Ruin Probabilities with Compounding Assets

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Abstract

We consider a classical surplus process modified by the action of a constant force of interest. We derive recursive algorithms for the calculation of the probability of ruin in finite time. We also discuss the numerical evaluation of the probability of ultimate ruin using methods proposed by de Vylder (1996) and Sundt and Teugels (1995). Finally, we consider the question of recovery from ruin.

Keywords: finite time ruin; recursive calculation; constant force of interest; ultimate ruin; recovery from ruin.

1. Introduction

In this paper we present and discuss some algorithms to calculate the probability of ruin in both finite and infinite time for a classical surplus process modified by the inclusion of (deterministic) interest on the insurer's surplus. In previous studies (Dickson and Waters (1991), Dickson et al (1995)) algorithms have been presented to calculate ruin probabilities for the classical surplus process. In these studies, the basic idea is that the probability of ruin for a classical surplus process can be approximated by a discrete time ruin probability for an integer-valued surplus process. We shall retain elements of this approach in this paper.

In the case of finite time, we will approximate the probability of ruin in continuous time by a probability of ruin in discrete time. However, due to the fact that income is not a linear function of time in the modified process, we cannot base our algorithm on an integer-valued process. Instead we must construct an algorithm based on integral equations. In the case of infinite time ruin, we discuss the numerical evaluation of the probability of ruin using methods proposed by de Vylder (1996) and Sundt and Teugels (1995).

In the final section we consider the question of recovery from ruin when the insurer must borrow funds at some fixed rate of interest until the surplus process returns to zero. We present algorithms to calculate the probability of the surplus process returning to zero without passing below specified levels, and discuss the circumstances under which these algorithms may usefully be applied.

2. The Probability of Ruin in Finite Time

In this and the following section we will consider a surplus process for which aggregate claims are generated by a compound Poisson process. The Poisson parameter for the number of claims per annum is λ and the individual claim amount distribution function is $F(x)$. We assume that $F(0) = 0$. We denote by μ the mean individual claim amount. Premium income is received continuously at constant rate P per annum, where $P = (1 + \theta)\lambda\mu$ for some premium loading factor θ. We assume that the insurer earns interest at a constant force of interest δ per annum. Hence, starting with an initial surplus of U, the insurer's surplus will have accumulated to $Ue^{\delta t} + P(e^{\delta t} - 1)/\delta$ by time t, assuming no claims occur in this time interval.

Let $\psi_h(U; t)$ denote the probability that the insurer's surplus, starting from $U\ (\geq 0)$ at time 0, is negative at one or more of the times $h; 2h; \ldots; t - h; t$. We will always assume that U is an integer and that t is an integer multiple of h. In this section we present algorithms for the calculation of $\psi_h(U; t)$. This is a probability of ruin in finite and discrete time for our surplus process. However, by choosing a sufficiently small value for h, we can regard it as an approximation to the probability of ruin in continuous time, denoted $\psi(U; t)$.

Now suppose that the insurer's initial surplus, U, the time interval, t, and the step size, h, are fixed. Let $G(x)$ denote the distribution function of aggregate claims in a time interval of length h, and for $x > 0$ let $g(x)$ be the corresponding density function. Let $w = Ue^{\delta h} + P(e^{\delta h} - 1)/\delta$, so that w represents the highest possible level of the insurer's surplus at time h. We assume that any claims are paid at the end of a time interval. This will result in an understatement of the probability of ruin. Then, writing $t = nh$, where n is an integer, we have

$$\psi_h(U; nh) = \psi_h(U; h) + G(0)\psi_h(w; (n-1)h) + \int_0^w g(x)\psi_h(w - x; (n-1)h)\,dx \quad (2.1)$$

and

$$\psi_h(U; h) = \int_w^\infty g(x)\,dx = 1 - G(w) \quad (2.2)$$

In order to apply these formulae we need to evaluate both $G(x)$ and $g(x)$. Let us assume that μ is large (we can always achieve this simply by rescaling the process), and let us approximate $F(x)$ by a discrete distribution with probability function $\{f_k\}_{k=0}^{\infty}$. In all our examples, we will use the discreritisation method of De Vylder and Goovaerts (1988) so that the discrete distribution has the same mean as $F(x)$. As an approximation to $G(x)$, we can calculate a discrete distribution $H(x)$, with probability function $\{h_k\}_{k=0}^{\infty}$, using Panjer's (1981) recursion formula. As we require a continuous distribution (for $x > 0$) in formulae (2.1) and (2.2), we will create such a distribution, denoted $J(x)$, from $H(x)$. We set

$$J(x) = H(x) \quad \text{for } 0 \le x < \tfrac{1}{2}$$

Next, for $0 \le f \le \frac{1}{2}$, and $x = 1, 2, 3, \ldots$, set

$$J(x + f) = H(x - 1) + (f + \tfrac{1}{2})h_x$$

and for $\frac{1}{2} \le f \le 1$, and $x = 0, 1, 2, \ldots$, set

$$J(x + f) = H(x) + (f - \tfrac{1}{2})h_{x+1}$$

Thus we have created a distribution $J(x)$ as an approximation to $H(x)$, and hence to $G(x)$. Note that $J(x)$ is a piecewise continuous function and that $j(x) = h_x$ on the interval $(x - \frac{1}{2}, x + \frac{1}{2})$. A consequence of this is that the means of the distributions $G(x)$ and $J(x)$ are identical.

