Some explicit solutions for the joint density of the time of ruin and the deficit at ruin

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Abstract

Using probabilistic arguments we obtain an integral expression for the joint density of the time of ruin and the deficit at ruin. For the classical risk model, we obtain the bivariate Laplace transform of this joint density and invert it in the cases of individual claims distributed as Erlang(2) and as a mixture of two exponential distributions. As a consequence, we obtain explicit solutions for the density of the time of ruin.

1 Introduction

In recent years a number of new explicit solutions for finite time ruin problems have appeared in the literature. See, for example, Drekic and Willmot (2003), Dickson and Willmot (2005), Dickson et al (2005) and Garcia (2005). In this paper we extend results given in some of these papers by considering the joint distribution of the time of ruin and the deficit at ruin in the classical risk model. In particular, we derive explicit solutions for the joint density of these quantities for two individual claim amount distributions - Erlang(2) and a mixture of two exponential distributions.

Let $\{U(t)\}_{t\geq 0}$ denote a surplus process and let T denote the time of ruin. Define

$$W(u,y,t) = \Pr(T \leq t,\ |U(T)| \leq y \mid U(0) = u),$$

to be the defective joint distribution function of the time of ruin and the deficit at ruin, and let

$$w(u,y,t) = \frac{\partial^2}{\partial y \partial t} W(u,y,t)$$

be the defective joint density.

By conditioning on the time and the amount of the first drop of the surplus process below its initial level we have

$$\begin{aligned} W(u,y,t) &= \int_{\tau=0}^{t}\int_{x=0}^{u} w(0,x,\tau)W(u-x,y,t-\tau)dxd\tau \\ &\quad + \int_{\tau=0}^{t}\int_{x=u}^{u+y} w(0,x,\tau)dxd\tau. \end{aligned}$$

Taking the partial derivative of $W(u,y,t)$ with respect to t we obtain

$$\begin{aligned} \frac{\partial}{\partial t}W(u,y,t) &= \int_{\tau=0}^{t}\int_{x=0}^{u} w(0,x,\tau)\frac{\partial}{\partial t}W(u-x,y,t-\tau)dxd\tau \\ &\quad + \int_{x=u}^{u+y} w(0,x,t)dx, \end{aligned}$$

and then taking the partial derivative with respect to y we obtain

$$w(u,y,t) = \int_{\tau=0}^{t}\int_{x=0}^{u} w(0,x,\tau)w(u-x,y,t-\tau)dxd\tau + w(0,u+y,t). \quad (1)$$

Further, we define

$$w(u,t) = \int_{0}^{\infty} w(u,y,t)dy$$

to be the defective density of T, and let $\psi(u,t) = \Pr(T \le t)$. Next, let

$$\bar{\psi}(u,t) = 1 - \psi(u,t) = 1 - \int_{0}^{t} w(u,s)ds$$

denote the survival probability to time t from initial surplus u. Then

$$\bar{\psi}(u,t) = \int_{\tau=0}^{t}\int_{y=0}^{u} w(0,y,\tau)\bar{\psi}(u-y,t-\tau)dyd\tau + \bar{\psi}(0,t),$$

and differentiation with respect to t gives

$$\begin{aligned} -w(u,t) &= \int_{0}^{u} w(0,y,t)dy \\ &\quad - \int_{0}^{t}\int_{0}^{u} w(0,y,\tau)w(u-y,t-\tau)dyd\tau - w(0,t), \end{aligned}$$

or, equivalently,

$$w(u,t) = \int_{0}^{t}\int_{0}^{u} w(0,y,\tau)w(u-y,t-\tau)dyd\tau + \int_{u}^{\infty} w(0,y,t)dy.$$

All the above formulae apply equally to a Sparre Andersen risk model and to the classical risk model. Our aim in this paper is to apply these formulae to the classical risk model to obtain explicit solutions for quantities such as $w(u,y,t)$ and $\psi(u,t)$. Our approach will be to use Laplace transforms.

2 Background

In the classical risk model, claims occur as a Poisson process, and we let λ denote the parameter of this process. Let F denote the distribution function of individual claim amounts and let f denote the density function. Let $g(\bullet, t)$ denote the density of aggregate claims over $(0, t)$ so that

$$g(x,t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{n*}(x),$$

where f^{n*} denotes the n-fold convolution of f with itself. Let c denote the premium income per unit which we assume to be received continuously, and we assume that c exceeds the expected aggregate claim amount per unit time.

For the classical risk model, Lundberg's fundamental equation is

$$\lambda + \delta - ct = \lambda \int_0^\infty e^{-tx} f(x) dx,$$

where $\delta > 0$, and Gerber and Shiu (1998) show that there is a unique positive solution to this equation, which we denote by ρ. Further, from formula (2.26) of Gerber and Shiu (1998) we have

$$\int_0^\infty \int_0^\infty e^{-sy-\delta t} w(0,y,t) dy dt = \frac{\lambda}{c} \int_0^\infty e^{-\rho t} \int_t^\infty e^{-s(y-t)} f(y) dy dt. \quad (2)$$

A key result in our analysis is the Laplace transform relationship given by Dickson and Willmot (2005). They show that for two functions A and B, if

$$\tilde{A}(\rho) \stackrel{def}{=} \int_0^\infty e^{-\rho t} A(t) dt = \tilde{B}(\delta) \stackrel{def}{=} \int_0^\infty e^{-\delta t} B(t) dt$$

then

$$B(t) = c e^{-\lambda t} A(ct) + \int_0^{ct} \frac{x}{t} g(ct - x, t) A(x) dx. \quad (3)$$

Cheung et al (2006) obtain general expressions for $w(u, y, t)$ for two types of individual claim amount distribution – a combination of exponential distributions and mixed Erlang distributions. Specifically, they show that if $f(x) = \sum_{i=1}^n p_i \, \alpha_i \exp\{-\alpha_i x\}$ where $\sum_{i=1}^n p_i = 1$ and each $p_i \neq 0$ then

$$w(u,y,t) = \sum_{i=1}^{n} h_i(u,t) \; \alpha_i \exp\{-\alpha_i y\}, \quad (4)$$

where the Laplace transform of $h_i(u, t)$ is known for $i = 1, 2, ..., n$. Similarly, if

$$f(x) = \sum_{i=1}^{n} q_i \; \frac{\beta^i x^{i-1} \exp\{-\beta x\}}{\Gamma(i)}$$

where $\sum_{i=1}^{n} q_i = 1$ and each $q_i \geq 0$ then

$$w(u,y,t) = \sum_{i=1}^{n} \eta_i(u,t) \sum_{k=n+1-i}^{n} q_k \frac{\beta^{k-n+i} y^{k-n+i-1} \exp\{-\beta y\}}{\Gamma(k-n+i)} \tag{5}$$

where the Laplace transform of $\eta_i(u,t)$ is known for $i = 1, 2, ..., n$. In this paper we consider special cases of these distributions – an Erlang(2) distribution and a mixture of two exponential distributions – and we obtain closed form solutions for the joint density of the time of ruin and the deficit at ruin. A consequence of this is that we obtain explicit formulae for finite time ruin probabilities which are of a different form to those presented by Garcia (2005) for these individual claim amount distributions.

