

# LAGUERRE SERIES FOR ASIAN AND OTHER OPTIONS

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## Abstract

This paper has four goals: (a) relate ladder height distributions to option values; (b) show how Laguerre expansions may be used in the computation of densities, distribution functions and option prices; (c) derive some new results on the integral of geometric Brownian motion over a finite interval; (d) apply the preceding results to the determination of the distribution of the integral of geometric Brownian motion and the computation of Asian option values. The usual fixed-strike options on the average are treated, as well as options with payoffs expressed in terms of one over the average of the underlying security, which this author calls “reciprocal Asian options.” In all cases the underlying asset is represented by geometric Brownian motion, the averages are performed continuously, and the options are of European type.

OPTION PRICING; ASIAN OPTION; RECIPROCAL ASIAN OPTION; EXPONENTIAL FUNCTIONAL OF BROWNIAN MOTION

## 1. Introduction

Asian options have payoffs which depend on the average value of the underlying asset over a certain time period. A number of papers have appeared, which propose various methods for pricing these options. We give a partial list of those papers. A first class of papers suggest some mathematical approximation to the distribution of the average; these methods were presented in the following papers: Turnbull and Wakeman (1991), Levy (1992), Ritchken *et al.* (1993), Bouaziz *et al.* (1994), Zhang (1995), Neave (1997). A second class of papers relates to simulation, and ways to improve them in the case of options on the average: Kemna and Vorst (1990), Curran (1994), Boyle *et al.* (1997), Vázquez-Abad and Dufresne (1998), Fu *et al.* (1999). A third class of papers applies numerical partial differential equation techniques: Rogers and Shi (1995), Alziary *et al.* (1997). Finally, some authors have either derived or used exact (or approximate) integral transforms of Asian option values: Carverhill and Clewlow (1990), De Schepper *et al.* (1992), Geman and Yor (1993), Fu *et al.* (1999). Yor has written several papers on integral functionals of Brownian motion, with various and important results concerning the integral of Brownian motion. We only cite Yor (1992a, 1992b, 1992c) and Carmina *et al.* (1997) (see the latter for other references).

This paper has four goals: (a) relate ladder height distributions to option values;

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(b) show how Laguerre expansions may be used in the computation of densities, distribution functions and option prices; (c) derive some new results on the integral of geometric Brownian motion over a finite interval; (d) apply the preceding results to the determination of the distribution of the integral of geometric Brownian motion and the computation of Asian option values. The usual fixed-strike options on the average are treated, as well as options with payoffs expressed in terms of one over the average of the underlying security, which this authors calls “reciprocal Asian options.” In all cases the underlying asset is represented by geometric Brownian motion, the averages are performed continuously, and the options are of European type.

The first part of the paper describes a general method for obtaining exact series expansions for distributions or option values, based on the generalized Laguerre polynomials (also called Laguerre-Sonine polynomials, as Laguerre defined the family  $\{L_n(x)\} = \{L_n^0(x)\}$  in 1879, while it is Sonine who defined the more general families  $\{L_n^a(x)\}$  for  $a > -1$  in 1880.) This method is far from new in applied mathematics, but does not seem to have been used often in financial applications. The result is a series of functions, which approximates the quantity chosen (density, distribution function of option value) when a finite number of terms is used. It will be seen that the method is directly applicable in situations where the Laplace transform of the distribution of the underlying asset (or some simple function of it) is an analytic function in a neighbourhood of the origin. The Laguerre series obtained then converge to the true density function, distribution function or option value.

The requirement regarding the Laplace transform would appear to exclude Asian options with an underlying asset which is geometric Brownian motion, as in this case the Laplace transform of the average is infinite for negative arguments. The trick we use is to deal with one over the average, which we call the *reciprocal average*; it turns out that the Laplace transform of the reciprocal average is analytic in a neighbourhood of the origin, so that an exact series for its distribution can be found. A series expression for the distribution of the average itself can then be derived, and the same is done for options on the average. Dealing with the reciprocal average has the added benefit of pricing options with payoffs depending on one over the average.

We consider the classical Black and Scholes model, with a single risky asset, represented by a geometric Brownian motion, and a risk-free asset with constant rate of interest. As Geman & Yor (1993) point out, the problem of pricing European-type options on the average may be reduced to computing the function

$$x \mapsto \mathbb{E} \left[ 2A_t^{(\mu)} - x \right]_+, \quad (1)$$

with

$$A_t^{(\mu)} = \int_0^t e^{2\mu s + 2W_s} ds, \quad t \geq 0, \quad \mu \in \mathbb{R}. \quad (2)$$

Here  $W$  is standard Brownian motion, and  $\mu, t$  are normalizations, respectively, of the risk-free interest rate and the length of time over which the average is computed; this normalization is an immediate consequence of the scaling property of Brownian motion, and effectively expresses any average option in a “canonical” setting, where the volatility is 2. (The mysterious “2” put in front of  $A$  above serves to simplify some of the formulas, as will be seen below.) Similarly, pricing options with payoffs expressed in terms of the reciprocal average reduces to evaluating the function

$$x \mapsto \mathbf{E} \left[ \frac{1}{2A_t^{(\mu)}} - x \right]_+ . \quad (3)$$

Section 2 describes how Laguerre polynomials may be used to compute probability density functions, distribution functions, and option values for a distribution satisfying certain conditions; ladder height distributions provide simple relationships between the distribution of a random variable  $X$  and the functions  $\mathbf{E}(X - x)_+$  and  $\mathbf{E}(1/X - x)_+$ . Section 3 is a proof that the Laplace transform of  $1/(2A_t^{(\mu)})$  exists for negative arguments, which implies (i) that the Laguerre series used later converge to the exact values, and (ii) that the distribution of  $1/(2A_t^{(\mu)})$  is determined by its moments. Section 4 calculates the moments of  $1/(2A_t^{(\mu)})$ . Section 5 applies the previous results to the determination of the probability density function of  $2A_t^{(\mu)}$ . The pricing of reciprocal Asian options is done in Section 6, while Section 7 deals with ordinary Asian options. Sections 5, 6, 7 contain numerical examples, which compare analytical and simulation results.

We recall some known facts.

**Theorem A.** (Bougerol, 1983) *Let  $\{V_t, W_t\}$  be two-dimensional standard Brownian motion starting at 0. Then*

$$\int_0^t e^{V_s} dW_s \stackrel{\mathcal{L}}{=} \sinh(V_t),$$

where “ $\stackrel{\mathcal{L}}{=}$ ” means “has the same distribution as”, and  $\sinh(x) = (e^x - e^{-x})/2$ .

**Theorem B.** (Dufresne, 1989) *Let  $\alpha_k = k\mu + k^2$ . For  $n = 0, 1, 2, \dots$ ,*

$$\mathbf{E}(2A_t^{(\mu)})^n = n! \sum_{k=0}^n e^{2\alpha_k t} \left[ \prod_{\substack{j=0 \\ j \neq k}}^n (\alpha_k - \alpha_j) \right]^{-1}.$$

(In the event that  $\alpha_j = \alpha_k$  for some  $j \neq k$ , this expression has to be modified, as terms of the form  $te^{\alpha_j t}$  make their appearance; see the original paper for details.)

