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Modern Landmarks in Actuarial Science

Inaugural Professorial Address

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INAUGURAL PROFESSORIAL ADDRESS

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Introduction

For tonight's lecture, I have chosen to explore a part of our profession's recent past. Actuarial science has changed greatly since I first set out on the path to becoming an actuary almost twenty-five years ago. One of the drivers of change is, of course, research. Research is fundamental to the long term security and survival of a profession such as ours. In this lecture, I intend to discuss actuarial research that I believe has made a real difference to our profession - to the way we practise, to the conduct of research and to the material we teach. I will cut across the practice areas, so I hope there is something of interest for all of you. There will be some mathematics, some history, some trivia, a problem, and some homework! For those of you who turned up tonight expecting an exciting hour of risk theory, I apologise, but there is some of that in the talk too.

The stimulus for this lecture may surprise you. It is an old professional exam question on probability. In April 1985, actuarial students were faced with the following problem.

A boy is collecting picture cards featuring famous actuaries from packets of breakfast cereal. There are five different cards which make up the complete set. During the time that the special packets are available, his family buys ten packets.

On the assumption that each packet is independently equally likely to contain any one of the five cards, determine, to three decimal places, the probability that he collects at least one complete set of cards.

I haven't given the whole question here. It's a question which can be answered using Waring's result, which was the first part of the question. For me the question conjures up interesting images. Can you imagine kids in school-yards trading actuary cards rather than footy cards? What sort of country would Australia be if kids grew up dreaming of becoming CEO of a life office or principal of a consulting firm rather than the likes of Wayne Carey or Brad Fittler? The unlucky country?

However, I think a more interesting question is the following: who are the five actuaries? If the question had asked this, and, why are they famous, then perhaps that one exam could have sufficed to determine who would be a Fellow and who would not!

What makes an actuary famous? Members of our profession are rarely in the public eye, let alone the tabloid press. I suspect that actuaries will only ever be famous within their own ranks - probably for a significant contribution to the discipline. I'm sure we could all rattle off a few names central to the history of actuarial science - for example, Graunt, de Witt and Price from the very early days. Each of these individuals was a pioneer in some sense, but are they really famous? Graunt gave us the concept of a life table. De Witt also gave use something that endures to the present day - the principle of equivalence, which he used to value annuities. Price was well-named - he priced insurance for the Equitable Society, the first life insurance company in the UK to calculate premiums on a scientific basis. I wonder how many of you would consider them to be famous actuaries, or actuaries at all for that matter!

Originally, I had planned to talk tonight about landmarks in actuarial science. As a starting point I browsed through Transactions of the Faculty of Actuaries, going back to 1913 when my collection starts. I was struck by three features of that journal going up to the early 1940s. First, I encountered names that became familiar to me from the textbooks I read when I first studied life contingencies and graduation theory - names like Hardy, Lidstone and Whittaker. Second, many of the discussions deal with computational problems, things that are no longer relevant with today's computing power. Third, there is a much more academic feel to the journal than there is in its present day successor, the British Actuarial Journal. Anyone reading those issues would be in no doubt that the actuarial profession was firmly based in the application of mathematics to insurance problems. It was both a learned and learning profession.

Similarly, as I browsed the contents of another journal with a great tradition, the Scandinavian Actuarial Journal, I encountered names familiar to me from PhD studies - Lundberg, Cramér, Segerdahl, Bohman and Esscher. It is a journal which has a different tradition to the British journals, and to this day, the focus remains on mathematical modelling in insurance. It is easy to pick the outstanding papers, the classics of their time, but like many of the great contributions in the British journals, time has passed them by. An example would be Bohman and Esscher's pair of papers from 1963-64 on approximate methods of calculating aggregate claims distributions. As a PhD student I thought these were wonderful papers, yet most of the techniques they describe have become largely redundant.

I decided therefore that I would concentrate tonight on what I see as significant developments in the modern era. I have chosen to discuss four papers from different areas of actuarial science which I believe have made significant contributions, not just to the discipline, but to the way that actuaries think, be it in actuarial practice, in academia, or in both. I will give a brief overview of each paper and explain why I think it is significant. I cannot say whether history will judge all the authors as famous. What I can say is that actuarial science is much the richer for their contributions.

I will go through the papers in chronological order. It would be an invidious task, indeed, to try to rank them.

Markov Models in Life Insurance

I will start with *Markov Chain Models in Life Insurance* by Jan Hoem, which was published in 1969 in Blätter der Deutshcen Gesellschaft für Versicherungsmathematik, which is the journal of the more scientific of the two German actuarial societies. Those of you who attended the International Congress in Helsinki in 1988 may have heard Jan Hoem give an invited lecture on this topic. Jan Hoem held the chair in Actuarial Mathematics at the University of Copenhagen, but sadly for actuarial science now spends his time working on demographic problems at the Max Planck Institute for Demographic Research in Germany.

In the introduction to his paper, Hoem alludes to the fact that there had been previous attempts to present the theory of life insurance mathematics in a probabilistic context. His approach is to present this theory in a general framework. I will illustrate this framework, and in particular how Hoem presents some well-known results as being trivially achievable within this context.

First of all, let's recall that a Markov chain is a stochastic process $\{X(t)\}_{t\geq 0}$ on a state space $\{1, 2, ...\}$ such that for all $s, t \geq 0$, and integers i, j and $x(u)$

$$\begin{aligned} & \Pr(X(t+s) = j | X(s) = i,\, X(u) = x(u),\, 0 \leq u < s) \\ = \; & \Pr(X(t+s) = j | X(s) = i). \end{aligned}$$

In life insurance mathematics, the simplest possible Markov chain model has two states - state 1 'alive' and state 2 'dead'. A slightly more complicated example which Hoem uses to illustrate ideas is a model for disability income insurance, with two live states, say 'able' and 'disabled', and a third state - 'dead'.

