}_+ \tag{4}$$

will govern that of the next state, given the dynamic structure of the model as embodied in (2).[5]

In order to proceed with the analysis of (2), one possibility is to use techniques from the classical theory of stochastic processes (see, for example, Shiryaev [18], Chapter 8). However, it is also possible to frame the same problem as a semidynamical system. The idea is to reinterpret the state space to be the collection of all densities on U. We call this set $D(L_1(U))$ for reasons that will become apparent. The other half of the pair is the operator (call it P) that associates current-period with next-period densities as in (4). This is the so-called L_1 approach to Markov processes.[6]

Evolution of the economy is now characterized by a sequence of densities $(P^n\psi)$ generated by iterating P on some initial density ψ. An equilibrium

[5] The intuition here is that the integral sums the probability of traveling to x_{t+1} through x_t for all $x_t \in U$, weighted by the likelihood of x_t occurring as the current state.

[6] Primary references are Hopf [9] and Foguel [6]. L_1 techniques have been applied extensively in mathematical biology. See, for example, Lasota and Mackey [14], Tyson and Hannsgen [21], Trycha [20], Lasota, Mackey and Trycha [16] and Lasota [13].

is a density $\bar{\psi}$ that is constant under the action of P, and such that nearby densities converge to $\bar{\psi}$ under iteration of P. At $\bar{\psi}$, events (subsets of U) occur with constant probability over time.

In Section 4 we show that $(D(L_1(U)), P)$ is a semidynamical system. In Section 5 we show that if p of (3) is everywhere positive then P is strongly contractive (Theorem 5.1), and that if there exists an everywhere positive $\bar{\psi} \in D(L_1(U))$ such that $P\bar{\psi} = \bar{\psi}$ then $(D(L_1(U)), P)$ is Lagrange stable (Theorem 5.3). These results are due to Lasota and Mackey [15], Baron and Lasota [2] and Lasota [13]. Together, strong contractiveness and Lagrange stability imply asymptotic stability of the system, as discussed in 2.1.

3 Contractive Systems and Stability

In this section we develop a condition for stability of semidynamical systems that is suitable for analyzing those systems which are generated by Markov models.

3.1 Strongly Contractive Systems

Definition 3.1. Semidynamical system (U, T) is called *contractive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in U. \tag{5}$$

The system is called *strongly contractive* if

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in U, \ x \neq y. \tag{6}$$

Suppose that strongly contractive system (U, T) has more than one fixed point. In particular, let distinct points x and y be stationary under T. Then

$$d(x, y) = d(Tx, Ty) < d(x, y). \tag{7}$$

Contradiction. Thus a strongly contractive system can have at most one fixed point.

The starting point for an investigation of the relationship between contractiveness and stability is usually the famous contraction mapping theorem of Banach, an immediate consequence of which is the asymptotic stability of every semidynamical system (U, T) where U is complete and T satisfies

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{8}$$

for some $\lambda < 1$.

For the semidynamical systems generated by stochastic growth models with unbounded shock, condition (6) is often very easy to verify, while (8) rarely holds. Unfortunately, the weaker condition (6) is not sufficient to guarantee the existence of a fixed point.[7] Nevertheless, stability results are available if we supplement (6) with other restrictions. The following theorem is familiar from analysis. (See, for example, Bollobás [3], Chapter 7, Exercise 3). A simple proof is provided in the appendix.

Theorem 3.1. *Let (U,T) be strongly contractive, and let U be a compactum. Then (U,T) is asymptotically stable.*

3.2 Lagrange Stability

For the systems that we wish to study, the assumption of compactness of the state space is not satisfied. Following Lasota [13], we develop a slightly weaker condition to pair with strong contractiveness and thereby obtain stability results.

Definition 3.2. Semidynamical system (U,T) is called *Lagrange stable* if the trajectory of x is precompact for every $x \in U$.[8]

Given semidynamical system (U,T) and $x \in U$, let $\Gamma(x)$ denote the set $\{T^n x\}$ consisting of all points in the trajectory $(T^n x)$, as well as all limit points of the same (i.e. all points to which a subsequence of the trajectory converges). Alternatively, $\Gamma(x)$ is the closure of $\{T^n x\}$, the (unordered) set of all points in $(T^n x)$. It is clear that the trajectory of x is precompact if and only if $\Gamma(x)$ is a compact subset of U.