3 Erlang(2,β) claims

Let us first consider the case when the individual claim amount distribution is Erlang(2) with scale parameter β, so that $f(x) = \beta^2 x e^{-\beta x}$. Then by applying equation (2) we obtain

$$\frac{\lambda}{c} \int_0^\infty e^{-\rho t} \int_t^\infty e^{-s(y-t)} f(y) dy dt = \frac{\lambda}{c} \left[\frac{1}{\rho+\beta} \left(\frac{\beta}{\beta+s} \right)^2 + \frac{\beta}{(\rho+\beta)^2} \frac{\beta}{\beta+s} \right],$$

so that

$$w(0,y,t) = h(0,t)\beta^2 y e^{-\beta y} + k(0,t)\beta e^{-\beta y}$$

where

$$\tilde{h}(0,\delta) \stackrel{def}{=} \int_0^\infty e^{-\delta t} h(0,t) dt = \frac{\lambda}{c} \frac{1}{\rho+\beta}$$

and

$$\tilde{k}(0,\delta) \stackrel{def}{=} \int_0^\infty e^{-\delta t} k(0,t) dt = \frac{\lambda}{c} \frac{\beta}{(\rho+\beta)^2}.$$

Further, from equation (5) with $n = 2$ and $q_2 = 1$ we have (with a change of notation)

$$w(u,y,t) = h(u,t)\beta^2 y e^{-\beta y} + k(u,t)\beta e^{-\beta y} \tag{6}$$

where $h(u,t)$ and $k(u,t)$ are functions that we will now identify.

Inserting formula (6) in equation (1) we get

$$
\begin{aligned}
& h(u,t)\beta^2 y e^{-\beta y} + k(u,t)\beta e^{-\beta y} \\
= & \int_0^t \int_0^u h(0,\tau)\beta^2 x e^{-\beta x}(h(u-x,t-\tau)\beta^2 y e^{-\beta y} + k(u-x,t-\tau)\beta e^{-\beta y}) dx d\tau \\
& + \int_0^t \int_0^u k(0,\tau)\beta e^{-\beta x}(h(u-x,t-\tau)\beta^2 y e^{-\beta y} + k(u-x,t-\tau)\beta e^{-\beta y}) dx d\tau \\
& + h(0,t)\beta^2 (y+u) e^{-\beta(y+u)} + k(0,t)\beta e^{-\beta(u+y)},
\end{aligned}
$$

giving

$$
h(u,t) = \int_0^t \int_0^u \left(h(0,\tau)\beta^2 x e^{-\beta x} + k(0,\tau)\beta e^{-\beta x}\right) h(u-x,t-\tau) dx d\tau + h(0,t) e^{-\beta u}
$$

and

$$
\begin{aligned}
k(u,t) &= \int_0^t \int_0^u \left(h(0,\tau)\beta^2 x e^{-\beta x} + k(0,\tau)\beta e^{-\beta x}\right) k(u-x,t-\tau) dx d\tau \\
& \quad + h(0,t)\beta u e^{-\beta u} + k(0,t) e^{-\beta u}. \qquad (7)
\end{aligned}
$$

Then defining

$$
\tilde{\tilde{h}}(s,\delta) = \int_0^\infty \int_0^\infty e^{-su-\delta t} h(u,t) dt du
$$

with a similar definition for the bivariate Laplace transform of $k(u,t)$, we obtain

$$
\tilde{\tilde{h}}(s,\delta) = \left(\tilde{h}(0,\delta)\left(\frac{\beta}{\beta+s}\right)^2 + \tilde{k}(0,\delta)\frac{\beta}{\beta+s}\right)\tilde{\tilde{h}}(s,\delta) + \tilde{h}(0,\delta)\frac{1}{\beta+s}
$$

giving

$$
\begin{aligned}
\tilde{\tilde{h}}(s,\delta) &= \frac{\tilde{h}(0,\delta)\frac{1}{\beta+s}}{1 - \tilde{h}(0,\delta)\left(\frac{\beta}{\beta+s}\right)^2 - \tilde{k}(0,\delta)\frac{\beta}{\beta+s}} \\
&= \tilde{h}(0,\delta)\frac{1}{\beta+s}\sum_{n=0}^\infty \left(\tilde{h}(0,\delta)\left(\frac{\beta}{\beta+s}\right)^2 + \tilde{k}(0,\delta)\frac{\beta}{\beta+s}\right)^n \\
&= \frac{1}{\beta}\sum_{n=0}^\infty \sum_{r=0}^n \binom{n}{r} \tilde{h}(0,\delta)^{r+1}\tilde{k}(0,\delta)^{n-r}\left(\frac{\beta}{\beta+s}\right)^{n+r+1}. \qquad (8)
\end{aligned}
$$

Considering formula (8), the term in s inverts to

$$
\frac{\beta^{n+r+1} u^{n+r} e^{-\beta u}}{(n+r)!}
$$

while

$$\tilde{h}(0,\delta)^{r+1}\tilde{k}(0,\delta)^{n-r} = \left(\frac{\lambda}{c}\right)^{n+1} \frac{\beta^{n-r}}{(\rho+\beta)^{2n-r+1}}$$

is the Laplace transform with transform parameter ρ of

$$\left(\frac{\lambda}{c}\right)^{n+1} \frac{\beta^{n-r} t^{2n-r} e^{-\beta t}}{(2n-r)!}$$

and hence by formula (3) is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda}{c}\right)^{n+1} \beta^{n-r} \left(c e^{-\lambda t} \frac{(ct)^{2n-r} e^{-\beta ct}}{(2n-r)!} + \int_0^{ct} \frac{y}{t} g(ct-y,t) \frac{y^{2n-r} e^{-\beta y}}{(2n-r)!} dy \right). \quad (9)$$

Consider first the integral term in formula (9). Inserting for g, it becomes

$$\begin{aligned}
& \sum_{m=1}^{\infty} e^{-\lambda t} \frac{\lambda^m t^{m-1}}{m!} \int_0^{ct} \frac{\beta^{2m}(ct-y)^{2m-1} e^{-\beta(ct-y)}}{(2m-1)!} \frac{y^{2n-r+1} e^{-\beta y}}{(2n-r)!} dy \\
= \quad & e^{-(\lambda+\beta c)t} \sum_{m=1}^{\infty} \frac{\left(\lambda\beta^2\right)^m t^{m-1}}{m!} \int_0^{ct} \frac{(ct-y)^{2m-1} y^{2n-r+1}}{(2m-1)!(2n-r)!} dy \\
= \quad & e^{-(\lambda+\beta c)t} (ct)^{2n-r} \sum_{m=1}^{\infty} \frac{\left(\lambda\beta^2\right)^m t^{m-1} (ct)^{2m+1}}{m!} \frac{2n-r+1}{(2m+2n-r+1)!}
\end{aligned}$$

and the summand at $m=0$ is $c/(2n-r)!$, so that we can combine the terms in formula (9) to get

$$\left(\frac{\lambda}{c}\right)^{n+1} \beta^{n-r} e^{-(\lambda+\beta c)t} (ct)^{2n-r} \sum_{m=0}^{\infty} \frac{\left(\lambda\beta^2\right)^m t^{m-1} (ct)^{2m+1}}{m!} \frac{2n-r+1}{(2m+2n-r+1)!}. \quad (10)$$