Replacing $G(x)$ and $g(x)$ by $J(x)$ and $j(x)$ respectively in (2.1) and (2.2) we can calculate $\psi_h(U; nh)$. First, let us assume that $w = [w] + f$, where $[w]$ is the integer part of w and $f \ge 1/2$. Then

$$\begin{aligned}
\psi_h(U; nh) &= \psi_h(U; h) + J(0)\psi_h(w; (n-1)h) + \int_{1/2}^{w} j(x)\psi_h(w - x; (n-1)h)\,dx \\
&= \psi_h(U; h) + J(0)\psi_h(w; (n-1)h) \\
&\quad + \sum_{k=1}^{[w]} \int_{k-1/2}^{k+1/2} j(x)\psi_h(w - x; (n-1)h)\,dx \\
&\quad + \int_{[w]+1/2}^{w} j(x)\psi_h(w - x; (n-1)h)\,dx
\end{aligned}$$

with the understanding that the summation is 0 if $[w]=0$ and the final integral is 0 if $f=1/2$. If we now allow for the fact that $j(x)$ is constant over the intervals $(k-1/2, k+1/2)$, where k is an integer, (and equals h_k), we have

$$\begin{aligned}\psi_h(U;nh) &= \psi_h(U;h) + J(0)\psi_h(w;(n-1)h) \\ &\quad + \sum_{k=1}^{[w]} h_k \int_{k-1/2}^{k+1/2} \psi_h(w-x;(n-1)h)\,dx \\ &\quad + h_{[w]+1} \int_{[w]+1/2}^{w} \psi_h(w-x;(n-1)h)\,dx\end{aligned}$$

We then replace the integral terms in the summation by the value of the integrand at the mid point of the range of integration (and evaluate the final integral in a similar fashion), giving

$$\begin{aligned}\psi_h(U;nh) &= \psi_h(U;h) + J(0)\psi_h(w;(n-1)h) + \sum_{k=1}^{[w]} h_k\psi_h(w-k;(n-1)h) \\ &\quad + h_{[w]+1}(f-\tfrac{1}{2})\psi_h(\tfrac{1}{2}(f-\tfrac{1}{2});(n-1)h) \qquad (2.3)\end{aligned}$$

Note that we require values of $\psi_h(x;(n-1)h)$ where, in general, x will not be an integer. Assuming we know this function for integer values of the surplus, we will calculate $\psi_h(x;(n-1)h)$ by linear interpolation as

$$\psi_h(x;(n-1)h) = \phi\psi_h([x]+1;(n-1)h) + (1-\phi)\psi_h([x];(n-1)h)$$

where $\phi = x-[x]$. Thus, the approach to calculating $\psi_h(U;nh)$ involves calculating $\psi_h(k;h)$ for integer k, from which we can calculate $\psi_h(k;2h)$ for integer k, and so on.

Similarly, if $w=[w]+f$ where $f<1/2$, we find that if $[w]=0$,

$$\psi_h(U;nh) = \psi_h(U;h) + J(0)\psi_h(w;(n-1)h)$$

and if $[w]>0$,

$$\begin{aligned}\psi_h(U;nh) &= \psi_h(U;h) + J(0)\psi_h(w;(n-1)h) + \sum_{k=1}^{[w]-1} h_k\psi_h(w-k;(n-1)h) \\ &\quad + h_{[w]}(f+\tfrac{1}{2})\psi_h(\tfrac{1}{2}(f+\tfrac{1}{2});(n-1)h) \qquad (2.4)\end{aligned}$$

with the understanding that the summation is zero if $[w]=1$.

It is clear that there are a number of approximations involved in using the above procedure to approximate values of $\psi(U;t)$. Two points are immediately clear: first, the value of n should be large so that the calculated discrete time ruin probability will give a reasonable approximation to the continuous time ruin probability, and, second, the value of $f\mu$ should be large so that $J(x)$ will be a reasonable approximation to $G(x)$. In all our examples we will link these two by setting $n=t$, the number of periods into which we split each year, equal to $(1+\theta)\lambda\beta$. This is the same approach used in Dickson and Waters' (1991) algorithm to calculate survival/ruin probabilities when $\delta = 0$. A consequence of this choice is that for $\delta = 0$, the surplus can increase by at most 1 in any time period. For $\delta > 0$, the maximum increase in the surplus in a unit time period will therefore be a little more than 1. In all our examples in this section, we have set $\beta = 100$.