**Theorem C.** (Dufresne, 1990) *For any  $\mu > 0$ ,  $\frac{1}{2A_\infty^{(-\mu)}} \sim \text{Gamma}(\mu, 1)$ .*

**Theorem D.** (Yor, 1992c) *Let  $T_\lambda$  be an exponentially distributed random variable, independent of  $W$ , with mean  $1/\lambda$ . Then*

$$2A_{T_\lambda}^{(\mu)} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta},$$

where  $\alpha = \frac{\mu}{2} + \frac{1}{2}\sqrt{2\lambda + \mu^2}$ ,  $\beta = \alpha - \mu$ , and the random variables  $B_{1,\alpha} \sim \text{Beta}(1, \alpha)$  and  $G_\beta \sim \text{Gamma}(\beta, 1)$  are independent.

Theorem A is used in the proof of Theorem 6 in Section 3. It is not known whether the moments of  $2A_t^{(\mu)}$  (Theorem B) determine uniquely its distribution. The similarity of the moments with those of the lognormal distribution, which is not determined by its moments, gives the intuition that the same might hold for  $2A_t^{(\mu)}$ . The limit distribution in Theorem C has probably little value for option pricing, as the transformed (“canonical”) time duration of options is usually rather small (see Section 5). However, the result does lead to the idea that the distribution  $1/(2A_t^{(\mu)})$  could be expressed as a combination of gamma distributions, which is precisely what is done in this paper. Theorem D is a double transform. An equivalent formula is:

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \mathbb{E}(2A_t^{(\mu)} - q)_+ dt &= \frac{1}{\lambda} \mathbb{E}(2A_{T_\lambda}^{(\mu)} - q)_+ = \frac{1}{\lambda} \mathbb{E}\left(\frac{B_{1,\alpha}}{G_\beta} - q\right)_+ \\ &= \frac{1}{\lambda} \int_0^1 \int_0^{u/q} \left(\frac{u}{x} - q\right) \frac{x^{\beta-1}}{\Gamma(\beta)} e^{-x} dx d_u (1-u)^\alpha \\ &= \frac{1}{\lambda} \int_0^1 \int_0^{u/q} \frac{x^{\beta-2}}{\Gamma(\beta)} e^{-x} dx d_u \frac{(1-u)^{\alpha+1}}{\alpha+1} \\ &= \frac{1}{\lambda(\alpha+1)\Gamma(\beta)} \int_0^1 u^{\beta-2} (1-u)^{\alpha+1} e^{-u/q} du. \end{aligned}$$

This is the formula derived in Geman & Yor (1993) for the Laplace transform of European call options on the average. This formula is not used in this paper, as we prefer to apply Theorem D to obtain the density, then use ladder height arguments (Section 2) to obtain option values.

The following abbreviations are used: “p.d.f.” for “probability density function,” “d.f.” for “distribution function,” and “Leb” for “Lebedev (1972)”.

## 2. Laguerre expansions for densities, distribution functions and options

For each  $n = 0, 1, 2, \dots$ ,  $a > -1$ , the generalized Laguerre polynomials are defined as (Leb, pp. 76-77)

$$L_n^a(x) = \frac{x^{-a}}{n!} e^x \frac{d}{dx^n} (x^{n+a} e^{-x}) = \sum_{k=0}^n \frac{\Gamma(n+a+1)}{\Gamma(k+a+1)} \frac{(-x)^k}{k!(n-k)!}.$$

The polynomials  $\{L_n^a(x); n = 0, 1, \dots\}$  are orthogonal on  $\mathbb{R}_+$ , with respect to the weight function  $x^a e^{-x}$ .

The following theorem may be found in Leb, p. 88. (A function is said to be “smooth” if it has a continuous first derivative; it is “piecewise smooth” if, in each finite interval  $[a, b]$ , it is smooth except for a finite number of points, where the function or its derivative may be discontinuous, but the right and left limits exist and are finite.)

**Theorem E.** *Suppose  $g : (0, \infty) \mapsto \mathbb{R}$  is piecewise smooth, and that*

$$\int_0^\infty x^a e^{-x} g^2(x) dx < \infty. \quad (4)$$

Then

$$\frac{1}{2}[g(x-) + g(x+)] = \sum_{n=0}^{\infty} c_n L_n^a(x), \quad 0 < x < \infty,$$

with

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^\infty x^a e^{-x} g(x) L_n^a(x) dx.$$

Consider a non-negative random variable  $X$ , and denote  $F_X(x) = \mathbf{P}(X \leq x)$  its d.f., and  $f_X(x) = dF_X(x)/dx$  its p.d.f., if it exists. There is a neat relationship between the so-called “ladder height distributions” (which is of importance in queueing theory and in risk theory) and call options; this relationship is useful in deriving the Laguerre expansions below. The ladder height p.d.f. and d.f. associated with  $X$  are defined as

$$\bar{f}_X(x) = \frac{\mathbf{P}(X > x)}{\mathbf{E} X}, \quad \bar{F}_X(x) = \int_0^x \bar{f}_X(y) dy,$$

(where we implicitly assume that  $0 < \mathbf{E} X < \infty$ ). We denote  $\bar{X}$  a random variable with p.d.f.  $\bar{f}_X$ , and  $\overline{\overline{X}}$  one with the “iterated” ladder height p.d.f.  $\overline{\overline{f}}_X$ .

**Theorem 1.** *If the indicated moments of  $X$  exist,*

$$\begin{aligned} \mathbf{E}(X-x)_+ &= (\mathbf{E} X^2/2) \bar{f}_X(x) \\ \mathbf{E} \bar{X}^p &= \frac{\mathbf{E} X^{p+1}}{(p+1)\mathbf{E} X}, & p > -1 \\ \mathbf{E} \overline{\overline{X}}^p &= \frac{2\mathbf{E} X^{p+2}}{(p+1)(p+2)\mathbf{E} X^2}, & p > -1. \end{aligned}$$

**Proof.** We have

$$\begin{aligned} \mathbb{E} X \mathbb{E} \bar{X}^p &= \int_0^\infty x^p \int_x^\infty dF_X(y) dx = \int_0^\infty \int_0^y x^p dx dF_X(y) = \frac{\mathbb{E} X^{p+1}}{(p+1)} \\ \mathbb{E}(X-x)_+ &= \int_x^\infty (y-x) dF_X(y) = - \int_x^\infty (y-x) d(1-F_X(y)) \\ &= \int_x^\infty (1-F_X(y)) dy = \mathbb{E} X \int_x^\infty \bar{f}(y) dy \\ &= \mathbb{E} X(1-\bar{F}_X(x)) = \mathbb{E} X \mathbb{E} \bar{X} \bar{f}_X(x) \\ &= (\mathbb{E} X^2/2) \bar{f}_X(x). \quad \square \end{aligned}$$

Thus the Laguerre expansion for the d.f.  $F_X$  and call function  $\mathbb{E}(X-x)_+$  can be found in the same way as the expansion for p.d.f.'s (Theorem 2 below). Observe that ladder height distributions are always absolutely continuous, and that the iterated ladder height probability density function  $\bar{f}_X$  always has a continuous first derivative.