For ease of computation, we now write our solution in terms of special functions. We first note that for integer j,

$$(2m+j)! = 4^m j! \left(\frac{j+2}{2}\right)_m \left(\frac{j+1}{2}\right)_m$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol, so that we can rewrite formula (10) as

$$\begin{aligned}
& c\left(\frac{\lambda}{c}\right)^{n+1} \beta^{n-r} e^{-(\lambda+\beta c)t} (ct)^{2n-r} \sum_{m=0}^{\infty} \frac{\left(\lambda\beta^2 c^2 t^3\right)^m}{m!} \frac{2n-r+1}{4^m (2n-r+1)! \left(\frac{2n-r+3}{2}\right)_m \left(\frac{2n-r+2}{2}\right)_m} \\
= \quad & c\left(\frac{\lambda}{c}\right)^{n+1} \frac{\beta^{n-r} e^{-(\lambda+\beta c)t} (ct)^{2n-r}}{(2n-r)!} \, {}_0F_2\left(\frac{2n-r+2}{2}, \frac{2n-r+3}{2}; \frac{\lambda\beta^2 c^2 t^3}{4}\right)
\end{aligned}$$

where

$$ {}_0F_q(C_1, C_2, \dots C_q; Z) = \sum_{m=0}^{\infty} \frac{1}{(C_1)_m (C_2)_m ... (C_q)_m} \frac{Z^m}{m!} $$

is the generalised hypergeometric function.

Finally,

$$ \begin{aligned} h(u,t) &= \frac{c}{\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \frac{\beta^{n+r+1} u^{n+r} e^{-\beta u}}{(n+r)!} \left(\frac{\lambda}{c}\right)^{n+1} \frac{\beta^{n-r} e^{-(\lambda+\beta c)t} (ct)^{2n-r}}{(2n-r)!} \\ &\qquad \times {}_0F_2\left(\frac{2n-r+2}{2}, \frac{2n-r+3}{2}; \frac{\lambda\beta^2 c^2 t^3}{4}\right) \\ &= \lambda e^{-\beta u-(\lambda+\beta c)t} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \left(\beta^2 u\right)^n \sum_{r=0}^{n} \binom{n}{r} \frac{u^r}{(n+r)!} \frac{(ct)^{2n-r}}{(2n-r)!} \\ &\qquad \times {}_0F_2\left(\frac{2n-r+2}{2}, \frac{2n-r+3}{2}; \frac{\lambda\beta^2 c^2 t^3}{4}\right) \end{aligned} $$

with

$$ h(0,t) = \lambda e^{-(\lambda+c\beta)t} \; {}_0F_2\left(1, \frac{3}{2}; \frac{\lambda\beta^2 c^2 t^3}{4}\right). $$

We can obtain a solution for $k(u,t)$ in a similar manner. From equation (7) we obtain

$$ \tilde{\tilde{k}}(s,\delta) = \left(\tilde{h}(0,\delta)\left(\frac{\beta}{\beta+s}\right)^2 + \tilde{k}(0,\delta)\frac{\beta}{\beta+s}\right)\tilde{\tilde{k}}(s,\delta) + \tilde{h}(0,\delta)\frac{\beta}{(\beta+s)^2} + \tilde{k}(0,\delta)\frac{1}{\beta+s} $$

giving

$$ \begin{aligned} \tilde{\tilde{k}}(s,\delta) &= \frac{1}{\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \tilde{h}(0,\delta)^{r+1} \tilde{k}(0,\delta)^{n-r} \left(\frac{\beta}{\beta+s}\right)^{n+r+2} \\ &\quad + \frac{1}{\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \tilde{h}(0,\delta)^{r} \tilde{k}(0,\delta)^{n-r+1} \left(\frac{\beta}{\beta+s}\right)^{n+r+1}. \end{aligned} \quad (11) $$

The first double summation in formula (11) is identical in form to $\tilde{\tilde{h}}(s,\delta)$, but the power of $\beta/(\beta+s)$ is one higher. Hence inversion gives

$$ \begin{aligned} &\lambda\beta u e^{-\beta u-(\lambda+\beta c)t} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \left(\beta^2 u\right)^n \sum_{r=0}^{n} \binom{n}{r} \frac{u^r}{(n+r+1)!} \frac{(ct)^{2n-r}}{(2n-r)!} \\ &\qquad \times {}_0F_2\left(\frac{2n-r+2}{2}, \frac{2n-r+3}{2}; \frac{\lambda\beta^2 c^2 t^3}{4}\right). \end{aligned} $$

In the second double summation in formula (11), inversion of the term involving s gives

$$\frac{\beta^{n+r+1}u^{n+r}e^{-\beta u}}{(n+r)!}$$

while

$$\tilde{h}(0,\delta)^r\tilde{k}(0,\delta)^{n-r+1}=\left(\frac{\lambda}{c}\right)^{n+1}\frac{\beta^{n-r+1}}{(\rho+\beta)^{2n-r+2}}$$

is the Laplace transform with transform parameter ρ of

$$\left(\frac{\lambda}{c}\right)^{n+1}\frac{\beta^{n-r+1}t^{2n-r+1}e^{-\beta t}}{(2n-r+1)!}$$

and hence is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda}{c}\right)^{n+1}\beta^{n-r+1}\left(ce^{-\lambda t}\frac{(ct)^{2n-r+1}e^{-\beta ct}}{(2n-r+1)!}+\int_0^{ct}\frac{y}{t}g(ct-y,t)\frac{y^{2n-r+1}e^{-\beta y}}{(2n-r+1)!}dy\right). \tag{12}$$

Inserting for g, the integral term in expression (12) becomes

$$\begin{aligned}
&\sum_{m=1}^{\infty}e^{-\lambda t}\frac{\lambda^m t^{m-1}}{m!}\int_0^{ct}\frac{\beta^{2m}(ct-y)^{2m-1}e^{-\beta(ct-y)}}{(2m-1)!}\frac{y^{2n-r+2}e^{-\beta y}}{(2n-r+1)!}dy\\
=\;& e^{-(\lambda+\beta c)t}\sum_{m=1}^{\infty}\frac{\left(\lambda\beta^2\right)^m t^{m-1}}{m!}\int_0^{ct}\frac{(ct-y)^{2m-1}y^{2n-r+2}}{(2m-1)!(2n-r+1)!}dy\\
=\;& e^{-(\lambda+\beta c)t}(ct)^{2n-r+2}\sum_{m=1}^{\infty}\frac{\left(\lambda\beta^2\right)^m t^{m-1}(ct)^{2m}}{m!}\frac{2n-r+2}{(2m+2n-r+2)!}
\end{aligned}$$

and as the summand at $m=0$ is $1/t(2n-r+1)!$, we can combine the terms in expression (12) to get

$$\left(\frac{\lambda}{c}\right)^{n+1}\beta^{n-r+1}e^{-(\lambda+\beta c)t}(ct)^{2n-r+2}\sum_{m=0}^{\infty}\frac{\left(\lambda\beta^2\right)^m t^{m-1}(ct)^{2m}}{m!}\frac{2n-r+2}{(2m+2n-r+2)!}. \tag{13}$$