In general, suppose a model has N states. Hoem denotes the state of the policy at time t by $S(t)$, numbering the states so that $S(0) = 1$. He defines

$$P_{jk}(s,t) = \Pr(S(t) = k | S(s) = j), \quad \text{for } t \geq s$$

as the transition probabilities for the process, and transition intensities

$$\mu_{jk}(s) = \lim_{\Delta s \to 0+} P_{jk}(s, s + \Delta s)/\Delta s$$

with

$$\mu_j(s) = \sum_{k\neq j} \mu_{jk}(s).$$

In the familiar actuarial notation of the two state model, if a life is aged x at time s,

$$P_{11}(s,t) = {}_t p_x = 1 - P_{12}(s,t),$$

and $\mu_{12}(s)$ corresponds to the force of mortality at age x. Hoem assumes the existence of these transition intensities which are central to the modelling. He does not explain how to find the transition probabilities from the transition intensities, but this is a standard problem in Markov chain theory which can be solved using Kolmogorov's forward equations.

Hoem considers the situation when a benefit of $B_{jk}(t)$ is payable on transition from state j to state k at time t, and when a benefit at rate $B_j(t)$ is payable at time t if the process is in state j. He does not impose restrictions on the signs of these, so that we can think of *negative* benefits as premium payments. He defines $V_\nu(s,\tau,u)$ to be the actuarial value at time s of the above benefit payments in the time interval $[\tau,u]$, where $s \leq \tau < u$, given that the process is in state ν at time s. In particular, when $s = \tau$, he denotes this value by $V_\nu(s,u)$

Defining

$$\bar{P}_\nu(s,t) = \exp\{-\int_s^t \mu_\nu(r)dr\}$$

to be the probability that the process remains in state ν throughout $[s,t]$, Hoem shows that

$$V_\nu(s,u) = \int_s^u e^{-\delta(t-s)}\bar{P}_\nu(s,t)\left\{B_\nu(t) + \sum_{j\neq\nu}\mu_{\nu j}(t)\left[B_{\nu j}(t) + V_j(t,u)\right]\right\}dt \tag{1}$$

where δ is the force of interest. This equation has a simple heuristic explanation, by considering the time spent continuously in state ν from time s.

The first well-known result we can derive is Thiele's differential equation. Hoem differentiates (1) giving

$$\frac{\partial}{\partial s}V_\nu(s,u) = \delta V_\nu(s,u) - B_\nu(s) - \sum_{j\neq\nu}\mu_{\nu j}(s)\left[B_{\nu j}(s) + V_j(s,u) - V_\nu(s,u)\right].$$

For example, in the two state model consider a whole life policy issued s years ago to a life then aged x, with unit sum assured and net annual premiums of $\pi = \bar{A}_x/\bar{a}_x$. Then, with $\nu = 1$, the above formula becomes

$$\frac{\partial}{\partial s} V_1(s,\infty) = \delta V_1(s,\infty) + \pi - \mu_{12}(s)\left[1 - V_1(s,\infty)\right]$$

or, in standard actuarial notation,

$$\frac{\partial}{\partial s}\ {}_s\bar{V}_x = \delta\ {}_s\bar{V}_x + \pi - \mu_{x+s}\left(1 - \ {}_s\bar{V}_x\right).$$

Hoem then shows that the type of identity that arises when prospective and retrospective reserves are equal can easily be derived. Again considering negative benefit payments as premiums, it is clear that $V_\nu(s,u)$ is the prospective reserve at time s provided that the policy terminates no later than u (which, as above, can be infinite). By partitioning the time period, he observes that for $t < u$

$$V_1(0,t,u) = V_1(0,u) - V_1(0,t).$$

Under the principle of equivalence, which does not have to be imposed, we have $V_1(0,u) = 0$. This observation leads to well-known identities, but does not define a retrospective reserve. Considering the same whole life policy as before, so that $V_1(0,u) = 0$, we get the well-known identity

$$V_1(0,t,u) = e^{-\delta t}\ {}_tp_x\ {}_t\bar{V}_x = \pi\bar{a}_{x:\overline{t|}} - \bar{A}^{1}_{x:\overline{t|}} = -V_1(0,t).$$

Rather than defining a retrospective reserve, this approach simply offers an alternative method of calculating the prospective reserve. In the paper, Hoem discuss retrospective reserves, and by considering what is effectively an infinite portfolio of identical policies in the two state model, he defines the retrospective reserve at time t as

$$V_1(t,u) - \frac{V_1(0,u)}{e^{-\delta t}P_{11}(0,t)}$$

so that there is equality of prospective and retrospective reserves only if the principle of equivalence applies. A significant aspect here is that Hoem applies some simple probability theory to reach his definition. This is in sharp contrast to the rather artificial method usually taught in a course on life contingencies. The discussion of retrospective reserves is interesting and has led to much further study of this topic. Indeed, different definitions of retrospective reserves exist under this model.

The significance of Hoem's paper is not that it provides a general framework from which many known results can easily be derived, but that it provides a flexible tool for modelling in life insurance. Today's insurance products, such as trauma insurance and disability income insurance, are products based on living, not necessarily in the same condition, compared to traditional insurance products under which life and death are the only discriminators. As a profession we need to be able to model such situations, and Markov models are a natural tool.

Indeed, since Hoem's paper we have seen a number of applications of Markov models. For example, in North America, a simple model proposed in the late 1980s for the spread of AIDS consisted of six states, representing a possible path to death from AIDS for a healthy individual through states such as HIV+. In this model, transition intensities were constant, and movements were in one direction only. For such a model it is very easy to work out the probability that an individual is in a given state at a given time because time spent in each state is an exponentially distributed random variable.