In view of the fact that we will use this value for β, it is also clear that a great deal of computational effort is required to produce approximations to $\psi(U; t)$. We therefore introduce a simple truncation procedure to reduce the number of calculations required. The most straightforward way to do this is to introduce truncation of the density $j(x)$.

In the construction of $J(x)$, we first calculated the distribution $H(x)$. Let ε be a suitably small number and let k_0 be such that

$$H(k_0) \geq 1 - \varepsilon > H(k_0 - 1)$$

Since $J(k_0 + 1/2) = H(k_0)$ it is clear that $J(k_0 + 1/2) \geq 1 - \varepsilon$. Set $h_x = 0$ for $x = k_0 + 1; k_0 + 2; \ldots$, so that $j(x) = 0$ for $x > k_0 + 1/2$. Next, define

$$j^{\varepsilon}(x) = \begin{cases} j(x) & \text{for } x < k_0 + 1/2 \\ 0 & \text{for } x > k_0 + 1/2 \end{cases}$$

$$\psi_h^{\varepsilon}(U;h) = \begin{cases} \psi_h(U;h) & \text{for } w < k_0 + 1/2 \\ 0 & \text{for } w \geq k_0 + 1/2 \end{cases}$$

and for $n = 2; 3; 4; \ldots$

$$\psi_h^{\varepsilon}(U;nh) = \psi_h^{\varepsilon}(U;h) + J(0)\psi_h^{\varepsilon}(w;(n-1)h) + \int_{1/2}^{w} j^{\varepsilon}(x)\psi_h^{\varepsilon}(w - x;(n-1)h)\,dx$$

Then, it is straightforward to show that

$$\psi_h(U;nh) - 2n\varepsilon \leq \psi_h^{\varepsilon}(U;nh) \leq \psi_h(U;nh)$$

for $n = 1; 2; 3; \ldots$. (See de Vylder and Goovaerts (1988).)

The advantage of this approach is that we can cut down the number of calculations required. We will use formulae corresponding to (2.3) and (2.4). The only real difference from these formulae will be that the upper limit of summation could be reduced to k_0 and the final term could be zero. In principle, we could also introduce truncation of the ruin probabilities by setting ruin probabilities smaller than ε to be zero. However, in view of the large number of additional calculations we would require in a programme - to check when calculated ruin probabilities had fallen below ε - we have not introduced such a procedure.

Example 1: Let the individual claim amount distribution be exponential with mean 1, let $\lambda = 1$ and let the premium loading factor, θ, be 10%. Table 1 shows approximations to $\psi(U; t)$ when $\delta = 0$, calculated from the algorithm described above and denoted by A in the table, and exact values, taken from Seal (1978) and denoted by E in the table. We have chosen ϵ such that that the maximum error introduced into our calculations by truncation is 10^{-4}.

Table 1: Values of $\psi(U; t)$ when $\alpha = 0$ - exponential claims

		t = 1	t = 5	t = 10	t = 20	t = 40
U = 0	A	0.4619	0.7184	0.7845	0.8311	0.8632
	E	0.4634	0.7196	0.7854	0.8318	0.8638
U = 5	A	0.0138	0.1024	0.1903	0.2953	0.3951
	E	0.0138	0.1027	0.1906	0.2956	0.3954
U = 10	A	0.0003	0.0092	0.0318	0.0820	0.1572
	E	0.0003	0.0092	0.0319	0.0821	0.1573

It is clear from Table 1 that the algorithm produces good approximations, especially when the ruin probability is small. Note, however, that when $\alpha = 0$ the algorithm is different to that given by Dickson and Waters (1991, Section 8) and does not produce approximations that are quite as accurate for all values of U. On the basis of Table 1 we would expect the algorithm to produce good approximations to $\psi(U; t)$ when $\alpha > 0$. Tables 2 to 4 show some calculated values for $\psi(U; t)$ when U = 0, 5 and 10 respectively. It is difficult to determine the extent to which figures in Tables 2, 3, 4 and 5 understate the true values as a result of our assumption that claims are paid at the end of each time interval of length h. However, the indications from Section 3.4.2 below are that this understatement is of no significance.