Lemma 3.1. *Let (U,T) be a semidynamical system. $\Gamma(x)$ is invariant under T for every $x \in U$.*

Proof. Take $u \in \Gamma(x)$. If u is a point in the trajectory then so is Tu. On the other hand, if u is the limit of some subsequence $(T^{n_k} x)$ then it follows from the continuity of T that Tu is the limit of the subsequence $(T^{n_k+1} x)$. Hence $Tu \in \Gamma(x)$. □

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[7] For example, consider $U = \mathbb{R}_+$, $T: x \mapsto x + e^{-x}$.

[8] A set $A \subset U$ is said to be *precompact* (or *relatively compact*) if every sequence in A has a convergent subsequence. (A is compact if, in addition, the limit of the sequence is in A.) A sequence (x_n) in U is said to be precompact whenever $\{x_n\}$ is a precompact subset of U. This is equivalent to the statement that every subsequence of the original sequence (regarded as an ordered sequence now, not a set) has a convergent subsequence.

3.3 A Convergence Result

We now have the following key result, which was proved by A. Lasota in the context of Hausdorff space using Liapunov methods ([13], Theorem 2.1). Here we provide a new proof based on Theorem 3.1.

Theorem 3.2. *Semidynamical system (U, T) is asymptotically stable if it is Lagrange stable and strongly contractive.*

Proof. Fix $x \in U$. Since $\Gamma(x)$ is invariant, $(\Gamma(x), T)$ is itself a strongly contractive semidynamical system on a compactum, and, by Theorem 3.1, has a unique fixed point $\bar{x}$ with $T^n x \to \bar{x}$. Moreover, (U, T) has at most one fixed point by strong contractiveness. The result follows. □

4 Markov Operators

In this section we show how Markov processes can be interpreted as semidynamical systems. These results are by no means new. Alternative references from the mathematical literature are Hopf [9] and Foguel [6].

Let an economic model be given, and suppose that X denotes the set of all possible values that may be taken by the state variable(s). Intuitively, a (first order) Markovian economic model is a collection of probability densities on X—one for each possible value of the current state. Knowledge of the current state therefore uniquely identifies the likelihood of events $E \subset X$ in the following period.[9]

Formally, let $\mathcal{X}$ be a σ-algebra of subsets of X containing all events of interest to the economist, and let μ be a σ-finite measure on $(X, \mathcal{X})$. Intuitively, μ can be thought of as some kind of "natural" measure on the state space, such as the Lebesgue measure on the Borel subsets of $\mathbb{R}$, or the counting measure on the collection of all subsets of $\mathbb{N}$.[10] As usual, $L_1(X, \mathcal{X}, \mu)$ will denote the Banach space of μ-summable functions on X with norm

$$\|f\| \stackrel{\text{df}}{=} \int |f| d\mu.$$

It is understood that functions in $L_1(X, \mathcal{X}, \mu)$ are defined only up to the complement of a μ-null set, and "almost everywhere" notation is suppressed

[9] We have chosen to work with density functions rather than probability measures. To date, operator-based treatments of Markov processes in economics have been embedded in a space of probability measures (see, for example, Futia [7], or Stokey, Lucas and Prescott [19]). In practice, however, there are several disadvantages with a set function approach. See Hopf [9].

[10] The counting measure is the measure that assigns to each subset of $\mathbb{N}$ the number of elements it contains.

throughout. $L_1(X, \mathcal{X}, \mu)$ has the usual partial ordering, whereby $f \leq g \iff f(x) \leq g(x)$, $\forall x \in X$.

4.1 A Definition of Hopf

Although previous work in economics has adopted probability measures (probability transition functions) as the primitive of Markov theory, the following operator-theoretic definition of B. Hopf (Hopf [9], Definition 2.1) is more suitable for our purposes.

Definition 4.1. Let M be a normed linear space with partial order. A *Markov operator* on M is a linear operator P mapping M into itself such that P is both positive and norm-preserving for positive elements of M.[11] A *Markov process* is a pair (M, P), where M is a partially ordered normed linear space and P is a Markov operator on M.

The Markov operator for our model is constructed as follows.

Definition 4.2. A *stochastic kernel* for measure space $(X, \mathcal{X}, \mu)$ is a nonnegative, real-valued and $\mathcal{X}^2$-measurable function p on X^2 such that

$$\int p(x, y)\mu(dy) = 1, \quad \forall x \in X. \tag{9}$$

In other words, $y \mapsto p(x, y)$ is a density for each $x \in X$. Intuitively, this function can be thought of as the conditional density of the state in the next period, given that the current location is x.