A more topical application appeared in British Actuarial Journal in 1997. In a paper on the potential impact of genetic testing and adverse selection on life insurance, Angus Macdonald uses a Markov model to represent the different routes that an individual may take to buying insurance, with or without a genetic test. For this model, as for the AIDS model, there is insufficient data to parameterise the model with a great degree of confidence. Nevertheless, the usefulness of such a model to a life insurance company is that it can identify the key parameters in the model. For example, Macdonald's study concluded that the biggest threat to insurers from adverse selection - represented by individuals being able to conceal a positive genetic test from an insurer - comes from these adverse selectors effecting a policy with an above average sum assured.

Markov models are an essential part of the modern actuary's toolkit. Their applications are widespread.

Martingale Methods

The second paper that I wish to discuss is *Martingales in Risk Theory* by Hans Gerber, Professor of Actuarial Mathematics in Lausanne. This paper appeared in the Mitteilungen der Vereinigung Schweizererischer Versicherungsmathematiker - the Bulletin of the Swiss Association of Actuaries - in 1973. Some of you may think that this is rather a strange choice. Hans Gerber has an international reputation, particularly for his research in ruin theory, and I must admit that of all his work, this is not my favourite paper. Those of you who are familiar with his substantial body of work will know of the delights of such papers as *Mathematical Fun with Ruin Theory*, characterised by elegant mathematics and witty observations. However, I believe that *Martingales in Risk Theory* is significant, not so much because of its content, but because it firmly planted the idea in the academic actuarial community that martingales are an important tool for actuaries. Since the publication of this paper, there have been many actuarial papers applying martingale techniques to problems in life insurance, general insurance and financial economics.

So, first of all, what is a martingale? Without being rigorous, I'll define a martingale as a stochastic process $\{X_t\}_{t\geq 0}$ such that $E(|X_t|) < \infty$ and for $t > \tau$, $E(X_t|F_\tau) = X_\tau$ where the filtration $\{F_t\}_{t\geq 0}$ can generally be thought of as the information gained by observing the process. We can think of a martingale as a generalisation of a fair game. As a simple example, a process $\{B_t\}_{t\geq 0}$ is standard Brownian motion if it has stationary and independent increments and if $B_t \sim N(0,t)$ for all $t > 0$, with $B_0 = 0$. We can use these properties to show that standard Brownian motion is a martingale since for $t > \tau$,

$$\begin{aligned} E(B_t|B_\tau) &= E(B_t - B_\tau + B_\tau|B_\tau) \\ &= E(B_t - B_\tau|B_\tau) + E(B_\tau|B_\tau) \\ &= B_\tau. \end{aligned}$$

The background knowledge required to understand Gerber's paper is

(i) what a stopping time is, an example of which will be given shortly, and

(ii) the optional stopping theorem, which says that for a stopping time, T, $E(X_T) = E(X_s)$ for all s.

In his paper Gerber considers ruin theory. He uses martingale techniques to derive bounds for both finite and infinite time ruin probabilities for different types of risk processes, assuming each process has independent increments. For convenience, I will consider only the classical risk model, a model to which I will refer again later. In this model, the insurer's surplus process $\{U(t)\}_{t\geq 0}$ is defined by

$$U(t) = u + ct - S(t)$$

where

- $u = U(0)$ is the surplus at time 0,
- c is the premium income per unit time, assumed to be received continuously,

- $S(t) = \sum_{i=1}^{N(t)} X_i$ denotes the aggregate claim amount up to time t,
- $\{N(t)\}_{t \geq 0}$ is a Poisson process with parameter λ, which counts the number of claims,
- $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, independent of $\{N(t)\}$, where X_i denotes the amount of the ith claim,
- $c > \lambda E(X_i)$.

He defines T to be the time of ruin and considers the stopped process $\{\tilde{U}(t)\}_{t \geq 0}$ given by

$$\tilde{U}(t) = \begin{cases} U(t) \text{ if } t < T \\ U(T) \text{ if } t \geq T. \end{cases}$$

The time of ruin is a stopping time because the decision to stop the surplus process on the first occasion it becomes negative can be made based on the history of the process up to that point in time.

The key idea is to find a function $v(x,t)$ such that $\{v(\tilde{U}(t), \min(t,T))\}_{t \geq 0}$ is a martingale with respect to $\{U(t)\}_{t \geq 0}$. This leads to bounds for the finite time ruin probability

$$\begin{aligned} \psi(u,t) &= \Pr(U(s) < 0 \text{ for some } s,\ 0 < s \leq t) \\ &\leq \frac{v(u,0)}{E[v(U(T),T)|U(0) = u,\ T \leq t]} \end{aligned}$$

and the infinite time ruin probability

$$\begin{aligned} \psi(u) &= \Pr(U(s) < 0 \text{ for some } s > 0) \\ &\leq \frac{v(u,0)}{E[v(U(T),T)|U(0) = u,\ T < \infty]} \end{aligned}$$

He shows that if

$$v(x,t) = \frac{e^{-rx}}{E[\exp\{-r(U(t) - u)\}]}$$

then $\{v(U(t),t)\}_{t \geq 0}$ is a martingale with respect to $\{U(t)\}_{t \geq 0}$.

Further, in the special case when there exists a unique positive constant R such that

$$E[\exp\{-R(U(t) - u)\}] = 1 \tag{2}$$

for all $t > 0$, then R is the adjustment coefficient, and

$$\psi(u) = \frac{e^{-Ru}}{E\left[\exp\{-RU(T)\}|T < \infty\right]}.$$

An immediate consequence of this result is the famous Lundberg inequality, $\psi(u) \leq e^{-Ru}$.