Table 2: Approximations to $\psi(0; t)$ - Exponential claims

α	t = 1	t = 5	t = 10	t = 20	t = 40
0.01	0.4615	0.7151	0.7785	0.8209	0.8467
0.02	0.4611	0.7119	0.7725	0.8107	0.8308
0.03	0.4607	0.7085	0.7664	0.8007	0.8159
0.04	0.4602	0.7052	0.7604	0.7908	0.8020
0.05	0.4598	0.7019	0.7544	0.7812	0.7893
0.06	0.4594	0.6986	0.7484	0.7719	0.7777
0.07	0.4590	0.6952	0.7425	0.7629	0.7670
0.08	0.4586	0.6919	0.7366	0.7542	0.7571
0.09	0.4581	0.6886	0.7308	0.7459	0.7479
0.1	0.4577	0.6852	0.7250	0.7380	0.7393

Table 3: Approximations to $\psi(5; t)$ - Exponential claims

δ	$t = 1$	$t = 5$	$t = 10$	$t = 20$	$t = 40$
0.01	0.0135	0.0970	0.1757	0.2633	0.3354
0.02	0.0133	0.0919	0.1619	0.2339	0.2836
0.03	0.0131	0.0869	0.1491	0.2073	0.2403
0.04	0.0128	0.0822	0.1371	0.1837	0.2048
0.05	0.0126	0.0778	0.1260	0.1628	0.1761
0.06	0.0124	0.0735	0.1158	0.1446	0.1528
0.07	0.0122	0.0695	0.1063	0.1287	0.1337
0.08	0.0119	0.0657	0.0977	0.1150	0.1180
0.09	0.0117	0.0621	0.0899	0.1031	0.1049
0.1	0.0115	0.0587	0.0827	0.0928	0.0939

Table 4: Approximations to $\psi(10; t)$ - Exponential claims

δ	$t = 1$	$t = 5$	$t = 10$	$t = 20$	$t = 40$
0.01	0.00030	0.0081	0.0264	0.0625	0.1072
0.02	0.00028	0.0072	0.0218	0.0472	0.0722
0.03	0.00027	0.0063	0.0179	0.0355	0.0488
0.04	0.00026	0.0056	0.0148	0.0268	0.0336
0.05	0.00025	0.0049	0.0122	0.0202	0.0238
0.06	0.00025	0.0043	0.0100	0.0154	0.0172
0.07	0.00024	0.0038	0.0083	0.0119	0.0128
0.08	0.00023	0.0034	0.0069	0.0093	0.0097
0.09	0.00022	0.0030	0.0057	0.0073	0.0076
0.1	0.00021	0.0027	0.0048	0.0059	0.0060

From Tables 2 to 4 we can see that the algorithm is producing ruin probabilities which have the correct characteristic: as the force of interest increases, the probability of ruin in a fixed time period decreases.

Example 2: Let the individual claim amount distribution be Pareto(3,2) (so that the distribution has mean 1), let $\lambda = 1$ and let the premium loading factor, θ, be 10%. Tables 5 to 7 show approximations to $\psi(U; t)$ for $U = 0$, 5 and 10. We have again chosen ε such that the maximum error introduced by the truncation procedure is 10^{-4}. We have chosen a shorter time period than in Example 1, and calculation times for Tables 5 to 7 were fairly short. However, due to the nature of the individual claim amount distribution, computation times for a larger value of t can be quite long as the truncation procedure may have little effect (i.e. k_0 may have a very high value). In such circumstances, a less accurate calculation may be a more realistic target if computer time is an issue.

Table 5: Approximations to Ã(0; t) - Pareto claims

±	t = 1	t = 5	t = 10
0	0.4201	0.6663	0.7396
0.025	0.4191	0.6587	0.7253
0.05	0.4181	0.6512	0.7110
0.075	0.4170	0.6436	0.6969
0.1	0.4160	0.6360	0.6830

Table 6: Approximations to Ã(5; t) - Pareto claims

±	t = 1	t = 5	t = 10
0	0.0264	0.1304	0.2212
0.025	0.0258	0.1190	0.1915
0.05	0.0251	0.1085	0.1652
0.075	0.0245	0.0989	0.1423
0.1	0.0239	0.0901	0.1228

Table 7: Approximations to Ã(10; t) - Pareto claims

±	t = 1	t = 5	t = 10
0	0.0054	0.0349	0.0754
0.025	0.0052	0.0302	0.0586
0.05	0.0051	0.0262	0.0457
0.075	0.0049	0.0227	0.0359
0.1	0.0047	0.0198	0.0286

In each of the above two examples, we have simply illustrated the application of our algorithm. We chose values for ± which would illustrate the behaviour of Ã(U; t) as a function of ±. It is, however, clear that for most real life insurance portfolios the interest rates in these examples are far too high. In our ¯nal example below we apply the algorithm with the same range of values for ±, but with a much larger expected number of claims per unit time. In such a situation, the principles involved in applying the algorithm are the same - but computer run times are much larger!

Example 3: Let the individual claim amount distribution be exponential with mean 1, let ¸ = 100 and let the premium loading factor, µ, be 10%. Table 8 shows approximations to Ã(10; t). We have chosen " such that the maximum error introduced in our calculations by truncation is 10^{-4}.