**Theorem 2. (Laguerre expansion for p.d.f.'s)** *Let  $a > -1$ ;  $b, d \in \mathbb{R}$ . Suppose an absolutely continuous random variable  $X$  has a distribution over  $\mathbb{R}_+$  which satisfies the conditions:*

- (i)  $f_X(x)$  is piecewise smooth and uniformly bounded,
- (ii)  $\mathbb{E} X^{a-2b} e^{(2d-1)X} < \infty$ ;

(Condition (ii) is satisfied if  $\mathbb{E} X^{a-2b} < \infty$  and there exists  $\delta > 2d-1$  such that  $\mathbb{E} e^{\delta X} < \infty$ .) Then  $f_X(x)$  has the converging expansion

$$\frac{1}{2}[f_X(x-) + f_X(x+)] = x^b e^{-dx} \sum_{n=0}^{\infty} c_n L_n^a(x), \quad 0 < x < \infty, \quad (5)$$

with

$$\begin{aligned} c_n &= \frac{n!}{\Gamma(n+a+1)} \mathbb{E} X^{a-b} e^{(d-1)X} L_n^a(X) \\ &= \sum_{k=0}^n \frac{n!(-1)^k}{\Gamma(k+a+1)k!(n-k)!} \mathbb{E} X^{a-b+k} e^{(d-1)X}, \quad n = 0, 1, \dots \end{aligned} \quad (6)$$

**Proof.** Let  $g(x) = x^{-b} e^{dx} f_X(x)$  in Theorem E. Then

$$\begin{aligned} \int_0^\infty x^a e^{-x} g^2(x) dx &= \int_0^\infty x^{a-2b} e^{(2d-1)x} f(x)^2 dx \\ &\leq \max(f) \int_0^\infty x^{a-2b} e^{(2d-1)x} f(x) dx < \infty. \quad \square \end{aligned}$$

From now on we concentrate on the case  $d = 1$  in (5) and (6), that is, on expansions of the form

$$h(x) = x^b e^{-x} \sum_{n=0}^{\infty} c_n L_n^a(x),$$

where  $h(x)$  is either the d.f., the put or the call option function associated with a random variable  $X$ .

**Theorem 3. (Laguerre expansion for d.f.'s)** *Let  $a, b$  be real numbers such that  $a > 2b_+ - 1$ , and suppose  $X$  is a non-negative random variable with  $\mathbf{E} \exp(dX) < \infty$  for some  $d > 1/2$ , with a piecewise smooth distribution function  $F_X(x)$ . Then*

$$\begin{aligned} 1 - \frac{1}{2}[F_X(x-) + F_X(x+)] &= x^b e^{-x} \sum_{n=0}^{\infty} c_n L_n^a(x), \quad 0 < x < \infty, \\ c_n &= \frac{n! \mathbf{E} X}{\Gamma(n+a+1)} \mathbf{E} \bar{X}^{a-b} L_n^a(\bar{X}) \\ &= \sum_{k=0}^n \frac{n!(-1)^k}{\Gamma(k+a+1)k!(n-k)!} \frac{\mathbf{E} X^{a-b+k+1}}{(a-b+k+1)}, \quad n = 0, 1, \dots \end{aligned} \quad (7)$$

**Proof.** Let  $g(x) = x^{-b} e^x \bar{f}_X(x) (\mathbf{E} X)$  in the proof of Theorem 2. By Markov's inequality,

$$\bar{f}_X(x) (\mathbf{E} X) = \mathbf{P}(X > x) = \mathbf{P}(e^{dX} > e^{dx}) \leq \mathbf{E} e^{dX} e^{-dx}.$$

If  $a - 2b > -1$ , then

$$\int_0^{\infty} x^a e^{-x} g^2(x) dx = \int_0^{\infty} x^{a-2b} e^x \bar{f}_X(x)^2 (\mathbf{E} X)^2 dx \leq \mathbf{E} e^{dX} \int_0^{\infty} x^{a-2b} e^{(1-2d)x} dx.$$

The constants  $\{c_n; n = 0, 1, \dots\}$  follow from Theorem E. □

**Theorem 4. (Laguerre expansion for options on X)** *Let  $a, b$  be real numbers such that  $a > 2b_+ - 1$ , and suppose  $X$  is a non-negative random variable with  $\mathbf{E} \exp(dX) < \infty$  for some  $d > 1/2$  and a piecewise continuous d.f.. Then*

$$\begin{aligned} \mathbf{E}(X-x)_+ &= x^b e^{-x} \sum_{n=0}^{\infty} c_n L_n^a(x), \quad 0 < x < \infty, \\ c_n &= \frac{n! (\mathbf{E} X^2 / 2)}{\Gamma(n+a+1)} \mathbf{E} \bar{X}^{a-b} L_n^a(\bar{X}) \\ &= \sum_{k=0}^n \frac{n!(-1)^k}{\Gamma(k+a+1)k!(n-k)!} \frac{\mathbf{E} X^{a-b+k+2}}{(a-b+k+1)(a-b+k+2)}, \quad n = 0, 1, \dots \end{aligned} \quad (8)$$

**Proof.** Let  $g(x) = x^{-b}e^x\mathbf{E}(X-x)_+ = x^{-b}e^x\overline{\overline{f}}_X(x)(\mathbf{E}X^2/2)$  in the proof of Theorem 2. We have

$$\begin{aligned}\mathbf{E}e^{d\overline{X}} &= \frac{1}{\mathbf{E}X} \int_0^\infty e^{dx} \int_x^\infty dF_X(y) dx \\ &= \int_0^\infty \left( \frac{e^{dy} - 1}{d\mathbf{E}X} \right) dF_X(y) dx = \frac{\mathbf{E}e^{dX} - 1}{d\mathbf{E}X} < \infty.\end{aligned}$$

By Markov's inequality,

$$\overline{\overline{f}}_X(x)(\mathbf{E}\overline{X}) = \overline{\overline{f}}_X(x)\mathbf{E}X^2/(2\mathbf{E}X) = \mathbf{P}(\overline{X} > x) = \mathbf{P}(e^{d\overline{X}} > e^{dx}) \leq \mathbf{E}e^{d\overline{X}}e^{-dx},$$

and the proof is finished as for Theorem 3.  $\square$

In Section 5 we consider options on  $1/X$ , where the the distribution of  $X = 1/(2A_t^{(\mu)})$  can be expressed as a Laguerre series. The p.d.f. and d.f. of  $1/X$  pose no problem, but for calls or puts, the following result is used.