Both these results were known before Gerber's paper, and can be derived without martingale techniques. However, from a pedagogical point of view, a significant aspect of the martingale approach in this problem is that it provides a natural interpretation for the adjustment coefficient. To many actuarial students, the adjustment coefficient is nothing more than the unique positive number R such that

$$\lambda + cR = \lambda E[\exp\{RX_i\}],$$

an equation which is difficult to motivate, even after showing its application in a martingale-free proof of Lundberg's inequality. The source of this equation is, however, clear from the martingale method when we insert the form of the compound Poisson moment generating function into equation (2).

The remainder of Gerber's paper, which is fairly short, is split into two parts. The first is devoted to finding bounds for ultimate ruin probabilities when the individual claim amount distribution has either an increasing or a decreasing failure rate. The second deals with a surplus process modified by a reflecting dividend barrier, so that the rate of growth of the surplus process is restricted. The trick here in finding bounds for the ruin probability is to find a suitable martingale.

A significant aspect of Gerber's paper is not so much the results he derives, but the fact that the results can be derived in such an elegant and insightful way using martingales. I do not wish to dwell on martingale techniques in ruin theory. Rather I wish to mention martingales in a different context. In his 1979 textbook on risk theory, Gerber wrote: "Martingales have not found a wide acceptance by actuaries. Hopefully, this will change. After all, fair games (and some say favourable games) are the bread and butter of insurance companies". There is little doubt in my mind that this change has occurred, certainly within the academic community. It would also appear that the importance of these ideas has finally reached professional actuarial education. Martingales, and one of their most important applications - the valuation of derivatives - are now in the Part I syllabus.

Actuarial students should now know that derivatives can be priced as an expected value under a martingale measure. The trick is to find that measure, but there are plenty of tools to help us in that task. As a final comment on why actuaries should be comfortable with such an approach to derivative pricing, when we consider a formula such as

$$e^{-rt}E[\max(0, S_t - k)]$$

then we see that the expectation is of exactly the same form as we would use in the calculation of a pure stop-loss premium.

Hans Gerber is one of the great researchers of our time, and *Martingales in Risk Theory* is one of his great contributions to the discipline.

The Panjer Recursion Formula

The third paper that I have selected is *Recursive Evaluation of a Family of Compound Distributions* by Harry Panjer. Harry may be known to some of you, not just because of his distinguished academic career at the University of Waterloo, but also because he has been very active in professional activities, most notably being President of the C.I.A. - that's the C.I.A. without intelligence, as the old joke goes. This paper has become famous for what is widely known as the Panjer recursion formula. Indeed, I suspect that this is the most cited paper in actuarial literature, certainly in recent times. The paper appeared in the ASTIN Bulletin in June 1981, and since then there has been a huge volume of actuarial literature involving recursive techniques. The paper is incredibly short - just five pages long and containing essentially one result - but its applications are far reaching, both in practical and theoretical problems.

Let's consider the problem the paper tackles in general terms. Suppose we have a random sum $S = \sum_{i=1}^{N} X_i$ where N is a discrete random variable and $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, independent of N. Let X_i be distributed on the positive integers with probability function f_j, $j = 1, 2, 3, ...$ and let N be distributed on the non-negative integers with probability function p_n, $n = 0, 1, 2, ...$. Then the probability function of S is g_j, $j = 0, 1, 2, ...$, where

$$g_0 = \Pr(S = 0) = p_0 = \Pr(N = 0)$$

and

$$g_j = \sum_{n=1}^{j} p_n f_j^{n*}$$

where f^{n*} is the probability function of $\sum_{i=1}^{n} X_i$, i.e. it is the n-fold convolution of f with itself.

The problem with applying this formula is that it is computationally intensive because of the convolutions. What Panjer showed was that if the probability function of N is of the form

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}$$

for $n = 1, 2, 3, \ldots$, where a and b are constants, then the probability function of S can be calculated recursively from the formula

$$g_j = \sum_{i=1}^{j} \left(a + \frac{bi}{j}\right) f_i g_{j-i}.$$

From a practical point of view, such a formula is trivial to program and the computer time required is far from prohibitive.

Panjer demonstrates that three well-known distributions satisfy the recursion formula for p - the Poisson, negative binomial (with geometric as a special case) and binomial distributions - and in the same edition of the ASTIN Bulletin, Sundt and Jewell prove that these are the only non-trivial distributions satisfying the recursion formula.

Why, then, is this result so famous? There are two reasons. First, it provides a numerical solution to two classical problems in risk theory, namely the calculation of an aggregate claims distribution, and the calculation of the probability of ultimate ruin in certain risk models. In the case of calculating aggregate claims distributions, we interpret S as the aggregate claim amount in some fixed period, say one year, N as the number of claims in that period, and X_i as the amount of the ith claim. Previously, aggregate claims distributions had been mostly approximated, by techniques described in the aforementioned papers by Bohman and Esscher such as the normal power method. The Panjer recursion formula allows us to dispense with such techniques, and to calculate the distribution of S precisely, given the distributions of N and X_i.

Of course, the recursion formula does not solve all the problems of calculating an aggregate claims distribution. Given a set of data we have to fit both counting and claim severity distributions, and if, as would be normal, we fit a continuous severity distribution, we need a method of replacing it with a discrete distribution to use in the recursion formula. There has been much research on methods of discretising distributions, as well as resolving such problems as what happens when the probability of no claims is so small that a computer treats its value as zero, so that the recursion formula gives a zero probability mass at all points.