Table 8: Approximations to $\psi(10; t)$ - exponential claims

δ	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$
0	0.1042	0.1835	0.2301	0.2603
0.025	0.1035	0.1816	0.2273	0.2567
0.05	0.1027	0.1798	0.2245	0.2530
0.075	0.1019	0.1780	0.2217	0.2494
0.1	0.1012	0.1761	0.2189	0.2459

3. The Probability of Ruin in Infinite Time

3.1. De Vylder's method

We denote by $\psi(U)$ and $\bar{\psi}(U)$ the probabilities of ultimate ruin and ultimate survival, respectively, for the surplus process described in the first paragraph of Section 2. De Vylder (1996, II.10.4.2, Theorem 18) proposed the following method for the calculation of $\bar{\psi}(U)$.

For any (integer) value of the initial surplus U, let $\tau(U)$ denote the time interval in which, if there were no claims, the surplus would increase to $U+1$, so that

$$U + 1 = U \exp\{\delta\tau(U)\} + P(\exp\{\delta\tau(U)\} - 1)/\delta$$

and hence

$$\tau(U) = \frac{1}{\delta} \log\left(\frac{U + 1 + P/\delta}{U + P/\delta}\right)$$

The probability that no claims occur in this time interval is $1 - \lambda\tau(U) + o(\lambda\tau(U))$, the probability that exactly one claim occurs is $\lambda\tau(U)) + o(\lambda\tau(U))$, and the probability of two or more claims is $o(\lambda\tau(U))$. Now replace the continuous individual claim amount distribution, which we denote $F(x)$, by the discrete distribution $\{f_k\}_{k=0}^{\infty}$ (as in Section 2). Then, conditioning on what happens in the time interval of length $\tau(U)$, we can write as an approximation

$$\bar{\psi}(U) = (1 - \lambda\tau(U))\bar{\psi}(U + 1) + \sum_{k=0}^{U+1} \lambda\tau(U) f_k \bar{\psi}(U + 1 - k)$$

and hence

$$\bar{\psi}(U + 1) = [1 - \lambda\tau(U)(1 - f_0)]^{-1} \left[\bar{\psi}(U) - \sum_{k=1}^{U+1} \lambda\tau(U) f_k \bar{\psi}(U + 1 - k)\right] \qquad (3.1)$$

Note that, as in Section 2, we are assuming that claims are paid at the end of the time interval of length $\tau(U)$.

In order to use (3.1) to calculate recursively $\bar{\psi}(1), \bar{\psi}(2), \ldots$ we need a value for $\bar{\psi}(0)$. De Vylder's method for estimating this is to define $\bar{\psi}_0(U)$ to be equal to $\bar{\psi}(U)/\bar{\psi}(0)$ and then to note that

- $\tilde{\psi}_0(U)$ satisfies (31).
- Formula (31) can be used to calculate recursively $\tilde{\psi}_0(U)$ for $U = 1, 2, \ldots$ starting from $\tilde{\psi}_0(0) = 1$.
- $\lim_{U \to \infty} \tilde{\psi}_0(U) = 1/\tilde{\psi}(0)$.

By calculating $\tilde{\psi}_0(U)$ for a sufficiently large value of U, we can estimate $\tilde{\psi}(0)$ and hence estimate $\tilde{\psi}(U)$ for $U = 1, 2, \ldots$.

The problems associated with this procedure are

- It is an approximate method since
  - it approximates a continuous time probability by a discrete time probability (with a variable time interval),
  - it uses a compound binomial approximation to a compound Poisson distribution,
  - it requires the recursive calculation of $\tilde{\psi}_0(U)$ to converge reasonably quickly so that a good estimate of $\tilde{\psi}(0)$ can be obtained.
- Formula (31) has every appearance of being numerically unstable. The authors did not experience any problems of instability when using this algorithm to compute the examples presented later in this section. However, this formula is almost identical in form to formula (72) in Dickson and Waters (1991), which proved to be numerically unstable. See Dickson et al (1995).
- It assumes claims are paid at the end of the interval of length $\tau(U)$.

3.2. A modification of De Vylder's method

An obvious modification of de Vylder's method is to drop the compound binomial approximation to the compound Poisson distribution inherent in (31). We again approximate the distribution $F(x)$ by $\{f_k\}_{k=0}^{\infty}$, and, for each integer value of the surplus U, let $h_k(U)$ denote the probability that aggregate claims in a time interval of length $\tau(U)$ equal k, for $k = 0, 1, 2, \ldots$. These probabilities can be calculated using Panjer's (1981) recursion formula since the aggregate claims distribution is compound Poisson with Poisson parameter $\lambda\tau(U)$ and (discretised) claim amount probabilities $\{f_k\}_{k=0}^{\infty}$. Formula (31) is then replaced by:

$$\tilde{\psi}(U+1) = h_0(U)^{-1}\left[\tilde{\psi}(U) - \sum_{k=1}^{U} h_k(U)\tilde{\psi}(U+1-k)\right] \qquad (32)$$

Note that we have made one small modification to formula (31). We have set the upper limit of summation in (32) to be U instead of $U+1$. This means that when

$\delta = 0$, (32) is of the same form as the algorithm given in Dickson et al (1995) (for the function denoted $\delta_d^*(u)$ in that paper). Formula (32) can be used in exactly the same way as (31) to calculate, or rather estimate, recursively $\psi(U)$, having first estimated $\psi(0)$. It suffers all the same drawbacks as (31) except that it does not use a compound binomial approximation to a compound Poisson distribution. However, (32) has the additional drawback that it requires far more computation time than (31) since a separate Panjer recursion procedure has to be carried out for each integer level of the initial surplus U.