**Theorem 5. (Laguerre expansion for options on  $1/X$ ).** *Let  $a, b$  be real numbers such that  $a > 2b_+ - 1$ , and let  $X$  be a positive random variable with  $\mathbf{E}1/X < \infty$  and  $\mathbf{E}e^{dX} < \infty$  for some  $d > 1/2$ . Let  $\widehat{X}$  be a random variable with p.d.f.*

$$\widehat{f}_X(x) = \frac{f_X(x)}{x\mathbf{E}(X^{-1})}.$$

Then

$$\mathbf{E}\left(x - \frac{1}{X}\right)_+ = x^{-b+1}e^{-1/x} \sum_{n=0}^{\infty} c_n L_n^a(1/x), \quad 0 < x < \infty,$$

where, for  $n = 0, 1, \dots$ ,

$$\begin{aligned}c_n &= \frac{n!\mathbf{E}(X^{-1})(\mathbf{E}\widehat{X}^2/2)}{\Gamma(n+a+1)} \overline{\overline{\overline{X}}}^{a-b} L_n^a(\overline{\overline{\overline{X}}}) \\ &= \sum_{k=0}^n \frac{n!(-1)^k}{\Gamma(k+a+1)k!(n-k)!} \frac{\mathbf{E}X^{a-b+k+1}}{(a-b+k+1)(a-b+k+2)}.\end{aligned}$$

**Proof.** Since  $\mathbf{E}1/X < \infty$ ,  $\widehat{f}_X$  qualifies as a p.d.f. Observe that  $\mathbf{E}e^{dX} < \infty$  implies  $\mathbf{E}e^{d\widehat{X}} < \infty$ . We get

$$\begin{aligned}\mathbf{E}\left(x - \frac{1}{X}\right)_+ &= x\mathbf{E}\frac{1}{X}\left(X - \frac{1}{x}\right)_+ \\ &= x \int_{1/x}^\infty \frac{f_X(y)}{y} \left(y - \frac{1}{x}\right) dy\end{aligned}$$



$$\begin{aligned}
&= x\mathbf{E}(X^{-1}) \int_{1/x}^{\infty} \widehat{f}_X(y) \left(y - \frac{1}{x}\right) dy \\
&= x\mathbf{E}(X^{-1})\mathbf{E}\left(\widehat{X} - \frac{1}{x}\right)_+ \\
&= x^{-b+1}e^{-1/x} \sum_{n=0}^{\infty} c_n L_n^a(1/x), \quad 0 < x < \infty,
\end{aligned}$$

from Theorem 4. The expression for  $c_n$  results from  $\mathbf{E}\widehat{X}^p = \frac{\mathbf{E}X^{p-1}}{\mathbf{E}X^{-1}}$ .  $\square$

### 3. The law of $1/(2A_t^{(\mu)})$ is determined by its moments

**Theorem 6.** Let  $\mu_+ = \max(\mu, 0)$ ,  $\mu_- = \max(-\mu, 0)$ . For any  $\mu, p \in \mathbb{R}$ ,  $t > 0$ ,  $\mathbf{E}(A_t^{(\mu)})^p < \infty$ . Also,  $\mathbf{E} \exp(s/(2A_t^{(\mu)})) < \infty$  for  $s < e^{-\mu-t}$ , and  $\mathbf{E} \exp(s/(2A_t^{(\mu)})) = \infty$  for  $s \geq e^{t\mu_+}$ . The distribution of  $1/(2A_t^{(\mu)})$  is determined by its moments.

**Proof.** The moments of  $A_t^{(\mu)}$  are all finite, since

$$t \exp(-2t\mu_- + 2 \min_{s \in [0,t]} W_s) \leq A_t^{(\mu)} \leq t \exp(2t\mu_+ + 2 \max_{s \in [0,t]} W_s),$$

and  $-\min_{s \in [0,t]} W_s \stackrel{\mathcal{L}}{=} \max_{s \in [0,t]} W_s \stackrel{\mathcal{L}}{=} |W_t|$  (Revuz & Yor, 1991, p.100). First, let  $\mu = 0$ ,  $A_t = A_t^{(0)}$ , and let  $\{V_t, W_t\}$  be two-dimensional standard Brownian motion starting at 0. Bougerol's identity (Theorem A) has the equivalent form (Yor, 1992a)

$$\mathbf{E} \frac{e^{-s/(2A_t)}}{\sqrt{2A_t}} = \frac{\exp\{-[\operatorname{arcsinh}(\sqrt{s})]^2/2t\}}{\sqrt{2t(1+s)}}, \quad s > 0. \quad (9)$$

The exponent may be rewritten

$$-[\operatorname{arcsinh}(\sqrt{s})]^2/(2t) = -s {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -s\right)^2/(2t)$$

where  ${}_2F_1$  denotes the Gauss hypergeometric function (Leb, p.238, Eq. (9.8.5)). The function on the right is analytic in  $\{|s| < 1\}$ . Thus, the following expansion has to hold for  $|s| < 1$ :

$$\frac{\exp\{-s {}_2F_1(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -s)^2/2t\}}{\sqrt{2t(1+s)}} = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \mathbf{E}(2A_t)^{-k-\frac{1}{2}}$$

For each fixed  $t > 0$ , there is  $n_0 < \infty$  such that  $\mathbf{E}(2A_t)^{-n_0} > 1$ , and, from Hölder's inequality,

$$\mathbf{E}(2A_t)^{-n} \leq \mathbf{E}(2A_t)^{-n-\frac{1}{2}}, \quad n \geq n_0.$$

If  $s \geq 1$ , then  $\mathbb{E} \exp(s/(2A_t)) = \infty$ , as  $\mathbb{E} (2A_t)^{-n} \geq \mathbb{E} (2A_t)^{-n+\frac{1}{2}}$  for all  $n$  large enough, and

$$\mathbb{E} \frac{e^{s/(2A_t)}}{\sqrt{2A_t}} = \infty, \quad s \geq 1,$$

by (9). Finally,  $\mathbb{E} \exp(s/(2A_t^{(\mu)})) < \infty$  for  $s < e^{-\mu-t}$ , and  $\mathbb{E} \exp(s/(2A_t^{(\mu)})) = \infty$  for  $s \geq e^{t\mu+}$  follow from

$$\frac{e^{-t\mu+}}{A_t^{(0)}} \leq \frac{1}{A_t^{(\mu)}} \leq \frac{e^{t\mu-}}{A_t^{(0)}}.$$

Hence, for any random variable  $Y$  with the same moments as  $1/2A_t^{(\mu)}$ , we can form the sum

$$\sum_{n=0}^N \frac{(-s)^n}{n!} \mathbb{E} Y^n = \sum_{n=0}^N \frac{(-s)^n}{n!} \mathbb{E} \left( \frac{1}{2A_t^{(\mu)}} \right)^n$$

which, by dominated convergence, tends to  $\mathbb{E} \exp(-sY)$  as  $N \rightarrow \infty$ , for any  $|s| < 1$ . Then  $\mathbb{E} \exp(-sY)$  and  $\mathbb{E} \exp(-s/(2A_t^{(\mu)}))$  are the same, since they are analytic where they exist, and coincide in  $\{0 \leq \operatorname{Re}(s) < 1\}$ .  $\square$

#### 4. The moments of $1/(2A_t^{(\mu)})$

The following results extend Theorem B.