The choice of counting distribution may appear restrictive for practical purposes. However, the ideas involved in deriving Panjer's recursion formula apply to other classes of counting distribution including the so-called $(a, b, 1)$ class where the probability function of N satisfies the recursion formula

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}$$

but for $n = 2, 3, 4, ...$ so that p_0 can be zero or an arbitrary value between 0 and 1, and Sundt's class where for some positive integer k and constants $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$,

$$p_n = \sum_{i=1}^{k} \left(a_i + \frac{b_i}{n}\right) p_{n-i}$$

for $n = 1, 2, 3, ...$. Between them, these classes contain a variety of counting distributions suitable as models for claim numbers. In each case, a simple Panjer-type recursion formula for the aggregate claims distribution exists. The Panjer recursion formula and its variants provide a wonderful means of calculating aggregate claims distributions.

The second problem from classical risk theory that the Panjer recursion formula solves, at least numerically, is the calculation of the probability of ultimate ruin. I will consider the classical risk model, that is I will assume that claims occur as a Poisson process, but the ideas also apply when claims occur as other types of renewal process. As mentioned earlier, the ultimate ruin probability is

$$\psi(u) = \Pr(U(s) < 0 \text{ for some } s > 0).$$

Whilst we can solve explicitly for $\psi(u)$ for certain types of individual claim amount distribution - for example, when individual claim amounts are exponentially distributed with mean 1, $\psi(u) = (1 - R) \exp\{-Ru\}$, where R is the adjustment coefficient - no closed form solutions are known for types of individual claim amount distribution such as Pareto or lognormal. For such cases, we can use the Panjer recursion formula to calculate lower and upper bounds for $\psi(u)$, and we can make these bounds as tight as we like, so that, to a certain number of decimal places, the bounds are the same, giving us a value for the ruin probability.

The connection between ultimate ruin probability and the Panjer recursion formula is that $\psi(u) = \Pr(L > u)$ where the random variable L is the maximum aggregate loss, defined by $L = \max\{S(t) - ct, t \geq 0\}$. Looking at a realisation of a surplus process, and the corresponding aggregate loss process

$\{L(t)\}_{t\geq 0}$ where $L(t) = S(t) - ct$, we see that we can write $L = \sum_{i=1}^{N} L_i$ where N denotes the number of record highs of the aggregate loss process and L_i denotes the amount of the ith increase in the record high of the aggregate loss process. Because each claim occurrence is a renewal point of the aggregate loss process, the probability that there is a further record high of the aggregate loss process is just the same as the probability that there will be a first record high, namely $\psi(0)$. Thus N has a geometric distribution and hence L has a compound geometric distribution. The distribution of L_i, say H, can be expressed in terms of the distribution function of X_i.

The final step in the calculation of bounds is to create discrete distributions A and B, distributed on the non-negative integers, such that

$$A(x) \leq H(x) \leq B(x).$$

This ordering is preserved under convolution, so that

$$A^{n*}(x) \leq H^{n*}(x) \leq B^{n*}(x)$$

and hence

$$1 - \sum_{n=0}^{\infty} \psi(0)^n (1-\psi(0)) B^{n*}(u) \leq \psi(u) \leq 1 - \sum_{n=0}^{\infty} \psi(0)^n (1-\psi(0)) A^{n*}(u).$$

As $\psi(0)$ is known for this model, the bounds can be calculated recursively, and in the case of the lower bound we have to modify the basic recursion formula to allow B to have a probability mass at zero. A simple rescaling of the monetary unit is the device we employ to tighten the bounds.

To illustrate the idea, suppose that the individual claim amount distribution is lognormal with mean 1 and variance 3, and that the premium is calculated with a loading of 30%. Then, using a monetary unit of 1/20th of the mean individual claim amount, we find, for example, $0.0379 \leq \psi(30) \leq 0.0394$, and we can sharpen these bounds using a monetary unit of 1/100th of the mean individual claim amount, giving $0.0386 \leq \psi(30) \leq 0.0389$, so that to three decimal places $\psi(30) = 0.039$. The Panjer recursion formula thus provides a very neat numerical solution to a classical problem.

The second reason why the Panjer recursion formula is important is that it can be used in the life insurance context to calculate the aggregate claims distribution for a life insurance portfolio over a fixed period such as one year. Suppose we have a portfolio of n life insurance policies, and suppose that for the ith life, the death benefit is an integer S_i and the mortality rate is q_i. If Y_i denotes the aggregate claim amount for the ith life in the period, then

Y_i has a compound binomial distribution. A problem with computing the aggregate claims distribution for the portfolio is that we cannot say what the distribution is of the sum of compound binomial random variables. However, if we approximate the distribution $b(1, q_i)$ by a Poisson(q_i) distribution, an approximation which is very good if q_i is small (as most mortality rates are), then we can approximate the distribution of Y_i by a compound Poisson distribution, and as the sum of compound Poisson random variables has a compound Poisson distribution, we can say that the distribution of $\sum_{i=1}^{n} Y_i$ is approximately compound Poisson. This then allows us to use the Panjer recursion formula to calculate the aggregate claims distribution.

Of course this is only an approximation, and other recursive methods do exist to calculate the aggregate claims distribution. However, the great advantages that the Panjer recursion formula has over other methods is that it is simple to apply, it is significantly more efficient than an exact calculation for a large portfolio in terms of computing time, and it is sufficiently accurate for practical purposes.

The Panjer recursion formula is undoubtedly one of the most significant contributions to actuarial science of the modern era.