Then we can write expression (13) as

$$\begin{aligned}
&\frac{1}{t}\left(\frac{\lambda}{c}\right)^{n+1}\beta^{n-r+1}e^{-(\lambda+\beta c)t}(ct)^{2n-r+2}\sum_{m=0}^{\infty}\frac{\left(\lambda\beta^2c^2t^3\right)^m}{m!}\frac{2n-r+2}{4^m(2n-r+2)!\left(\frac{2n-r+3}{2}\right)_m\left(\frac{2n-r+4}{2}\right)_m}\\
=\;&\frac{1}{t}\left(\frac{\lambda}{c}\right)^{n+1}\frac{\beta^{n-r+1}e^{-(\lambda+\beta c)t}(ct)^{2n-r+2}}{(2n-r+1)!}\,{}_0F_2\left(\frac{2n-r+3}{2},\frac{2n-r+4}{2};\frac{\lambda\beta^2c^2t^3}{4}\right),
\end{aligned}$$

giving

$$\begin{aligned}
k(u,t) &= \lambda\beta u e^{-\beta u-(\lambda+\beta c)t}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^{n}\left(\beta^2 u\right)^{n}\sum_{r=0}^{n}\binom{n}{r}\frac{u^r}{(n+r+1)!}\frac{(ct)^{2n-r}}{(2n-r)!}\\
&\qquad\times {}_0F_2\left(\frac{2n-r+2}{2},\frac{2n-r+3}{2};\frac{\lambda\beta^2c^2t^3}{4}\right)\\
&\quad+\frac{\beta e^{-\beta u-(\lambda+\beta c)t}}{t}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^{n+1}\left(\beta^2 u\right)^{n}\sum_{r=0}^{n}\binom{n}{r}\frac{u^r}{(n+r)!}\frac{(ct)^{2n-r+2}}{(2n-r+1)!}\\
&\qquad\times {}_0F_2\left(\frac{2n-r+3}{2},\frac{2n-r+4}{2};\frac{\lambda\beta^2c^2t^3}{4}\right).
\end{aligned}$$

Further,

$$k(0,t)=\lambda\beta cte^{-(\lambda+c\beta)t}\ {}_0F_2\left(\frac{3}{2},2;\frac{\lambda\beta^2c^2t^3}{4}\right),$$

and we can compute the density of the time of ruin as $w(u,t)=h(u,t)+k(u,t)$.

Figures 1 to 3 illustrate the application of these formulae. In each figure, $\beta=2$, $\lambda=1$ and $c=1.1$. The values of u are 0 in Figure 1, 10 in Figure 2 and 20 in Figure 3. All values were calculated using Mathematica, which includes the generalised hypergeometric function as a supplied function, and infinite sums were truncated at a suitably high level. Although it is clear from the above formula that $h(u,t)$ and $k(u,t)$ can be written in terms of Erlang densities, it is straightforward to integrate these functions numerically to obtain values of the joint distribution function and of the finite time ruin probability. Values of $W(10,t,y)$ are shown in Table 1 for a range of values for t and y for the same parameters as in Figures 1 to 3.

4 Mixed exponential claims

We now consider the case when individual claims have a mixed exponential distribution with density

$$f(x)=p\alpha e^{-\alpha x}+q\beta e^{-\beta x}$$

where $p+q=1$, $0<p<1$ and $\alpha<\beta$. Then from formula (4), we can write

$$w(u,y,t)=\eta(u,t)\alpha e^{-\alpha y}+\kappa(u,t)\beta e^{-\beta y} \tag{14}$$

and we will now determine $\eta(u,t)$ and $\kappa(u,t)$. From equation (2) we obtain

$$\tilde{\eta}(0,\delta)\stackrel{def}{=}\int_0^\infty e^{-\delta t}\eta(0,t)dt=\frac{\lambda}{c}\frac{p}{\rho+\alpha}$$

	$y=1$	$y=2$	$y=3$	$y\to\infty$
$t=10$	0.0107	0.0131	0.0136	0.0137
$t=20$	0.0360	0.0444	0.0460	0.0464
$t=30$	0.0603	0.0744	0.0771	0.0776
$t=40$	0.0806	0.0994	0.1030	0.1038
$t=50$	0.0972	0.1199	0.1243	0.1252
$t=60$	0.1109	0.1368	0.1418	0.1428
$t=70$	0.1222	0.1508	0.1563	0.1575
$t=80$	0.1318	0.1626	0.1685	0.1698
$t=90$	0.1399	0.1726	0.1789	0.1802
$t=100$	0.1469	0.1812	0.1878	0.1892

Table 1: Values of $W(10,t,y)$, Erlang(2) claims

and

$$\tilde{\kappa}(0,\delta) \overset{def}{=} \int_0^\infty e^{-\delta t}\kappa(0,t)dt = \frac{\lambda}{c}\frac{q}{\rho+\beta}.$$

The method of solution is essentially the same as in the previous section, but the details are more complicated. We first insert expression (14) into equation (1), giving

$$\begin{aligned}
&\eta(u,t)\alpha e^{-\alpha y} + \kappa(u,t)\beta e^{-\beta y} \\
= &\int_0^t\int_0^u \left(\eta(0,\tau)\alpha e^{-\alpha x} + \kappa(0,\tau)\beta e^{-\beta x}\right)\eta(u-x,t-\tau)\alpha e^{-\alpha y}dxd\tau \\
&+\int_0^t\int_0^u \left(\eta(0,\tau)\alpha e^{-\alpha x} + \kappa(0,\tau)\beta e^{-\beta x}\right)\kappa(u-x,t-\tau)\beta e^{-\beta y}dxd\tau \\
&+\eta(0,t)\alpha e^{-\alpha(u+y)} + \kappa(0,t)\beta e^{-\beta(u+y)}.
\end{aligned}$$

Then

$$\eta(u,t) = \int_0^t\int_0^u \left(\eta(0,\tau)\alpha e^{-\alpha x} + \kappa(0,\tau)\beta e^{-\beta x}\right)\eta(u-x,t-\tau)dxd\tau + \eta(0,t)e^{-\alpha u}.$$