Associate with a given kernel p the operator P, where

$$P\colon L_1(X, \mathcal{X}, \mu) \ni f \mapsto \int_X f(x)p(x, \cdot)\mu(dx) \in L_1(X, \mathcal{X}, \mu). \tag{10}$$

(This is just the operation in (4).) We can verify that P in (10) is a Markov operator by confirming that it satisfies the properties in Definition 4.1. The proof is straightforward and is omitted.

Thus every stochastic kernel defines a Markov operator. If p is a stochastic kernel describing the probabilistic structure of an economic model, P is the associated Markov operator, and φ is a known probability density allocating probabilities to current period events, then the image of φ under P is a probability measure governing the likelihood of next-period outcomes.

[11] For a normed linear space M, a partial order $\leq$ is usually defined in terms of a cone M_+, which is declared to be positive. For $x, y \in M$, $x \leq y$ iff $y - x \in M_+$. In this terminology, a Markov operator is a linear operator P such that $PM_+ \subset M_+$ and $\|Px\| = \|x\|$ whenever $x \in M_+$.

Let P be a Markov operator on $L_1(X, \mathcal{X}, \mu)$, and let $D(L_1(X, \mathcal{X}, \mu))$ be the collection of all densities on the same (i.e. the set of all nonnegative functions integrating to unity). $D(L_1(X, \mathcal{X}, \mu))$ inherits its metric from the norm topology of the function space. We show in the appendix that in this topology the operator P is bounded on bounded sets (Lemma A.1) and hence is continuous. Moreover, it is readily verified that any operator P satisfying the conditions of Definition 4.1 maps $D(L_1(X, \mathcal{X}, \mu))$ into itself. Hence $(D(L_1(X, \mathcal{X}, \mu)), P)$ is a semidynamical system.

Example 4.1. Consider the unit elastic optimal accumulation problem

$$\max \mathrm{E} \left[\sum_{t=0}^{\infty} \beta^t \log c_t \right] \tag{11}$$

$$\text{s.t. } x_{t+1} = \varepsilon_t x_t^{\alpha} - c_t, \tag{12}$$

$$0 < c_t < \varepsilon_t x_t^{\alpha},\ t = 0, 1, \ldots \tag{13}$$

where the control c_t denotes consumption, $x_t \in \mathbb{R}_{++}$ is a state variable representing capital per head, $x \mapsto x^{\alpha}$ is the production function, $\alpha \in (0,1)$, (ε_t) is a sequence of serially uncorrelated lognormal-(μ, σ^2) shocks, $\beta \in (0,1)$ is a discount factor and $x_0 > 0$ is given.

For such a specification, the optimal strategy $x_t \mapsto c_t$ and the transition relation (12) imply an economy that evolves according to the nonlinear rule

$$x_{t+1} = \alpha\beta\varepsilon_t x_t^{\alpha}. \tag{14}$$

(See, for example, Stokey, Lucas and Prescott [19], Chapter 9, Section 4.) By a change of variable argument, the stochastic kernel of the process defined by (14) is found to be

$$p\colon (x, y) \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y} \exp\left(- \frac{(\log(yx^{-\alpha}) - \log(\alpha\beta) - \mu)^2}{2\sigma^2} \right) \tag{15}$$

for positive real numbers x and y.

Formally, the process will be denoted

$$(D(L_1(\mathbb{R}_{++}, \mathcal{B}(\mathbb{R}_{++}), \lambda)), P), \tag{16}$$

where $\mathcal{B}(\mathbb{R}_{++})$ denotes the Borel subsets of the positive reals and λ is the Lebesgue measure on the same. P is the Markov operator associated with kernel (15).

A plot of (15) is shown in Figure 1. The figure is easy to interpret. For each x_t a density function runs parallel to the x_{t+1} axis. The density governs

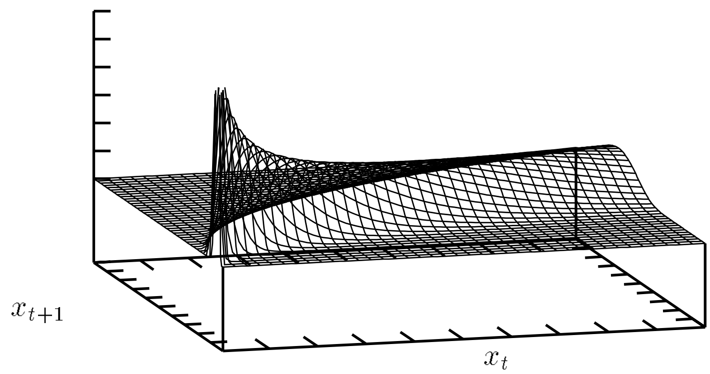

Figure 1: Stochastic kernel (15), lognormal shock.

the likelihood that capital per head takes values along that axis, given that the current state is x_t.