**Theorem 7.** For  $\mu \in \mathbb{R}$  and  $\operatorname{Re}(r) > -3/2$ ,

$$\mathbb{E}(2A_t^{(\mu)})^r = \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}} \int_0^\infty y e^{-y^2/2t} \psi_\mu(r, y) dy \quad (10)$$

where

$$\psi_\mu(r, y) = \frac{\Gamma(1+r)}{\Gamma(2+2r)} e^{-\mu y} (1 - e^{-2y})^{1+2r} {}_2F_1(\mu + 1 + 2r, 1+r, 2+2r; 1 - e^{-2y}) \quad (11)$$

$$\psi_\mu(-1, y) = \frac{\cosh[(\mu - 1)y]}{\sinh(y)}. \quad (12)$$

**Proof.** Theorem (D) is equivalent to

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \mathbb{E}(2A_t^{(\mu)})^r dt &= \frac{1}{\lambda} \mathbb{E}(2A_{T_\lambda}^{(\mu)})^r = \frac{\Gamma(\alpha+1)\Gamma(\beta-r)\Gamma(1+r)}{\lambda\Gamma(\beta)\Gamma(\alpha+1+r)} \\ &= \frac{\alpha\beta\Gamma(\alpha)\Gamma(\beta-r)\Gamma(1+r)}{\lambda\Gamma(\beta+1)\Gamma(\alpha+1+r)} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta-r)\Gamma(1+r)}{2\Gamma(\beta+1)\Gamma(\alpha+1+r)}, \end{aligned} \quad (13)$$

for  $-1 < r < \beta$ . Defining  $\psi_\mu(r, y)$  as in (11), we find (Oberhettinger & Badii, 1973, p.308, Eq. (10.11))

$$\int_0^\infty e^{-py} \psi_\mu(r, y) dy = \frac{\Gamma(\frac{\mu}{2} + \frac{p}{2}) \Gamma(-\frac{\mu}{2} - r + \frac{p}{2}) \Gamma(1 + r)}{2\Gamma(-\frac{\mu}{2} + 1 + \frac{p}{2}) \Gamma(\frac{\mu}{2} + 1 + r + \frac{p}{2})}.$$

This is the right-hand side of Eq. (13) above, with  $\sqrt{2\lambda + \mu^2}$  replaced with  $p$ . From the formula

$$\int_0^\infty \frac{e^{-pt-a/2t}}{t^{3/2}} dt = \left(\frac{2\pi}{a}\right)^{1/2} e^{-(2ap)^{1/2}},$$

we get

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \left[ \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}} \int_0^\infty y e^{-y^2/2t} \psi_\mu(r, y) dy \right] dt \\ = \frac{\Gamma(\frac{\mu}{2} + \frac{1}{2}\sqrt{2\lambda + \mu^2}) \Gamma(-\frac{\mu}{2} - r + \frac{1}{2}\sqrt{2\lambda + \mu^2}) \Gamma(1 + r)}{2\Gamma(-\frac{\mu}{2} + 1 + \frac{1}{2}\sqrt{2\lambda + \mu^2}) \Gamma(\frac{\mu}{2} + 1 + r + \frac{1}{2}\sqrt{2\lambda + \mu^2})}, \end{aligned}$$

which proves (10) for  $r > -1$ . However,  $\psi_\mu(r, y)$  is an analytic function of  $r$  in  $\{\text{Re}(r) > -3/2\}$  (apply the duplication formula to  $\Gamma(2 + 2r)$  and Eq. (9.6.12), p. 253, in Leb). The integral on the right-hand side of (10) is finite if

$$\int_0^\epsilon y(1 - e^{-2y})^{1+2r} dy < \infty \quad \iff \quad \text{Re}(r) > -3/2$$

(this is seen from the behaviour of  $\psi_\mu(r, y)$  near  $y = 0$  and  $y = \infty$ ). A uniform convergence argument with respect to  $r$  then shows that the right-hand side of (10) is an analytic function of  $r$  in the region  $\{\text{Re}(r) > -3/2\}$ . The two sides of (10) represent analytic functions of  $r$ , which coincide when  $\text{Re}(r) > -1$ , and thus they have to agree in the larger region  $\{\text{Re}(r) > -3/2\}$ . To prove (12), use the formula (Leb, p.249)

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, 1+a+b-c; 1-z) \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b, 1-a-b+c; 1-z), \end{aligned}$$

which holds for  $|\arg z| < \pi$ ,  $|\arg(1-z)| < \pi$ ,  $\alpha + \beta - \gamma \notin \mathbb{Z}$ . Temporarily excluding the cases  $\mu \in \mathbb{Z}$ , we find that  $\psi_\mu(r, y)$  is the sum of two terms. From the analyticity of  ${}_2F_1(\alpha, \beta, \gamma; z)/\Gamma(\gamma)$  with respect to  $\alpha$ ,  $\beta$ , and  $\gamma$  (Leb, p.245), we conclude that the first term equals  $e^{-\mu y}(1 - e^{-2y})^{1+2r}$  times

$$\left. \frac{\Gamma(-r-\mu)}{\Gamma(1-\mu)} {}_2F_1(\mu+1+2r, 1+r, 1+\mu+r; e^{-2y}) \right|_{r=-1} = 1.$$

As to the second term, we find  $e^{-\mu y}(1 - e^{-2y})^{1+2r}$  times

$$\frac{\Gamma(\mu + r)}{\Gamma(\mu + 1 + 2r)} {}_2F_1(1 - \mu, 1 + r, 1 - \mu - r; e^{-2y}) \Big|_{r=-1} = (e^{-2y})^{1-\mu}.$$

Thus

$$\psi_\mu(-1, y) = e^{-y\mu}(1 - e^{-2y})^{-1}(1 + e^{2(\mu-1)y}) = \frac{\cosh[(\mu - 1)y]}{\sinh(y)}.$$

This proves (10) for  $\mu \notin \mathbb{Z}$ . The result holds for  $\mu \in \mathbb{Z}$  by continuity.  $\square$

For  $r \leq -3/2$ ,  $\mathbf{E}(2A_t^{(\mu)})^r$  may be found by applying Ito's formula. The Laguerre expansions in the next sections require those expectations for  $r = -1, -2, \dots$

**Theorem 8.** *The moments of  $1/(2A_t^{(\mu)})$  may be found recursively as follows:*

$$\begin{aligned} \mathbf{E}(2A_t^{(\mu)})^{-k} &= \int_0^\infty \phi_\mu(k, t, y) \psi_\mu(-1, y) dy, \quad k = 1, 2, \dots, \\ \phi_\mu(1, t, y) &= \frac{ye^{-\mu^2 t/2 - y^2/2t}}{\sqrt{2\pi t^3}} \\ \phi_\mu(k, t, y) &= \frac{1}{2(1-k)} \frac{\partial}{\partial t} \phi_\mu(k-1, t, y) + (k - \mu - 1) \phi(k-1, t, y), \quad k = 2, 3, \dots \end{aligned}$$