The Wilkie Model

My final choice of significant paper is David Wilkie's *A Stochastic Investment Model for Actuarial Use*, which was presented to a sessional meeting of the Faculty of Actuaries in November 1984, and subsequently published in Transactions of the Faculty of Actuaries. Of the papers I have discussed tonight, I am sure that this is the one most familiar to members of the audience. The Wilkie model is well-known internationally, although if he were here tonight, I'm sure David would correct me and say that it is *a* Wilkie model. The paper might be regarded as the culmination of a number of years' work, stemming from Wilkie's membership of the British professions' Maturity Guarantees Working Party. This working party was established in the late 1970s, and reported to the profession at sessional meetings in London and Edinburgh in 1980. Incidentally, their report was considered to be so important that Issue 435 of the Journal of the Institute of Actuaries, in which it appeared, was published ahead of Issue 434! In their terms of reference, the working party was charged with the task of "recommending bases for reserving for investment performance guarantees under investment linked individual life assurance contracts". As part of this work they recommended the use of fairly simple time series models for both a share and a dividend index.

The model that Wilkie subsequently proposed was more ambitious in its scope. In the opening paragraph of the paper, Wilkie describes his model as "the minimum model that might be used to describe the total investments of a life office or pension fund". Let's recall briefly the workings of the model. It is based on four variables:

- an index of inflation (the Retail Prices Index in the UK),
- an index of share dividends (the FT-Actuaries All Share Index in the UK),
- the dividend yield on the share index, and
- the general level of fixed interest yields (measured by the yield on 2.5% Consols in the UK).

Wilkie's approach to modelling was to use a cascade structure in which the index of inflation was the key driver. The structure was

INFLATION
↓
↙ SHARE YIELD ↘
↙ ↘
SHARE DIVIDENDS F.I. YIELDS

Using data from just over 60 years, Wilkie proposes time series models for each of the variables, where, for example, the value of the dividend yield on the share index at time t depends on the value at time t of the inflation index and on a component of the model for the share yield. Each of the four time series models contains fixed parameters, for which Wilkie suggests appropriate values, and random white noise series. The model does not lend itself to analytical calculations, but can be implemented easily using simulation, even on a package like Excel. For example, the inflation series, denoted $Q(t)$, is given by

$$\nabla \ln Q(t) = QMU + QA(\nabla \ln Q(t-1) - QMU) + QSD.QZ(t)$$

If we take the values suggested in the paper for the constants QMU, QA and QSD then all we have to do to generate values of $\nabla \ln Q(t)$ is

(i) set a value for $\nabla \ln Q(0)$ to get started,

(ii) generate the required number of random drawings from the standard normal distribution as our observations of $QZ(t)$, and

(iii) apply the formula mechanically!

It's a very easy model to program - even if the notation Wilkie uses appears a little daunting at first.

It is inevitable that such a model would be subject to some criticism when first presented. At the meeting in Edinburgh, some speakers complained about choices of parameter values or the suitability of the time series models chosen. In the years since then there have been actuaries in other countries, including Australia, who have claimed that the Wilkie model is inappropriate to their local conditions and who have tried to produce their own stochastic interest rate model. Different questions have arisen -

- is it appropriate to use annual data?
- does the same cascade structure apply in different countries?

However, the important point about Wilkie's paper is that it firmly established in the minds of actuaries, not just in the UK, the need to use a stochastic interest rate model. That much has never been disputed.

Much of Wilkie's paper is concerned with describing the general features of the model, listing the equations and considerations for the choice of parameter values, detailing sensitivity tests that were used, and discussing how to apply the model. The final section of the paper touches on possible applications, including ideas like optimal portfolio selection, premium calculation for conventional life insurance, investigating minimum solvency levels, the costing of maturity guarantees on index-linked policies, and investigations on pension funds such as the effect of different investment strategies.

The paper does not give any numerical illustrations of these ideas, yet I believe that its applications are what makes the model such a valuable tool. One of the useful applications of the model is to questions of solvency. In a paper that should be familiar to anyone who has passed Part II, Mary Hardy describes the use of the Wilkie model in the assessment of life office solvency. She demonstrates that by using the Wilkie model, a different quality of information can be found from simulation. Although her life office model is a relatively simple one, it allows for both a dynamic investment strategy and a dynamic allocation of reversionary bonus. She considers six different types of office, with different investment and bonus strategies, projecting their asset-liability ratios for a period of years. A barrage of sensitivity tests shows that each office is quite secure under a deterministic investment model. By contrast, when equity and gilt yields from these calculations were replaced by simulated values from the Wilkie model, a totally different picture emerged. Based on 1,000 simulations of each office, all but one of the model offices was subject to a probability of insolvency ranging from 7% to 21%. The crucial point in this study is that the use of the Wilkie model allowed 1,000 scenarios to be tested, not just a handful of deterministic scenarios. The message that comes across loud and clear from this study is that stochastic modelling offers an insight which traditional techniques do not. Not a single deterministic scenario test suggested that any of the offices was in trouble.

It has to be recognised that such studies using the Wilkie model rely totally on simulation, which in turn relies on computing power. However, computing power has increased significantly since the model was first proposed, and even on a PC it is now an easy task to program, say, 1,000 realisations of each series over, say, fifty years. As far as I can see, there is now little to limit the application of this, or a similar, model. Perhaps it is time to move on to some of the other applications Wilkie suggests. Why, for example, do we still teach premium calculation on the assumption of a constant rate of interest over the duration of a policy?

At the discussion of Wilkie's paper in Edinburgh, the opening speaker congratulated the author on his paper and stated that he thought that it would "in time be recognised as a landmark of actuarial thought". He wasn't wrong. David Wilkie's numerous contributions to the actuarial profession in the UK have earned him gold medals from both the Institute and the Faculty, a rare achievement indeed. Yet of all his works, there is no doubt in my mind that this is the paper that really made him famous.