It is interesting to compare the figure with a kernel estimated from the actual growth data. See Quah [17], Figures 5 and 6. Even a casual inspection of the two kernels shows that the work-horse theoretical specification (15) differs from the estimated kernel in several respects.

5 Stability Properties of Markov Operators

In order to apply Theorem 3.2 we seek conditions that imply strong contractiveness and Lagrange stability of Markov processes. The exposition is based on Lasota [13].

5.1 Strongly Contractive Markov Operators

A sufficient condition for strong contractiveness of Markov operators on $L_1(X, \mathcal{X}, \mu)$ is stated and proved. For the purposes of the following discussion, the *support* of a real function f on X is defined to be that subset of X on which f differs from zero.

Theorem 5.1 (Lasota and Mackey). *For given measure space $(X, \mathcal{X}, \mu)$, let p be a stochastic kernel and let P be the associated Markov operator. If $p > 0$ on X^2 then $(D(L_1(X, \mathcal{X}, \mu)), P)$ is strongly contractive.*

Proof. Fix $\varphi, \psi \in D(L_1(X, \mathcal{X}, \mu))$, $\varphi \neq \psi$. It is clear that if f is any function in $L_1(X, \mathcal{X}, \mu)$ with $f \geq 0$ and $f > 0$ on a set of positive measure then $\operatorname{supp} Pf = X$.

$$\therefore \quad [\operatorname{supp} P(\varphi - \psi)^+] \cap [\operatorname{supp} P(\varphi - \psi)^-] = X. \tag{17}$$

$$\therefore \quad |P(\varphi - \psi)| < P(\varphi - \psi)^+ + P(\varphi - \psi)^- = P|\varphi - \psi| \text{ on } X. \tag{18}$$

$$\therefore \quad \|P\varphi - P\psi\| < \int_X P|\varphi - \psi| d\mu = \|\varphi - \psi\|. \tag{19}$$

□

Incidentally, Theorem 5.1 yields a new proof of the following classical result of A. A. Markov. The proof is a minor variation on a common (and more algebra intensive) contraction-based proof. See, for example, Stokey, Lucas and Prescott [19], Lemma 11.3.

Theorem 5.2 (Markov). *Let $\mathbf{P}$ define a finite Markov chain, $\mathbf{P} = (p_{ij})$. If $p_{ij} > 0$ for all i and all j then there exists a unique density $\bar{\varphi}$ such that $\bar{\varphi}\mathbf{P} = \bar{\varphi}$ and $\varphi\mathbf{P}^n \to \bar{\varphi}$ for every initial density φ.*

Proof. Strong contractiveness is by Theorem 5.1. In addition, the density space is compact for finite X. The result then follows from Theorem 3.1. □

5.2 Lagrange Stability of Markov Operators

Next we state and prove a sufficient condition of Lasota [13] for the Lagrange stability of Markov processes defined by integral Markov operators. Integral Markov operators are those operators P on $L_1(X, \mathcal{X}, \mu)$ that can be expressed in the form (10) for some stochastic kernel p.

Theorem 5.3 (Lasota). *Given measure space $(X, \mathcal{X}, \mu)$, let p be a stochastic kernel and let P be the associated Markov operator. If there exists a probability density $\bar{\varphi} \in D(L_1(X, \mathcal{X}, \mu))$ such that $\bar{\varphi}$ is invariant under P and positive on X then semidynamical system $(D(L_1(X, \mathcal{X}, \mu)), P)$ is Lagrange stable.*