3.3. Sundt and Teugels' method

By conditioning on the time and amount of the first claim, Sundt and Teugels (1995, formula (2)) derived the following integral equation for $\psi(U)$

$$\psi(U) = \frac{P}{P+\delta U}\psi(0) + \frac{1}{P+\delta U}\int_0^U \psi(U-y)\{\delta + \lambda[1-F(y)]\}dy \qquad (3.3)$$

By discretising the integral in (3.3), Sundt and Teugels derived the following bounds for $\psi(U)$ for $U = 1, 2, \ldots$

$$\psi^{(-)}(U) \le \psi(U) \le \psi^{(+)}(U)$$

where

$$\psi^{(-)}(U) = \frac{1}{P+\delta U}\left\{P\psi^{(-)}(0) + \sum_{k=1}^{U}\psi^{(-)}(U-k)\bar{f}_k\right\}$$

$$\psi^{(+)}(U) = \frac{1}{P+\delta U-\bar{f}_1}\left\{P\psi^{(+)}(0) + \sum_{k=1}^{U-1}\psi^{(+)}(U-k)\bar{f}_{k+1}\right\}$$

$$\bar{f}_k = \int_{k-1}^{k}[\delta + \lambda(1-F(y))]dy$$

The recursive calculation of $\psi^{(-)}(U)$ requires a starting value for $\psi^{(-)}(0)$. Sundt and Teugels do not provide such a value (and hence do not calculate bounds), but it is clear that this can be estimated by defining $\psi_0^{(-)}(U)$ to be equal to $\psi^{(-)}(U)/\psi^{(-)}(0)$ and then proceeding exactly as described in Section 31 above. The same procedure can be used to estimate $\psi^{(+)}(U)$. We will then approximate $\psi(U)$ as

$$\tfrac{1}{2}\left(\psi^{(-)}(U) + \psi^{(+)}(U)\right)$$

Such an approach works well when $\delta = 0$. (See, for example, Dickson et al (1995).)

34. Numerical examples

34 1. De Vylder's Method

Let us assume that $F(x) = 1 - \exp(-x/\mu)$, $x > 0$. Suppose we wish to estimate $\psi(U)$ when $\mu = 1$. We simply achieve this by rescaling our process, i.e. setting $\mu = 1/b > 1$

and calculating $\tilde{A}(\mathbf{b}U)$. Table 9 shows calculated values of $\tilde{A}(U)$ when $\lambda = 100$, $\mu = 0.1$ and $\delta = 0.01$, for different values of $\mathbf{b}$. In the final column we also give the exact value of $\tilde{A}(U)$, due to Segerdahl (1942). (See also Sundt and Teugels (1995).)

Table 9: Approximations to $\tilde{A}(U)$ - exponential claims

U	$\mathbf{b}$ = 20	$\mathbf{b}$ = 50	$\mathbf{b}$ = 100	$\mathbf{b}$ = 200	$\mathbf{b}$ = 400	Exact
0	0.0960	0.0935	0.0926	0.0922	0.0920	0.0918
10	0.6483	0.6428	0.6410	0.6401	0.6396	0.6391
20	0.8643	0.8604	0.8591	0.8584	0.8581	0.8578
30	0.9481	0.9459	0.9452	0.9448	0.9446	0.9444
40	0.9803	0.9792	0.9788	0.9786	0.9785	0.9784
50	0.9926	0.9921	0.9919	0.9918	0.9918	0.9917

From these calculations we observe that with a large value of $\mathbf{b}$ the method produces good approximations to $\tilde{A}(U)$, particularly for large U. However, for small values of $\mathbf{b}$ the approximations are not as good. As we shall see, this is in contrast to the other two methods which require only a small value of $\mathbf{b}$ in order to produce good approximations. In performing these calculations we have to select a point at which we assume $\tilde{A}_0(U)$ has converged. In this example we always chose this point such that $\tilde{A}_0(150\mathbf{b}) = 1$. Experiments with different stopping points in the recursive calculation did not produce significantly different numerical values.

As a second illustration of this method, let us consider the situation when $F(x) = 1 - (1+x)^{-2}$ (i.e. a Pareto(2,1) distribution), $\lambda = 1$, $\mu = 0.1$ and $\delta = 0.01$. Table 10 shows calculated values of $\tilde{A}(U)$ (we again use rescaling, with $\mathbf{b} = 100$) for different stopping points, denoted $W\mathbf{b}$, in the recursive calculation. These points are such that $\tilde{A}_0(W\mathbf{b}) = 1$.