Let

$$\tilde{\tilde{\eta}}(s,\delta) = \int_0^\infty\int_0^\infty e^{-su-\delta t}\eta(u,t)dtdu \quad \text{and} \quad \tilde{\eta}(0,\delta) = \int_0^\infty e^{-\delta t}\eta(0,t)dt,$$

with similar notation for the Laplace transforms of $\kappa(u,t)$ and $\kappa(0,t)$. Then

$$\tilde{\tilde{\eta}}(s,\delta) = \left(\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s} + \tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}\right)\tilde{\tilde{\eta}}(s,\delta) + \tilde{\eta}(0,\delta)\frac{1}{\alpha+s}$$

giving

$$\begin{aligned}
\tilde{\tilde{\eta}}(s,\delta) &= \frac{\tilde{\eta}(0,\delta)\frac{1}{\alpha+s}}{1-\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s}-\tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}} \\
&= \tilde{\eta}(0,\delta)\frac{1}{\alpha+s}\sum_{n=0}^{\infty}\left(\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s}+\tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}\right)^{n} \\
&= \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s}\right)^{k+1}\left(\tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}\right)^{n-k}
\end{aligned}$$

Inversion of

$$\left(\frac{\alpha}{\alpha+s}\right)^{k+1}\left(\frac{\beta}{\beta+s}\right)^{n-k}$$

gives

$$\begin{aligned}
&\int_0^u \frac{\alpha^{k+1}x^k e^{-\alpha x}}{\Gamma(k+1)}\frac{\beta^{n-k}(u-x)^{n-k-1}e^{-\beta(u-x)}}{\Gamma(n-k)}dx \\
=\ & \frac{\alpha^{k+1}\beta^{n-k}e^{-\beta u}}{\Gamma(k+1)\Gamma(n-k)}\int_0^u x^k e^{-(\alpha-\beta)x}(u-x)^{n-k-1}dx \\
=\ & \frac{\alpha^{k+1}\beta^{n-k}e^{-\beta u}}{\Gamma(k+1)\Gamma(n-k)}\int_0^1 (ut)^k e^{-(\alpha-\beta)ut}(u-ut)^{n-k-1}udt \\
=\ & e^{-\beta u}\left(\frac{\alpha}{\beta}\right)^k\frac{\alpha(\beta u)^n}{\Gamma(n+1)}\ {}_1F_1(k+1,n+1,(\beta-\alpha)u).
\end{aligned}$$

(See Abramowitz and Stegun (1965, p.505).)

Now consider inversion of $(\tilde{\eta}(0,\delta))^{k+1}(\tilde{\kappa}(0,\delta))^{n-k}$. We have

$$\begin{aligned}
&(\tilde{\eta}(0,\delta))^{k+1}(\tilde{\kappa}(0,\delta))^{n-k} \\
=\ & \left(\frac{\lambda}{c}\right)^{n+1}\left(\frac{p}{\rho+\alpha}\right)^{k+1}\left(\frac{q}{\rho+\beta}\right)^{n-k} \\
=\ & \left(\frac{\lambda}{c}\right)^{n+1}p^{k+1}q^{n-k}\int_0^\infty e^{-\rho t}\int_0^t \frac{x^k e^{-\alpha x}}{\Gamma(k+1)}\frac{(t-x)^{n-k-1}e^{-\beta(t-x)}}{\Gamma(n-k)}dxdt \\
=\ & \left(\frac{\lambda}{c}\right)^{n+1}p^{k+1}q^{n-k}\int_0^\infty e^{-\rho t}\frac{t^n e^{-\beta t}}{\Gamma(n+1)}\ {}_1F_1(k+1,n+1,(\beta-\alpha)t)dt.
\end{aligned}$$

By formula (3) this is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda}{c}\right)^{n+1}p^{k+1}q^{n-k}\left(ce^{-\lambda t}\frac{(ct)^n e^{-\beta ct}}{\Gamma(n+1)}\ {}_1F_1(k+1,n+1,(\beta-\alpha)ct)\right.$$
$$\left. +\int_0^{ct}\frac{y}{t}g(ct-y,t)\frac{y^n e^{-\beta y}}{\Gamma(n+1)}\ {}_1F_1(k+1,n+1,(\beta-\alpha)y)dy\right).$$

To evaluate the above integral we must insert for g and hence we need an expression for f^{n*}. Following Willmot and Woo (2006) we can write

$$f^{n*}(x) = \sum_{j=0}^{\infty} \gamma_{n,j} \; e(n+j, \beta; x)$$

where $e(n, \beta; x)$ denotes the Erlang(n) density with scale parameter β and

$$\gamma_{n,j} = q^n (1 - \alpha/\beta)^j \sum_{r=0}^{n} \binom{n}{r} \frac{(r)_j}{j!} \left(\frac{\alpha p}{\beta q}\right)^r.$$