Proof. By Lemma A.4 it is sufficient to find a set $A \subset L_1(X, \mathcal{X}, \mu)$ such that the closure of A contains $D(L_1(X, \mathcal{X}, \mu))$ and $(P^n f)$ is weakly precompact for every $f \in A$. Associate to each density φ the sequence (φ_k), where

$$\varphi_k \stackrel{\text{df}}{=} \min(\varphi, k\bar{\varphi}). \tag{20}$$

Since $\bar{\varphi} > 0$ it follows that $\varphi_k \uparrow \varphi$ $(k \to \infty)$ and hence $\varphi_k \to \varphi$ in $L_1(X, \mathcal{X}, \mu)$ (by the monotone convergence theorem of Lebesgue). If

$$A \stackrel{\text{df}}{=} \{\varphi_k \,:\, \varphi \in D(L_1(X, \mathcal{X}, \mu)),\ k \in \mathbb{N}\} \tag{21}$$

then the closure of A contains the density space as required. Moreover,

$$P^n \varphi_k \leq P^n k\bar{\varphi} = k\bar{\varphi}, \tag{22}$$

and therefore $(P^n \varphi_k)_{n \in \mathbb{N}}$ is weakly precompact for each $k \in \mathbb{N}$.[12] The result follows. □

6 An application to Brock-Mirman

In this section it is established that the economy of Example 4.1 is both strongly contractive and Lagrange stable.

Proposition 6.1. *The Brock-Mirman economy of Example 4.1 is strongly contractive.*

Proof. Immediate from (15) and Theorem 5.1. □

Proposition 6.2. *The Brock-Mirman economy of Example 4.1 is Lagrange stable.*

Proof. Let

$$\bar{\varphi} \colon x \mapsto \sqrt{\frac{1-\alpha^2}{2\pi\sigma^2}} \frac{1}{x} \exp\left(-\frac{(1-\alpha^2)(\log x - (\log(\alpha\beta) - \mu)/(1-\alpha))^2}{2\sigma^2} \right). \tag{23}$$

Some manipulation shows that $P\bar{\varphi} = \bar{\varphi}$ on $\mathbb{R}_{++}$. (Here P is the operator in (16).) Hence $\bar{\varphi}$ is stationary under P. Moreover, $\bar{\varphi}$ is everywhere positive, implying Lagrange stability of (16) by Theorem 5.3. □

[12] Let $(X, \mathcal{X}, \mu)$ be any measure space and let $\Omega \subset L_1(X, \mathcal{X}, \mu)$. If for some $g \in L_1(X, \mathcal{X}, \mu)$ we have $|f| \leq g$, $\forall f \in \Omega$, then Ω is weakly precompact in $L_1(X, \mathcal{X}, \mu)$. See Dunford and Schwartz [5].

Proposition 6.3. *Let a Brock-Mirman economy be as described in Example 4.1. The economy converges asymptotically to unique stationary probability density (23), independent of initial conditions.*

Proof. Immediate from Proposition 6.1, Proposition 6.2 and Theorem 3.2. □

7 Conclusion

Using a number of new results in the theory of Markov operators, we have established asymptotic convergence for a small class of noisy economies with unbounded shock. The two conditions for stability were strong contractiveness and Lagrange stability. For the model considered, strong contractiveness is immediate from positivity of the stochastic kernel. Lagrange stability was demonstrated by finding a positive solution to the functional equation $P\varphi = \varphi$, $\varphi \in D(L_1(X, \mathcal{X}, \mu))$. Solution of the functional equation is amenable to computational (numerical) methods when the Markov model P is estimated from data. As such, the techniques are particularly appropriate for applied analysis.

A Appendix

This appendix collects a number of technical results cited in the arguments above. First is a short proof of Theorem 3.1.

Proof. Uniqueness holds for any strong contraction. Regarding existence, consider the continuous function

$$g\colon U \ni x \mapsto d(Tx, x) \in \mathbb{R}. \tag{24}$$

Let p minimize g on U. By construction, $g(p) \geq 0$. Suppose that $g(p) > 0$. Then $Tp \neq p$ and $d(T^2p, Tp) < d(Tp, p)$, which implies that $g(Tp) < g(p)$. Contradiction. Hence $g(p) = 0$ and $Tp = p$.

Regarding convergence, take any $x \in U$. $d(T^n x, p)$ is clearly monotone decreasing and bounded below, implying convergence to some $\alpha \geq 0$. If $\alpha = 0$ then we are done. Suppose otherwise. By the compactness of U, the trajectory $(T^n x)$ possesses at least one convergent subsequence $(T^{n_j} x)$. If the limit of this subsequence is q then clearly $d(q, p) = \alpha$. By continuity, the image $(T^{n_j+1} x)$ of the subsequence under T is again convergent with limit Tq, and once again $d(Tq, p) = \alpha$. But now strong contractiveness implies that $d(Tq, p) < d(q, p)$. Contradiction. □

The next lemma proves that all Markov operators on L_1 are contractive.