Concluding Remarks

When I chose the papers about which to speak this evening, I did not set out to select any theme. The topics I have covered include Markov models, martingales, recursive techniques and time series models. Their applications range across the whole spectrum of actuarial work. I have classed each paper as a modern landmark in actuarial science. Apart from their impact on research and practice, what links the papers is that the topics they cover are all part of a modern actuarial education.

It is important to our profession that research thrives and that the outstanding ideas from that research find their way not just into actuarial practice, but also into actuarial education. Much of the research I have discussed tonight relies heavily on ideas from other areas, most notably probability theory. As a profession we must be able to harness key ideas from other areas and apply them to our own problems. We cannot afford to be introspective, nor can we afford to reduce our skill set in an ever competitive world. I firmly believe that actuaries who understand the sort of modelling I have described tonight are well placed to face the challenges of the future. We are a profession whose roots lie in the application of mathematics to financial problems. As the nature of financial problems changes, it is imperative that we provide future generations of actuaries with the skills required to face these challenges.

Recent experience in the UK has highlighted the need for these skills. Earlier, I mentioned the Equitable Society as the first company to price insurance on a scientific basis in the UK. One wonders if it still does. In December last year, it closed its doors to new business. Its downfall was the result of its inability to price and reserve for annuities with investment guarantees. Are their actuaries to blame, or is the profession as a whole? I think that the profession must shoulder some of the blame, and I hope the lesson has been learnt. With an up-to-date professional education based on stochastic modelling and with a more rigorous program of Continuing Professional Education, such a disaster need not happen again.

The syllabus today for Part I of our professional education is very different from what it was, say, twenty years ago. For years, the standard technical actuarial education revolved around compound interest, life contingencies and graduation. Although these topics are still in the Part I syllabus, there is much less emphasis on them, and they are taught in a very different way. For example, basic life contingencies is taught within the framework of future lifetime of an individual being a random variable, and standard actuarial functions such as A_x are presented as expected values of random variables.

The emphasis in Part I is now very much on the applications of modern probability theory. In particular, the new Subject 103 - Stochastic Modelling - introduces students to many of the concepts I have mentioned tonight. Indeed, we might summarise the content of Subject 103 as:

- martingales,
- Markov chains,
- Markov processes and their applications,
- time series models,
- Gauss-Wiener processes (including Brownian motion),
- stochastic simulation.

I have mentioned something from each of these topics tonight, and given some of the applications that are involved in subsequent Part I subjects. In my view, Subject 103 is crucial to the survival and advancement of our profession.

The actuarial program at the University of Melbourne has undergone a radical restructuring in the past two years. Starting this year, honours graduates will have seen most of what I have discussed tonight by the end of their studies. It is not just important to the profession that universities conduct research, but that they keep their courses fresh by incorporating the best of recent research. At this university, we do not seek to mirror professional education in our courses. Rather, our aim is to be ahead of it. Uniquely among the Australian universities, Melbourne has positioned its actuarial program as a four year honours program. This gives us the flexibility to teach subjects at an appropriate level, and to expand on topics given a superficial treatment in a professional education. I make no apologies for the fact that the program has a strong theoretical basis. I am a firm believer in the adage that there is nothing as practical as good theory. Three of the papers I have discussed tonight might be considered theoretical in nature, but their practical applications are manyfold. Our program provides graduates with a toolkit which will allow them to tackle financial problems with confidence, not only now, but in the years ahead.

And so, I will conclude by setting your homework. The complete set of Actuary Cards in the exam question contained five famous actuaries. I have suggested four names. Who would you choose to complete the set, and why?

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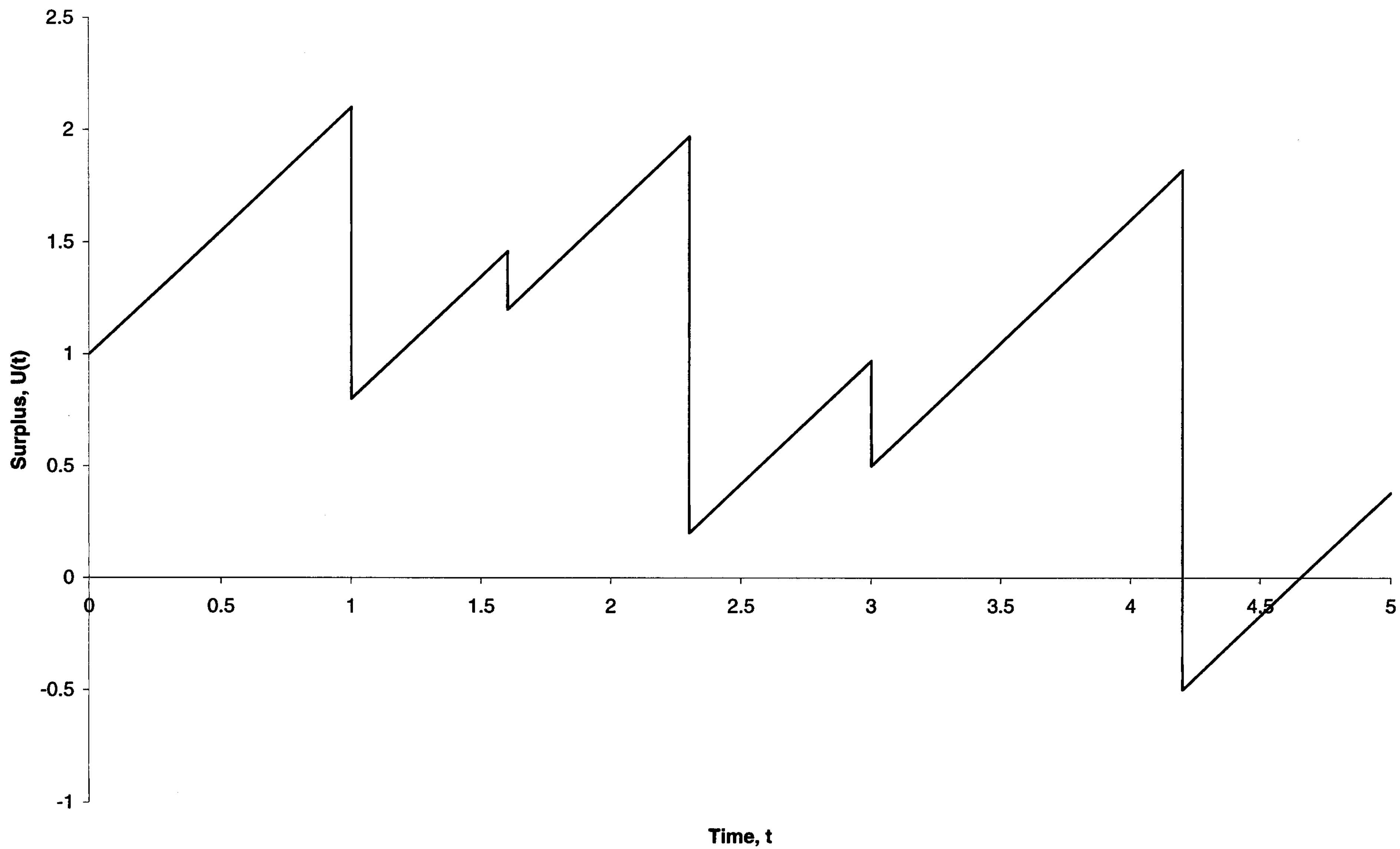