**Proof.** Let  $m_\mu(k, t) = \mathbf{E}(2A_t^{(\mu)})^{-k}$ ,  $k = 1, 2, \dots$ . By time reversal (Dufresne, 1989, 1990, Carmona *et al.*, 1997),  $2A_t^{(\mu)}$  has the same distribution as  $X_t^{(\mu)}$ , where

$$X_t^{(\mu)} = 2e^{2\mu t + 2W_t} \int_0^t e^{-2\mu s - 2W_s} ds$$

is a process with Ito differential

$$dX_t^{(\mu)} = [(2\mu + 2)X_t^{(\mu)} + 2]dt + 2X_t^{(\mu)} dW_t.$$

From this equation, we find

$$\frac{d}{dt} \mathbf{E}(X_t^{(\mu)})^{1-k} = [2(1-k)\mu + 2(1-k)^2] \mathbf{E}(X_t^{(\mu)})^{1-k} + 2(1-k) \mathbf{E}(X_t^{(\mu)})^{-k},$$

which implies

$$m_\mu(k, t) = \frac{1}{2(1-k)} \frac{\partial}{\partial t} m_\mu(k-1, t) + (k - \mu - 1) m_\mu(k-1, t), \quad k = 2, 3, \dots$$

The result follows by differentiating under the integral sign, which is allowed by dominated convergence.  $\square$

## 5. The distributions of $2A_t^{(\mu)}$ and $1/(2A_t^{(\mu)})$

**Theorem 9.** *The p.d.f.'s of  $2A_t^{(\mu)}$  and  $1/(2A_t^{(\mu)})$  exist, and are denoted  $f_\mu(t, x)$ ,  $g_\mu(t, x)$ , respectively. For  $0 < t, x < \infty$ ,*

$$f_\mu(t, x) = c^{a+1} x^b e^{-cx} \sum_{n=0}^{\infty} a_n(t) L_n^a(cx) \quad (14)$$

$$g_\mu(t, y) = f_\mu(t, 1/y)/y^2 = c^{a+1} y^{-b-2} e^{-c/y} \sum_{n=0}^{\infty} a_n(t) L_n^a(c/y),$$

where  $a > -1$ ,  $0 < c < e^{-\mu-t}$ , and

$$a_n(t) = \frac{n!}{\Gamma(n+a+1)} \mathbb{E} L_n^a\left(\frac{c}{2A_t^{(\mu)}}\right)$$

$$= \sum_{k=0}^n \frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!} \mathbb{E}(2A_t^{(\mu)})^{-(a-b+k)}.$$

**Proof.** In Theorem 2, let  $d = 1$ ,  $X = c/(2A_t^{(\mu)})$ ,  $a > -1$  and  $0 < c < e^{-\mu-t}$ . Condition (ii) of Theorem 2 is satisfied, from Theorem 6. We only need to check that  $f_\mu(t, x)$  exists and has (as a function of  $x$ ) a bounded continuous derivative. We recall Proposition 2, p.527, in Yor (1992a):

$$\begin{aligned} \mathbb{P}(A_t^{(\mu)} \in du \mid W_t + \mu t = x) &= a_t(x, u) du \\ &= \frac{\sqrt{2\pi t}}{u} \exp\left(\frac{x^2}{2t} - \frac{1}{2u}(1 + e^{2x})\right) \theta_{e^x/u}(t) du, \end{aligned}$$

where

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} \exp\left(\frac{\pi^2}{t}\right) \int_0^\infty dy \exp(-y^2/2t) \exp(-r \cosh y) (\sinh y) \sin\left(\frac{\pi y}{t}\right).$$

Thus, the p.d.f. of  $A_t^{(\mu)}$  is the result of integrating  $a_t(x, u)$  with respect to the normal p.d.f. (in  $u$ ) with mean  $\mu t$  and variance  $t$ . It can be checked, using the Dominated Convergence Theorem, that the double integral obtained is a uniformly bounded function of  $u$ , which has a continuous derivative.

The formulas for the p.d.f.'s of  $1/(2A_t^{(\mu)})$  result from

$$f_\mu(t, x) = cf_X(cx). \quad \square$$

**Theorem 10.** If  $F_\mu(t, x)$  and  $G_\mu(t, x)$  denote the d.f.'s of  $1/(2A_t^{(\mu)})$  and  $2A_t^{(\mu)}$ , respectively, then, for  $0 < t, x < \infty$ ,

$$F_\mu(t, x) = 1 - c^{a+1} x^b e^{-cx} \sum_{n=0}^{\infty} a_n(t) L_n^a(cx), \quad 0 < t, x < \infty$$

$$G_\mu(t, y) = c^{a+1} y^{-b} e^{-c/y} \sum_{n=0}^{\infty} a_n(t) L_n^a(c/y), \quad 0 < t, y < \infty,$$

where  $a > 2b_+ - 1$ ,  $0 < c < 2e^{-\mu-t}$ , and

$$a_n(t) = \sum_{k=0}^n \frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!} \frac{\mathbb{E}(2A_t^{(\mu)})^{-(a-b+k+1)}}{(a-b+k+1)}.$$

**Proof.** In Theorem 3, let  $X = c/(2A_t^{(\mu)})$ ,  $a > 2b_+ - 1$  and  $0 < c < 2e^{-\mu-t}$ . Then  $d = (c + 2e^{-\mu-t})/4c > 1/2$ , and

$$cd = \frac{c + 2e^{-\mu-t}}{4} < e^{-\mu-t}$$

so that  $\exp(dX) < \infty$ . The proof of Theorem 9 shows that  $1/(2A_t^{(\mu)})$  has a continuous p.d.f., hence its d.f. has a continuous derivative. The formulas given result from

$$\mathbb{P}\left(\frac{1}{2A_t^{(\mu)}} \leq x\right) = \mathbb{P}\left(\frac{c}{2A_t^{(\mu)}} \leq cx\right), \quad \mathbb{P}\left(2A_t^{(\mu)} \leq y\right) = \mathbb{P}\left(\frac{1}{2A_t^{(\mu)}} \geq \frac{1}{y}\right). \quad \square$$

{*Figure 1 approximately here*}

*Example 1.* The solid line Figure 1 represents the density of  $2A_t^{(1)}$ , obtained from one million simulation runs; the integral is approximated by the corresponding sum over 200 equally spaced points. In this and in the other illustrations, the parameters chosen were  $a = b = 0$ ,  $c = 1$ . (Theorem 9 only guarantees convergence for  $c < 1$ , but the numerical results are very close for  $.7 < c < 1.5$ .) The dashed lines represent the three-term series (long dashes) and five-term series (short dashes). Over the range shown ( $0 < x < 1$ ) and with the same order of magnification, the ten-term series curve and the simulations curve cannot be distinguished.  $\square$

## 6. Reciprocal Asian Options

Options with payoffs

$$\left(\frac{1}{Avg} - \frac{1}{K}\right)_+ \quad (\text{call})$$

$$\left(\frac{1}{K} - \frac{1}{Avg}\right)_+ \quad (\text{put})$$

exist in the currency option market, where  $Avg$  is an average currency rate (in practice discretely computed), and  $K$  is the exercise price.