Lemma A.1. *Let P be a Markov operator on $L_1(X, \mathfrak{X}, \mu)$. Then the system $(L_1(X, \mathfrak{X}, \mu), P)$ is contractive.*

Proof. By linearity and positivity,

$$|Pf| = |Pf^+ - Pf^-| \leq Pf^+ + Pf^- = P|f|. \tag{25}$$

Integration obtains

$$\|Pf\| \stackrel{\text{df}}{=} \int |Pf| d\mu \leq \int P|f| d\mu = \|f\|. \tag{26}$$

An application of linearity yields (5). □

The next few results pertain to the proof of Theorem 5.3. The proof of the first lemma is that of Lasota [13], Proposition 3.2.

Lemma A.2. *Let M be a complete metric space, U any subset of M and A a subset of M dense in U (in the sense that U is contained in the closure of A). If (U, T) is a contractive semidynamical system and trajectory $(T^n a)$ is precompact for every $a \in A$ then (U, T) is Lagrange stable.*

Proof. Fix $x \in U$ and let (a_m) be a sequence in A converging to x. We construct an index set $I \subset \mathbb{N}$ such that $(T^n a_m)_{n \in I}$ is convergent for every $m \in \mathbb{N}$ as follows. Take a convergent subsequence of the trajectory of a_1 and denote the corresponding index set I_1. Now take a convergent subsequence of $(T^n a_2)_{n \in I_1}$ and denote the corresponding index set I_2. Continuing in this manner we create a decreasing sequence of index sets

$$I_1 \supset I_2 \supset I_3 \supset \cdots . \tag{27}$$

To construct I take the first element of I_1, the second element of I_2, the third element of I_3, and so on. I so constructed has the property stated above.

Since each $(T^n a_m)_{n \in I}$ is convergent it is also Cauchy. But then $(T^n x)_{n \in I}$ is also Cauchy, because if ε is any positive number then for all $m \in \mathbb{N}$,

$$d(T^n x, T^k x) \leq d(T^n x, T^n a_m) + d(T^n a_m, T^k a_m) + d(T^k a_m, T^k x) \tag{28}$$
$$\leq 2d(Tx, Ta_m) + d(T^n a_m, T^k a_m) \tag{29}$$
$$\leq 2d(Tx, Ta_m) + \varepsilon \tag{30}$$

for n, k suitably chosen.

$$\therefore \quad d(T^n x, T^k x) \leq \varepsilon \tag{31}$$

by continuity of T and the fact that m is arbitrary.

As M is complete $(T^n x)_{n\in I}$ is also convergent, and we have shown that $(T^n x)_{n\in\mathbb{N}}$ has a convergent subsequence. It is clear that the same result holds if we substitute $\mathbb{N}'$ for $\mathbb{N}$, where $\mathbb{N}'$ is any increasing subset of $\mathbb{N}$. The result follows. □

Lemma A.3. *For given measure space $(X, \mathcal{X}, \mu)$, let p be a stochastic kernel and let P be the associated integral Markov operator. If Λ is a weakly precompact subset of $L_1(X, \mathcal{X}, \mu)$ then $P\Lambda$ is strongly precompact.*

Proof. A straightforward proof is available in the appendix to Lasota [13]. □

Corollary A.1. *For given measure space $(X, \mathcal{X}, \mu)$, let p be a stochastic kernel and let P be the associated integral Markov operator. If $(P^n f)$ is a weakly precompact trajectory in $L_1(X, \mathcal{X}, \mu)$ for arbitrary summable function f then it is also a strongly precompact trajectory.*

As a result, we now have the following condition for the Lagrange stability of integral Markov processes.

Lemma A.4. *Given measure space $(X, \mathcal{X}, \mu)$ let p be a stochastic kernel and let P be the associated Markov operator. $(D(L_1(X, \mathcal{X}, \mu)), P)$ is Lagrange stable whenever there exists a set $A \in L_1(X, \mathcal{X}, \mu)$ such that A is dense in $D(L_1(X, \mathcal{X}, \mu))$ and $(P^n f)$ is weakly precompact for every $f \in A$.*

Proof. Immediate from Corollary A.1 and Lemma A.2. □

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