Table 10: Approximations to $\tilde{A}(U)$ - Pareto claims

U	150 $\mathbf{b}$	300 $\mathbf{b}$	450 $\mathbf{b}$	600 $\mathbf{b}$
0	0.1837	0.1834	0.1833	0.1833
5	0.5394	0.5384	0.5382	0.5381
10	0.7018	0.7006	0.7003	0.7003
15	0.8002	0.7987	0.7985	0.7984
20	0.8626	0.8611	0.8608	0.8607
25	0.9033	0.9016	0.9014	0.9013

The main message from Table 10 is that this method of calculating $\tilde{A}(U)$ can be very sensitive to the choice of W. While the method can safely be applied to individual claim amount distributions for which the adjustment coefficient exists, we would suggest that great care must be exercised in selecting the stopping point for the recursive calculation in the case when the adjustment coefficient does not exist. We remark now that this is also a feature of the other methods of calculating $\tilde{A}(U)$.

3.4.2 Modified de Vylder Method and Sundt and Teugels' Method

We will show the results of these two methods together as they are very similar. Again we use $F(x) = 1 - \exp(-x/\mu)$, $x > 0$, as our illustration with $\lambda = 100$, $\theta = 0.1$ and varying δ. Table 11 shows approximations to $\psi(U)$ (with obvious notation), in each case with the scaling factor $\beta = 20$ and $\psi(150\beta) = 1$. Exact values are also shown.

We can see from Table 11 that each method produces very good approximations to $\psi(U)$. In terms of computed values there is little to choose between the two methods. The only real difference we experienced was in computational time. The Modified de Vylder method is slower (for reasons mentioned earlier) but is not prohibitively slow. There is therefore no reason to prefer the apparent computational convenience of de Vylder's method, which in any case needs a very large value of β to compete with these methods. We also experimented with other stopping points for the recursions for these methods, but the calculated values were once again virtually unaffected.

Table 11: Approximations to $\psi(U)$ - exponential claims

U	Method	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.04$	$\delta = 0.05$
0	S&T	0.0918	0.0927	0.0935	0.0943	0.0951
	MdV	0.0918	0.0927	0.0935	0.0943	0.0951
	Exact	0.0918	0.0927	0.0935	0.0943	0.0951
5	S&T	0.4270	0.4308	0.4345	0.4380	0.4414
	MdV	0.4269	0.4307	0.4344	0.4379	0.4413
	Exact	0.4269	0.4307	0.4344	0.4379	0.4413
10	S&T	0.6392	0.6443	0.6492	0.6540	0.6585
	MdV	0.6391	0.6442	0.6492	0.6539	0.6585
	Exact	0.6391	0.6443	0.6492	0.6539	0.6585
15	S&T	0.7732	0.7786	0.7837	0.7886	0.7933
	MdV	0.7732	0.7786	0.7837	0.7886	0.7932
	Exact	0.7732	0.7786	0.7837	0.7886	0.7933
20	S&T	0.8577	0.8627	0.8674	0.8719	0.8760
	MdV	0.8577	0.8627	0.8674	0.8719	0.8761
	Exact	0.8578	0.8628	0.8675	0.8719	0.8761
25	S&T	0.9109	0.9152	0.9192	0.9229	0.9264
	MdV	0.9110	0.9153	0.9192	0.9230	0.9264
	Exact	0.9110	0.9153	0.9193	0.9230	0.9264

Our conclusion is that either of these methods can easily be applied to compute $\psi(U)$ when the individual claim amount distribution is such that the adjustment coefficient exists.

An interesting point to note about the results in Table 11 is that the Modified de Vylder method seems to work very well, and at least as well as Sundt and Teugels' method, despite the fact that the former assumes claims are paid at the end of a time interval whereas the latter does not. This indicates that the corresponding assumption may not have had a significant effect on the numerical results in Tables 2 to 9.

4. Recovery from ruin

In this section we consider, for the same surplus process as in Sections 2 and 3, the distribution of the maximal negative surplus before recovery once ruin has occurred. More specifically, we derive an algorithm for the approximate calculation of the probability $\pi(-y; -z)$, where

$$\pi(-y; -z) = \Pr[\text{surplus reaches } 0 \text{ without going below } -z \mid \text{current surplus is -y}]$$

and where y and z are such that $-P/\delta < -z \leq -y < 0$. The probability $\pi(-y; -z)$ is the probability that the surplus recovers to 0 without going below $-z$, given that ruin has occurred and that the current surplus is $-y$. Note that if the surplus reaches $-P/\delta$, recovery is impossible since the interest payments exceed the premium income. The calculation of $\pi(-y; -z)$ is a generalisation of a problem studied by Picard (1994), the generalisation being that our model incorporates interest on the surplus.