Further, let us write

$${}_1F_1(k+1, n+1, (\beta - \alpha)y) = \sum_{i=0}^{\infty} d_{n,k,i} \; y^i$$

where

$$d_{n,k,i} = \frac{(k+1)_i}{(n+1)_i} \frac{(\beta - \alpha)^i}{i!}.$$

Then

$$\begin{aligned}
&\int_0^{ct} \frac{y}{t} g(ct-y,t) \frac{y^n e^{-\beta y}}{\Gamma(n+1)} \; {}_1F_1(k+1, n+1, (\beta-\alpha)y) dy \\
= \; & \sum_{m=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^m}{m!} \int_0^{ct} \sum_{j=0}^{\infty} \gamma_{m,j} \; e(m+j, \beta; ct-y) \frac{y}{t} \frac{y^n e^{-\beta y}}{n!} \sum_{i=0}^{\infty} d_{n,k,i} \; y^i dy \\
= \; & \sum_{m=1}^{\infty} e^{-(\lambda+\beta c)t} \frac{\lambda^m t^{m-1}}{m!} \sum_{j=0}^{\infty} \gamma_{m,j} \sum_{i=0}^{\infty} d_{n,k,i} \int_0^{ct} \frac{\beta^{m+j} (ct-y)^{m+j-1}}{\Gamma(m+j)} \frac{y^{n+i+1}}{n!} dy \\
= \; & \frac{1}{n!} \sum_{m=1}^{\infty} e^{-(\lambda+\beta c)t} \frac{\lambda^m t^{m-1}}{m!} \sum_{j=0}^{\infty} \gamma_{m,j} \sum_{i=0}^{\infty} d_{n,k,i} \frac{\beta^{m+j}}{\Gamma(m+j)} \int_0^{ct} (ct-y)^{m+j-1} y^{n+i+1} dy \\
= \; & \frac{1}{n!} \sum_{m=1}^{\infty} e^{-(\lambda+\beta c)t} \frac{\lambda^m t^{m-1}}{m!} \sum_{j=0}^{\infty} \gamma_{m,j} \sum_{i=0}^{\infty} d_{n,k,i} \frac{\beta^{m+j}}{\Gamma(m+j)} \\
& \times (ct)^{m+j+n+i+1} \frac{\Gamma(m+j)\Gamma(n+i+2)}{\Gamma(m+j+n+i+2)} \\
= \; & \frac{c^{n+1}}{n!} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \sum_{j=0}^{\infty} \gamma_{m,j} (\beta c)^{m+j} \sum_{i=0}^{\infty} \frac{d_{n,k,i} \; c^i}{(n+i+2)_{m+j}} e^{-(\lambda + c\beta)t} t^{2m+j+n+i} \\
= \; & \frac{c^{n+1}}{n!} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \sum_{j=0}^{\infty} \gamma_{m,j} (\beta c)^{m+j} \sum_{i=0}^{\infty} \frac{d_{n,k,i} \; c^i}{(n+i+2)_{m+j}}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\Gamma(2m+j+n+i+1)}{(\lambda+c\beta)^{2m+j+n+i+1}} e(2m+j+n+i+1, \lambda+c\beta; t) \\
= \;& \left(\frac{c}{\lambda+c\beta}\right)^{n+1} \sum_{m=1}^{\infty} \frac{1}{m!} \left(\frac{\lambda}{\lambda+c\beta}\right)^{m} \sum_{j=0}^{\infty} \gamma_{m,j} \left(\frac{c\beta}{\lambda+c\beta}\right)^{m+j} \\
&\times \sum_{i=0}^{\infty} \frac{d_{n,k,i}}{(n+i+2)_{m+j}} \left(\frac{c}{\lambda+c\beta}\right)^{i} \frac{\Gamma(2m+j+n+i+1)}{n!} e(2m+j+n+i+1, \lambda+c\beta; t)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& \frac{\Gamma(2m+j+n+i+1)}{n!} \\
= \;& \frac{\Gamma(2m+j+n+i+1)}{\Gamma(2m+j+n+1)} \frac{\Gamma(2m+j+n+1)}{\Gamma(2m+n+1)} \frac{\Gamma(2m+n+1)}{\Gamma(n+1)} \\
= \;& (2m+j+n+1)_i \; (2m+n+1)_j \; (n+1)_{2m}
\end{aligned}
$$

so that for computational purposes,

$$
\begin{aligned}
& \int_0^{ct} \frac{y}{t} g(ct-y, t) \frac{y^n e^{-\beta y}}{\Gamma(n+1)} \; {}_1F_1(k+1, n+1, (\beta-\alpha)y) dy \\
= \;& \left(\frac{c}{\lambda+c\beta}\right)^{n+1} \sum_{m=1}^{\infty} \frac{(n+1)_{2m}}{m!} \left(\frac{\lambda}{\lambda+c\beta}\right)^{m} \sum_{j=0}^{\infty} \gamma_{m,j} \left(\frac{c\beta}{\lambda+c\beta}\right)^{m+j} (2m+n+1)_j \\
&\times \sum_{i=0}^{\infty} \frac{d_{n,k,i} \; (2m+j+n+1)_i}{(n+i+2)_{m+j}} \left(\frac{c}{\lambda+c\beta}\right)^{i} e(2m+j+n+i+1, \lambda+c\beta; t).
\end{aligned} \tag{15}
$$

Further,

$$
\begin{aligned}
& \int_0^{t} c e^{-\lambda\tau} \frac{(c\tau)^n \, e^{-\beta c\tau}}{\Gamma(n+1)} \; {}_1F_1(k+1, n+1, (\beta-\alpha)c\tau) d\tau \\
= \;& \frac{c^{n+1}}{n!(\lambda+c\beta)^{n+1}} \sum_{j=0}^{\infty} \frac{(k+1)_j}{(n+1)_j} \frac{[(\beta-\alpha)c]^j}{j!(\lambda+c\beta)^j} \Gamma(n+j+1) E(n+j+1, (\lambda+c\beta)t)
\end{aligned} \tag{16}
$$

and

$$\int_0^t \int_0^{c\tau} \frac{y}{\tau} g(c\tau - y, \tau) \frac{y^n e^{-\beta y}}{\Gamma(n+1)} \ {}_1F_1(k+1, n+1, (\beta-\alpha)y) dy d\tau$$

$$= \left(\frac{c}{\lambda + c\beta}\right)^{n+1} \sum_{m=1}^{\infty} \frac{(n+1)_{2m}}{m!} \left(\frac{\lambda}{\lambda + c\beta}\right)^m \sum_{j=0}^{\infty} \gamma_{m,j} \left(\frac{c\beta}{\lambda + c\beta}\right)^{m+j} (2m+n+1)_j$$

$$\times \sum_{i=0}^{\infty} \frac{d_{n,k,i} \ (2m+j+n+1)_i}{(n+i+2)_{m+j}} \left(\frac{c}{\lambda + c\beta}\right)^i E(2m+j+n+i+1, \lambda + c\beta; t). \tag{17}$$

where $E(n, \beta; x)$ denotes the Erlang(n) distribution function with scale parameter β.

Now let the sum of expressions (16) and (17) be denoted by $\zeta_{n,k}(t)$. Then

$$H(u,t) \overset{def}{=} \int_0^t \eta(u,\tau) d\tau$$

$$= \frac{\lambda p e^{-\beta u}}{c} \sum_{n=0}^{\infty} \frac{(\lambda \beta q u/c)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\alpha p}{\beta q}\right)^k \ {}_1F_1(k+1, n+1, (\beta-\alpha)u) \zeta_{n,k}(t)$$

We can take the same approach to obtain $\kappa(u,t)$. We have

$$\kappa(u,t) = \int_0^t \int_0^u \left(\eta(0,\tau)\alpha e^{-\alpha x} + \kappa(0,\tau)\beta e^{-\beta x}\right) \kappa(u-x, t-\tau) dx d\tau + \kappa(0,t) e^{-\beta u}$$

which gives

$$\tilde{\tilde{\kappa}}(s,\delta) = \left(\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s} + \tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}\right) \tilde{\tilde{\kappa}}(s,\delta) + \tilde{\kappa}(0,\delta)\frac{1}{\beta+s},$$

and hence

$$\tilde{\tilde{\kappa}}(s,\delta) = \frac{1}{\beta} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\tilde{\kappa}(0,\delta)\frac{\beta}{\beta+s}\right)^{k+1} \left(\tilde{\eta}(0,\delta)\frac{\alpha}{\alpha+s}\right)^{n-k}.$$

Inversion of

$$\left(\frac{\beta}{\beta+s}\right)^{k+1} \left(\frac{\alpha}{\alpha+s}\right)^{n-k}$$

gives

$$e^{-\alpha u} \left(\frac{\beta}{\alpha}\right)^k \frac{\beta(\alpha u)^n}{n!} \ {}_1F_1(k+1, n+1, (\alpha-\beta)u), \tag{18}$$

and using the identity (see Abramowitz and Stegun (1965, p.504))

$$e^x \ {}_1F_1(a,b,-x) = \ {}_1F_1(b-a,b,x)$$

we can rewrite expression (18) as

$$e^{-\beta u}\left(\frac{\beta}{\alpha}\right)^k \frac{\beta(\alpha u)^n}{n!} \ {}_1F_1(n-k,n+1,(\beta-\alpha)u).$$