We now consider reciprocal average options with continuous averaging, assuming the Black-Scholes model of one risk-free asset  $e^{rt}$  and one risky asset, represented by a geometric Brownian motion. The model is complete, and both types of payoffs described above have finite expectation. Therefore the unique no-arbitrage price is the expected value of the discounted payoff, under the risk-neutral measure. Suppose that, under the risk neutral measure, the risky asset is  $S_0 e^{\nu t + \sigma W_t}$ , where  $W$  is standard Brownian motion, and that, moreover, the averaging period is  $[0, T]$ . Then the price at time 0 is

$$e^{-rT} \mathbf{E} \left( \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right]^{-1} - \frac{1}{K} \right)_+ . \quad (15)$$

This can be reformulated as follows. First, let  $s = \sigma^2 t/4$  in the integral:

$$\frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt = \frac{4S_0}{\sigma^2 T} \int_0^{\sigma^2 T/4} \exp((2\nu/\sigma^2)s + \sigma W_{4s/\sigma^2}) ds. \quad (16)$$

The scaling property of Brownian motion means that the two processes  $\{\sigma W_{4s/\sigma^2}; s \geq 0\}$  and  $\{2W_s; s \geq 0\}$  have the same distribution. Thus,

$$\frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \stackrel{\mathcal{L}}{=} \frac{2S_0}{\sigma^2 T} 2A_T^{(\mu)} \quad \text{with} \quad t = \frac{\sigma^2 T}{4}, \quad \mu = \frac{2\nu}{\sigma^2}. \quad (17)$$

**Theorem 11.** *The no-arbitrage price of the reciprocal Asian call option just described (see (15)) is equal to*

$$e^{-rT} \frac{\sigma^2 T}{2S_0} C^R(\mu, T, x),$$

where  $t = \frac{\sigma^2 T}{4}$ ,  $\mu = \frac{2\nu}{\sigma^2}$ ,  $x = \frac{2S_0}{\sigma^2 K T}$  and  $C^R(\mu, t, x) = \mathbf{E} \left( \frac{1}{2A_t^{(\mu)}} - x \right)_+$ .

The function  $C^R(\mu, t, x)$  has a converging Laguerre expansion

$$C^R(\mu, t, x) = c^{a+1} x^b e^{-cx} \sum_{n=0}^{\infty} a_n(t) L_n^a(cx), \quad 0 < x < \infty,$$

$$a_n(t) = \sum_{k=0}^n \frac{n! (-c)^k}{\Gamma(k+a+1) k! (n-k)!} \frac{\mathbf{E}(2A_t^{(\mu)})^{-(a-b+k+2)}}{(a-b+k+1)(a-b+k+2)}, \quad n = 0, 1, \dots$$

where  $a > 2b_+ - 1$  and  $0 < c < 2e^{-\mu-t}$ . The price of a reciprocal Asian put option with the same characteristics can be found from the put-call parity relationship

$$\begin{aligned} e^{-rT} \mathbb{E} \left( \frac{1}{K} - \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right]^{-1} \right)_+ &= e^{-rT} \mathbb{E} \left( \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right]^{-1} - \frac{1}{K} \right)_+ \\ &= e^{-rT} \left( \frac{1}{K} - \mathbb{E} \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right]^{-1} \right) \end{aligned}$$

where

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right]^{-1} &= \frac{\sigma^2 T}{2S_0} \mathbb{E}(1/2A_t^{(\mu)}) \\ &= \frac{\sigma^2 T}{2S_0} \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}} \int_0^\infty y e^{-y^2/(2t)} \frac{\cosh[(\mu-1)y]}{\sinh(y)} dy. \end{aligned}$$

**Proof.** Let  $X = c/(2A_t^{(\mu)})$  in Theorem 4. The series is obtained from

$$\mathbb{E} \left( \frac{1}{2A_t^{(\mu)}} - x \right)_+ = \frac{1}{c} \mathbb{E}(X - cx)_+. \quad \square$$

{**Figure 2 approximately here**}

*Example 2.* The solid line Figure 2 represents the value of  $C^R(1, .1, x)$ , obtained from one hundred thousands simulation runs with the geometric average as control variate; the integral is approximated by the corresponding sum over 200 equally spaced points. The dashed lines represent the three-term series (long dashes) and five-term series (short dashes). Over the range shown ( $0 < x < 8$ ) and with the same order of magnification, the seven-term series curve and the one obtained by simulations would be identical. More detailed computations are shown in the next section.  $\square$

## 7. Asian Options

We now turn to the more usual type of average options, again with continuous averaging and in the Black-Scholes framework with one risky asset. Suppose that, under the risk neutral measure, the risky asset is  $S_0 e^{\nu t + \sigma W_t}$ , where  $W$  is standard Brownian motion, and that, moreover, the averaging period is  $[0, T]$ . Then the price of a European put with strike  $K$  is, at time 0,

$$e^{-rT} \mathbb{E} \left( K - \left[ \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right] \right)_+. \quad (18)$$



The time scale is transformed as in the previous section; formulas (15) (16) and (17) apply verbatim. Observe that if the averaging period has already begun at the moment the valuation is performed, an obvious transformation of the payoff reduces the expression to the one above; the reader is referred to Geman & Yor (1993) for details.

**Theorem 12.** *The no-arbitrage price of the Asian put option just described (see (18)) is equal to*

$$e^{-rT} \frac{2S_0}{\sigma^2 T} P^A(\mu, t, x),$$

where  $t = \frac{\sigma^2 T}{4}$ ,  $\mu = \frac{2\nu}{\sigma^2}$ ,  $x = \frac{\sigma^2 KT}{2S_0}$  and  $P^A(\mu, t, x) = \mathbb{E} \left( x - 2A_t^{(\mu)} \right)_+$ .