We follow closely the methodology of Section 3.2. For any positive integer x, where $-P/\delta < -x$, recall that $\tau(-x)$ is the interval of time in which, if there were no claims, the surplus would increase from $-x$ to $-x+1$. Recall also that $h_k(-x)$ is the probability that, in a time interval of length $\tau(-x)$, the discretised aggregate claim amount is equal to k.

Let us now assume that y and z are integers. Conditioning on the aggregate claim amount in a time interval of length $\tau(-x)$, and assuming as in the previous sections that claims occurring in this time period are paid at the end of the time period, we have for $x = z, z-1, \ldots, 1$, we can write as an approximation

$$\pi(-x; -z) = \sum_{k=0}^{z-x} h_k(-x)\pi(-x+1-k; -z) \qquad (4.1)$$

Note that in the above formula we do not allow the surplus to fall to $-z$ even though the current surplus, $-x$, could be equal to $-z$. This is consistent with our treatment of 0 in the previous two sections: a surplus of 0 means "ruin" unless it is the initial surplus.

To solve (4.1), define for $x = 0, 1, \ldots, z$

$$\pi_0(-x; -z) = \pi(-x; -z)/\pi(-z; -z)$$

Then $\pi_0(-x; -z)$ satisfies (4.1) and $\pi_0(-z; -z) = 1$. Hence (4.1) can be used recursively to calculate $\pi_0(-z+1; -z), \pi_0(-z+2; -z), \ldots, \pi_0(0; -z)$. However

$$\begin{aligned} \pi_0(0; -z) &= \pi(0; -z)/\pi(-z; -z) \\ &= 1/\pi(-z; -z) \end{aligned}$$

Hence, we can calculate $\pi(-z; -z)$ and then $\pi(-x; -z)$ for $x = z - 1, z - 2, \ldots, 1$.

This is an approximate method since not only is it based on a discretisation of the individual claim amount distribution, but it also ignores what happens within the interval of time $\tau(-x)$ and considers only what happens at the end of this time interval. From the definition of $\tau(-x)$, it can be seen that this time interval is an increasing function of x and approaches ∞ as x approaches c/δ.

As an example of the application of this algorithm, let us consider the situation from the previous section where $F(x) = 1 - \exp(-x)$, $x > 0$, $\lambda = 100$ and $\theta = 0.1$. We again make use of a scaling factor, taken as $\beta = 20$. Table 12 shows values of $\pi(-y; 50)$ as a function of δ. We cannot compare these values with any exact results. We simply comment that $\pi(-y; -z)$ is a decreasing function of δ and a decreasing function of y. The results in Table 12 are consistent with these observations. With these parameter values the largest time period was $\tau(-50)$ which equalled 0.000465 when $\delta = 0.05$.

Table 12: Values of $\pi(-y; -z)$ - exponential claims

y	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.04$	$\delta = 0.05$
5	0.9940	0.9936	0.9931	0.9927	0.9921
10	0.9845	0.9835	0.9824	0.9812	0.9799
15	0.9697	0.9677	0.9656	0.9633	0.9609
20	0.9465	0.9431	0.9396	0.9359	0.9319
25	0.9102	0.9051	0.8996	0.8940	0.8880
30	0.8536	0.8463	0.8386	0.8306	0.8222
35	0.7655	0.7558	0.7458	0.7354	0.7247
40	0.6287	0.6173	0.6057	0.5938	0.5817
45	0.4166	0.4061	0.3955	0.3848	0.3741
50	0.0886	0.0854	0.0823	0.0792	0.0761

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5. References

De Vylder, F. (1996) Advanced Risk Theory. Editions de l'Université de Bruxelles.

De Vylder, F. and Goovaerts, M. J. (1988) Recursive calculation of finite time survival probabilities. Insurance: Mathematics and Economics 7, 1-8.

Dickson, D.C.M. and Waters, H.R. (1991) Recursive calculation of survival probabilities. ASTIN Bulletin 21, 199-221.

Dickson, D.C.M., Egídio dos Reis, A.D. and Waters, H.R. (1995) Some stable algorithms in ruin theory and their applications. ASTIN Bulletin 25, 153-175.

Panjer, H.H. (1981) Recursive calculation of a family of compound distributions. ASTIN Bulletin 12, 22-26.

Picard, P. (1994) On some measures of the severity of ruin in the classical Poisson model. Insurance: Mathematics and Economics 14, 107-115.

Seal, H.L. (1978) Survival Probabilities. John Wiley and Sons, New York.

Segerdahl, C-O. (1942) Über einige risikotheoretische Fragestellungen. Skandinavisk Aktuaritidskrift 61, 43-83.

Sundt, B. and Teugels, J.L. (1995) Ruin estimates under interest force. Insurance: Mathematics and Economics 16, 7-22.