Next, inversion of

$$(\tilde{\kappa}(0,\delta))^{k+1}\,(\tilde{\eta}(0,\delta))^{n-k}$$

yields

$$\begin{aligned}
&\left(\frac{\lambda}{c}\right)^{n+1} q^{k+1}p^{n-k}\int_0^\infty e^{-\rho t}\frac{t^n e^{-\alpha t}}{n!} \ {}_1F_1(k+1,n+1,(\alpha-\beta)t)dt \\
= &\left(\frac{\lambda}{c}\right)^{n+1} q^{k+1}p^{n-k}\int_0^\infty e^{-\rho t}\frac{t^n e^{-\beta t}}{n!} \ {}_1F_1(n-k,n+1,(\beta-\alpha)t)dt
\end{aligned}$$

which by formula (3) is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda}{c}\right)^{n+1} q^{k+1}p^{n-k}\left(ce^{-\lambda t}\frac{(ct)^n\, e^{-\beta ct}}{\Gamma(n+1)} \ {}_1F_1(n-k,n+1,(\beta-\alpha)ct)\right.$$
$$\left. + \int_0^{ct}\frac{y}{t}g(ct-y,t)\frac{y^n e^{-\beta y}}{\Gamma(n+1)} \ {}_1F_1(n-k,n+1,(\beta-\alpha)y)dy\right).$$

Now define

$$d^*_{n,k,i} = \frac{(n-k)_i}{(n+1)_i}\frac{(\beta-\alpha)^i}{i!}$$

Then by analogy with equation (15),

$$\begin{aligned}
&\int_0^{ct}\frac{y}{t}g(ct-y,t)\frac{y^n e^{-\beta y}}{\Gamma(n+1)} \ {}_1F_1(n-k,n+1,(\beta-\alpha)y)dy \\
= &\left(\frac{c}{\lambda+c\beta}\right)^{n+1}\sum_{m=1}^\infty \frac{(n+1)_{2m}}{m!}\left(\frac{\lambda}{\lambda+c\beta}\right)^m \sum_{j=0}^\infty \gamma_{m,j}\left(\frac{c\beta}{\lambda+c\beta}\right)^{m+j}(2m+n+1)_j \\
&\times\sum_{i=0}^\infty \frac{d^*_{n,k,i}\ (2m+j+n+1)_i}{(n+i+2)_{m+j}}\left(\frac{c}{\lambda+c\beta}\right)^i e(2m+j+n+i+1,\lambda+c\beta;t).
\end{aligned}$$

Further,

$$\begin{aligned}
&\int_0^t ce^{-\lambda\tau}\frac{(c\tau)^n\, e^{-\beta c\tau}}{\Gamma(n+1)} \ {}_1F_1(n-k,n+1,(\beta-\alpha)c\tau)d\tau \\
= &\frac{c^{n+1}}{n!(\lambda+c\beta)^{n+1}}\sum_{j=0}^\infty \frac{(n-k)_j}{(n+1)_j}\frac{[(\beta-\alpha)\,c]^j}{j!(\lambda+c\beta)^j}\Gamma(n+j+1)E(n+j+1,(\lambda+c\beta)t)
\end{aligned} \quad (19)$$

and

$$\int_0^t \int_0^{c\tau} \frac{y}{\tau} g(c\tau - y, \tau) \frac{y^n e^{-\beta y}}{\Gamma(n+1)} \; {}_1F_1(n-k, n+1, (\beta-\alpha)y) dy d\tau$$
$$= \left(\frac{c}{\lambda + c\beta}\right)^{n+1} \sum_{m=1}^{\infty} \frac{(n+1)_{2m}}{m!} \left(\frac{\lambda}{\lambda + c\beta}\right)^m \sum_{j=0}^{\infty} \gamma_{m,j} \left(\frac{c\beta}{\lambda + c\beta}\right)^{m+j} (2m+n+1)_j$$
$$\times \sum_{i=0}^{\infty} \frac{d^*_{n,k,i} \; (2m+j+n+1)_i}{(n+i+2)_{m+j}} \left(\frac{c}{\lambda + c\beta}\right)^i E(2m+j+n+i+1, \lambda + c\beta; t). \tag{20}$$

Letting $\zeta^*_{n,k}(t)$ denote the sum of expressions (19) and (20), we obtain

$$K(u,t) \stackrel{def}{=} \int_0^t \kappa(u,\tau) d\tau$$
$$= \frac{\lambda q e^{-\beta u}}{c} \sum_{n=0}^{\infty} \frac{(\lambda p \alpha u/c)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\beta q}{\alpha p}\right)^k \; {}_1F_1(n-k, n+1, (\beta-\alpha)u) \zeta^*_{n,k}(t),$$

and $\psi(u,t) = H(u,t) + K(u,t)$.

The above formulae for $\eta(u,t)$, $\kappa(u,t)$, $H(u,t)$ and $K(u,t)$ involve infinite sums whose summands comprise simple functions. In applying these formulae, we generally found it easier to compute $H(u,t)$ and $K(u,t)$ (although this may simply reflect the author's programming skills), and results based on these are shown in Tables 2 to 4. For small values of u and t, evaluation of $H(u,t)$ and $K(u,t)$ posed no problems and computer run times were short. For the larger values of u and t in Tables 2 to 4, computer run times very lengthy (several hours) and care has to be exercised in computer programs to avoid numerical underflow or overflow. To illustrate our formulae, we have chosen the same mixed exponential distribution and parameters as Garcia (2005), namely $p = 1 - q = \frac{1}{3}$, $\alpha = \frac{1}{2}$ and $\beta = 2$, with $\lambda = 1$ and $c = 1.1$.

	$H(0,t)$	$K(0,t)$	$\psi(0,t)$	$W(0,1,t)$	$W(0,3,t)$	$W(0,5,t)$
$t = 10$	0.4640	0.2863	0.7503	0.4301	0.6460	0.7122
$t = 20$	0.5142	0.2924	0.8066	0.4551	0.6911	0.7643
$t = 30$	0.5365	0.2950	0.8316	0.4662	0.7111	0.7875
$t = 40$	0.6497	0.2966	0.8463	0.4727	0.7229	0.8012
$t = 50$	0.5587	0.2976	0.8563	0.4771	0.7309	0.8104

Table 2: Values when $u = 0$, Mixed exponential claims

As it took several hours of computing time to produce the values in Table 4, we also calculated approximations to values of $W(20, y, t)$ using the

	$H(10,t)$	$K(10,t)$	$\psi(10,t)$	$W(10,1,t)$	$W(10,3,t)$	$W(10,5,t)$
$t = 10$	0.0644	0.0068	0.0712	0.0312	0.0568	0.0659
$t = 20$	0.1281	0.0142	0.1422	0.0626	0.1136	0.1317
$t = 30$	0.1754	0.0196	0.1950	0.0860	0.1558	0.1806
$t = 40$	0.2110	0.0237	0.2347	0.1035	0.1876	0.2174
$t = 50$	0.2387	0.0269	0.2656	0.1172	0.2123	0.2460