Figure 1: A realisation of the surplus process

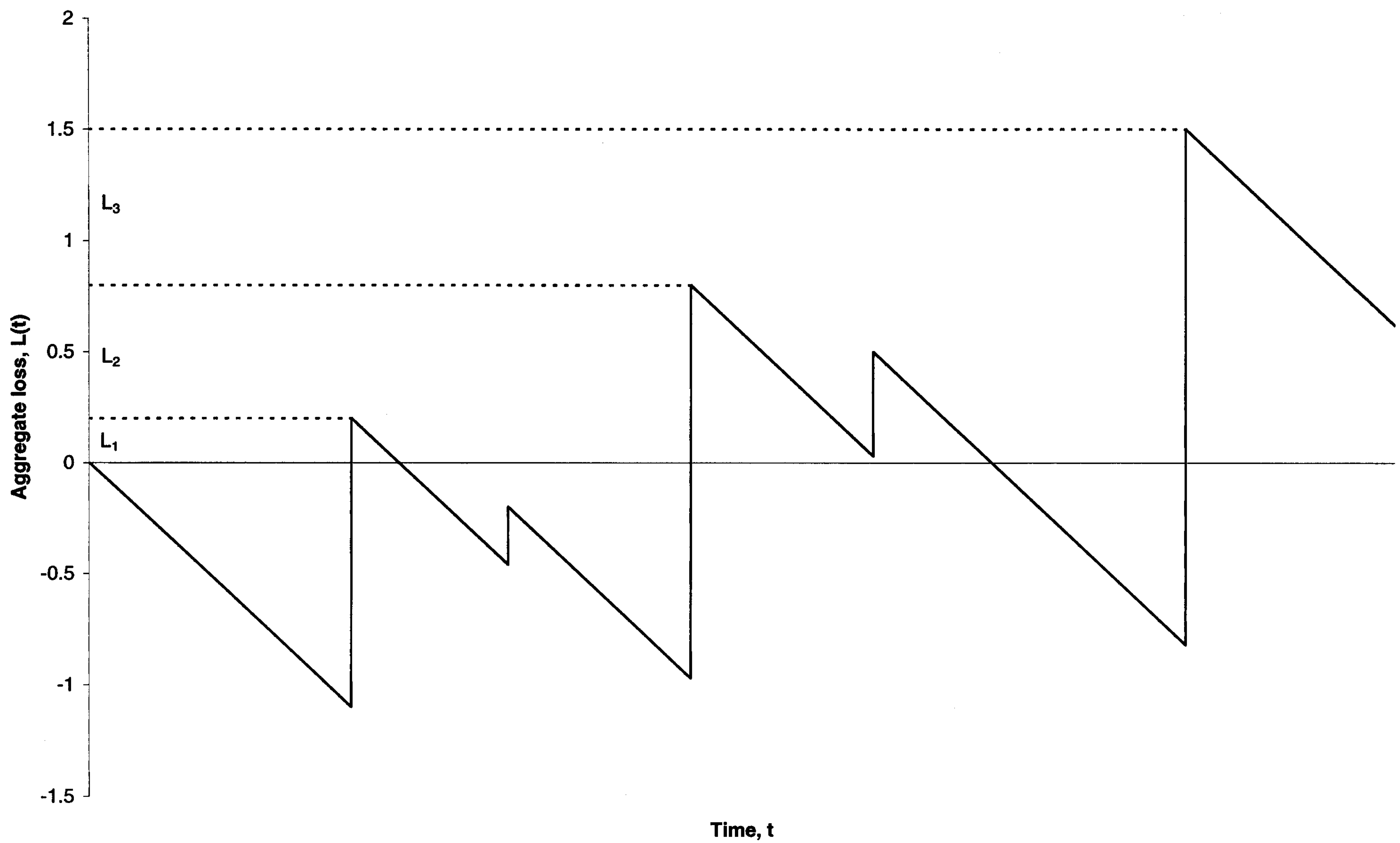

Figure 2: The realisation of the aggregate loss process corresponding to Figure 1

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