The function  $P^A(\mu, t, x)$  has a converging Laguerre expansion

$$P^A(\mu, t, x) = c^{a+1} x^{-b+1} e^{-c/x} \sum_{n=0}^{\infty} a_n(t) L_n^a(c/x), \quad 0 < x < \infty, \quad (18)$$

$$a_n(t) = \sum_{k=0}^n \frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!} \frac{\mathbb{E}(2A_t^{(\mu)})^{-(a-b+k+1)}}{(a-b+k+1)(a-b+k+2)}, \quad n = 0, 1, \dots$$

where  $a > 2b_+ - 1$  and  $0 < c < 2e^{-\mu-t}$ . The price of a Asian call option with the same characteristics can be found from the put-call parity relationship

$$e^{-rT} \mathbb{E} \left( \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt - K \right)_+ - e^{-rT} \mathbb{E} \left( K - \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right)_+$$

$$= e^{-rT} \mathbb{E} \left( \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt - K \right)$$

where

$$\mathbb{E} \left( \frac{1}{T} \int_0^T S_0 e^{\nu t + \sigma W_t} dt \right) = \frac{2S_0}{\sigma^2 T} \mathbb{E} 2A_t^{(\mu)} = \frac{2S_0}{\sigma^2 T} \frac{e^{2(\mu+1)t} - 1}{\mu + 1}.$$

**Proof.** Let  $X = c/(2A_t^{(\mu)})$  in Theorem 5. The series is obtained from

$$\mathbb{E} \left( 2A_t^{(\mu)} - x \right)_+ = c \mathbb{E} \left( \frac{1}{X} - x/c \right)_+. \quad \square$$

{Figure 3 approximately here}

*Example 3.* The solid line Figure 3 represents the value of

$$C^A(1, .1, x) = \mathbb{E} 2A_{.1}^{(1)} - x + P^A(1, .1, x),$$

obtained from one hundred thousands simulation runs with the geometric average as control variate; the integral is approximated with 200 equally spaced points. The dashed lines represent the three-term series (long dashes) and five-term series (short dashes). Over the range shown ( $0 < x < .6$ ) and with the same order of magnification, the eighth-term series curve and the one obtained by simulations are identical.  $\square$

*Example 4.* Here we compare the Laguerre series with Monte Carlo simulations and also with the numerical inversion of the Geman & Yor (1993) Laplace transform. We reproduce part of Table 4 of Fu *et al.* in our Table 1. The seven cases in Table 4 of Fu *et al.* are given, but they are reordered in increasing size of  $\tau = \sigma^2 T$ . In all cases the call option has a strike price of  $K = 2.0$ , and the average is computed during the whole period of  $T$  years. The columns  $r$ ,  $\sigma$ ,  $S_0$ , Euler, MC and STDERR are from Fu *et al.*. The column MC shows their Monte Carlo results with 10,000 replications and 100 averages per day. Twenty-one terms of the Laguerre series (18) were generated; the parameters were  $a = b = 0$ ,  $c = 1$ .  $N_1$  represents the number of terms required for the truncated Laguerre series to be inside the 95 % interval  $MC \pm 1.96$  STDERR;  $N_2$  is the number of terms it takes the truncated Laguerre series to reach its ultimate value plus or minus .001. The accuracy of the Laguerre series clearly increases with  $\tau$ . For  $\tau$  equal to .01 the series is very inaccurate, even after 21 terms.  $\square$

There is another example in Fu *et al.*, with  $\sigma = .2$  and  $T = .4$ ; the Laguerre series does not give accurate results in this case. These and other computations indicate that the series perform well when  $\tau > .08$  approximately.

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**Table 1**

*Numerical Example 4.*

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<u>Case</u>	<u>r</u>	<u><math>\sigma</math></u>	<u>T</u>	<u><math>S_0</math></u>	<u><math>\tau</math></u>	<u><math>\mu</math></u>	<u>Euler</u>	<u>MC</u>	<u>STDERR</u>	<u><math>N_1</math></u>	<u><math>N_2</math></u>	<u>LS</u>
1	.02	.10	1	2.0	.010	3	.0560	.0565	.0008	—	—	.0197
2	.18	.30	1	2.0	.090	3	.219	.220	.003	10	11	.2184
3	.0125	.25	2	2.0	.125	-6	.172	.172	.003	8	10	.1723
4	.05	.50	1	1.9	.250	-6	.194	.196	.004	4	6	.1932
5	.05	.50	1	2.0	.250	-6	.247	.249	.004	4	6	.2464
6	.05	.50	1	2.1	.250	-6	.307	.309	.005	4	6	.3062
7	.05	.50	2	2.0	.500	-6	.352	.348	.007	3	5	.3501

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## 8. Conclusion

The main advantage of the Laguerre series proposed for Asian options is of course accuracy, as the series converges to the true price. There is no randomness in the prices obtained, as in simulation; moreover, a whole curve (or surface) of prices is obtained in a single step. The programming done by the author using the software Mathematica was not very involved, about fifteen instructions to generate the series. The execution times and memory requirements are significantly dependent on the way the symbolic mathematics is programmed. For instance, generating  $n$  terms of the series in pure symbolic form uses more memory than generating the same  $n$  terms with specific numerical values for some (or all) of the parameters; but the first approach leads to faster and more flexible pricing. The author performed the numerical calculations shown in this paper on a MacIntosh PowerBook 3400c; generating the series for Figure 3 took 7.8 seconds of CPU time. The two dashed curves in the figure then took an extra 2.3 seconds. Generating a single price after the series is obtained in symbolic form required a few hundreds of a second. The 21-term series in Example 4 took about 150 seconds to generate; however, the same series with only 11 terms took only 15 seconds. By comparison, the simulation (in the programming language C) that computed the solid curve in the same figure took 169 seconds. (Of course all these execution times would be reduced on a faster machine.)

The Laguerre series have some disadvantages: (1) the series applies to continuous averaging only, with equal weights, (2) no analytical bound on the error introduced by truncating the series after  $n$  terms is available, and (3) they seem to perform well only for  $\tau = \sigma^2 T$  greater than about .08. More work is required to find out precisely why this is so, but at this point it appears that more terms are needed when  $\tau$  is small. The last point agrees with the work done by Fu *et al.*, who noticed the same phenomenon with regards to the numerical inversion of the Geman & Yor Laplace transform. Rogers & Shi (1995, p.1087) also report that the accuracy of their numerical PDE techniques improves as volatility increases. Whether something can be done to improve convergence for small  $\tau$ , such as choosing appropriate parameters  $a$ ,  $b$  and  $c$ , remains to be seen.

Thus, there is strong evidence that an important (if not the most important) factor to take into account when computing Asian option values is the magnitude of  $\tau = \sigma^2 T$ . Some methods do well for large  $\tau$ , others for small  $\tau$  (*e.g.* simulation). The normalization used in this paper (initially suggested by Geman & Yor (1993)) thus turns out to have more than just theoretical value.

## References

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*Captions for figures*

**Figure 1.** The probability density function of  $2A_{.1}^{(1)}$ . Solid line: simulation, long dashes: 3-term Laguerre series, short dashes: 5-term Laguerre series

**Figure 2.** Reciprocal Asian option values: the function  $C^R(1, .1, x) = \mathbf{E}(1/(2A_{.1}^{(1)} - x)_+)$ . Solid line: simulation, long dashes: 3-term Laguerre series, short dashes: 5-term Laguerre series

**Figure 3.** Asian option values: the function  $C^A(1, .1, x) = \mathbf{E}(2A_{.1}^{(1)} - x)_+$ . Solid line: simulation, long dashes: 3-term Laguerre series, short dashes: 5-term Laguerre series