Table 3: Values when $u = 10$, Mixed exponential claims

	$H(20,t)$	$K(20,t)$	$\psi(20,t)$	$W(20,1,t)$	$W(20,3,t)$	$W(20,5,t)$
$t = 10$	0.0041	0.0004	0.0045	0.0020	0.0036	0.0042
$t = 20$	0.0156	0.0017	0.0173	0.0076	0.0138	0.0160
$t = 30$	0.0303	0.0033	0.0336	0.0148	0.0268	0.0311
$t = 40$	0.0453	0.0050	0.0504	0.0222	0.0402	0.0466
$t = 50$	0.0596	0.0067	0.0663	0.0292	0.0530	0.0614

Table 4: Values when $u = 20$, Mixed exponential claims

algorithm of Dickson and Waters (1992). The approximations are shown in parenthesis in Table 5, and are based on a scaling factor of 100 in the algorithm. (A higher value of the scaling factor can be used to produce better approximations.) By contrast with the computer run time required to produce exact values, the approximations were computed in minutes, and are excellent.

	$W(20,1,t)$	$W(20,3,t)$	$W(20,5,t)$	$\psi(20,t)$
$t = 10$	0.0020	0.0036	0.0042	0.0045
	(0.0020)	(0.0036)	(0.0042)	(0.0045)
$t = 30$	0.0148	0.0268	0.0311	0.0336
	(0.0148)	(0.0268)	(0.0311)	(0.0336)
$t = 50$	0.0292	0.0530	0.0614	0.0663
	(0.0292)	(0.0529)	(0.0614)	(0.0663)

Table 5: Exact and approximate values when $u = 20$, Mixed exponential claims

5 Concluding remarks

The approach presented in this paper easily applies in the case of exponential individual claims. It produces the same solution for $w(u,t)$ as Drekic and Willmot (2003) give, although the transform inversion problem is quite different. Similarly, if

$$f(x) = p\ \beta e^{-\beta x} + (1-p)\ \beta^2 x e^{-\beta x}$$

where $0 < p < 1$, an explicit solution can be obtained for $w(u,y,t)$. The approach, however, becomes unwieldy for other individual claim amount distributions, for example Erlang(3), although it is possible to obtain further explicit solutions when $u = 0$. Nevertheless, this approach has allowed us to obtain formulae from which the joint distribution of the time of ruin and the deficit at ruin can be calculated. It should also apply to certain Sparre Andersen models with the individual claim amount distributions considered in this paper, although a first stage in developing this idea will be to find a transform relationship equivalent to (3) for these models.

We had mixed experiences in our computations. For Erlang(2) claims, programming of formulae is straightforward and computations can be done quickly. By contrast, for our example with mixed exponential claims, programming was more complicated and computer run times were lengthy. Our comparison of exact and approximate values of $W(u,y,t)$ in Table 3 suggests that while it is appealing to have an explicit solution to a problem, in practice it may be more appropriate to compute an approximate solution.

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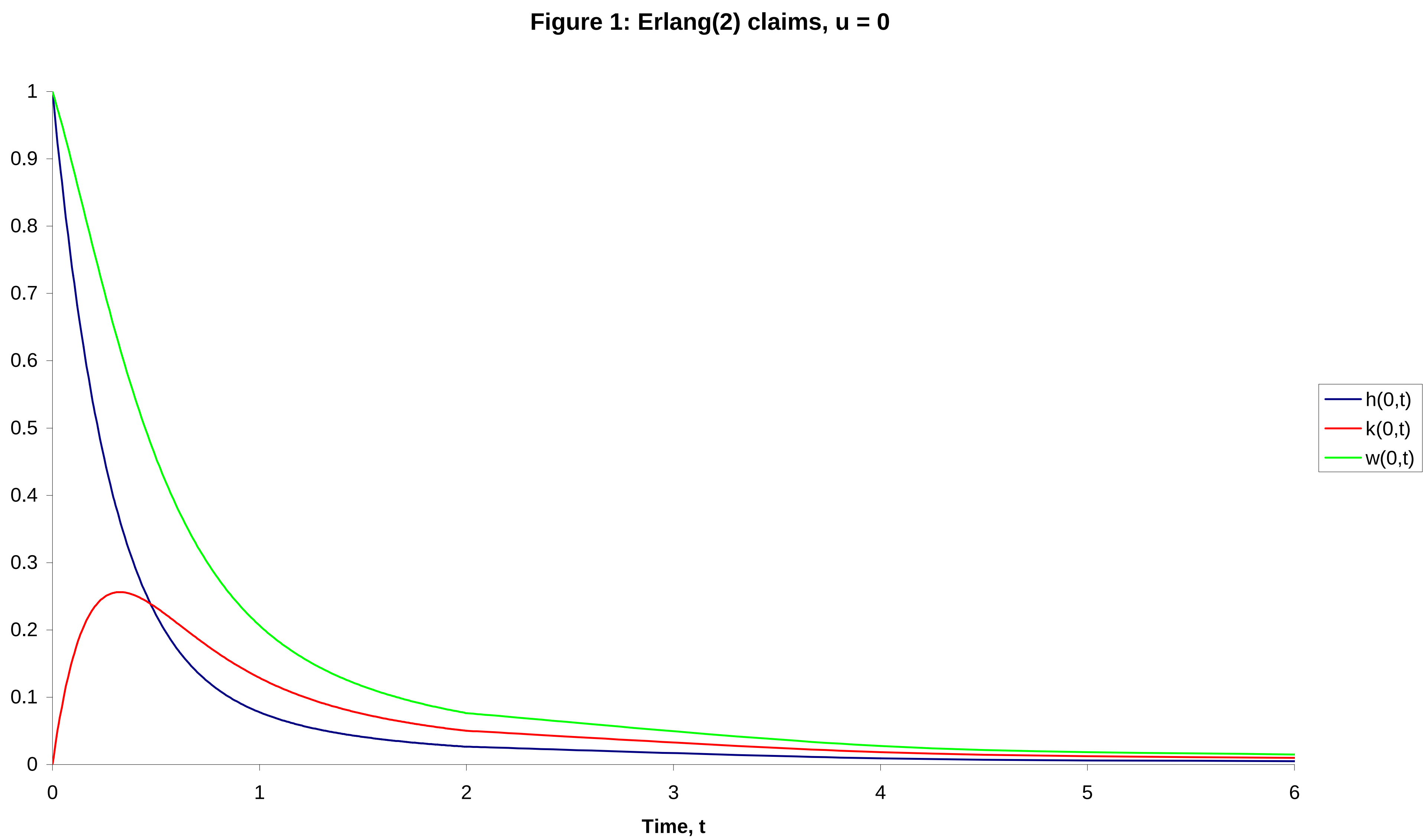
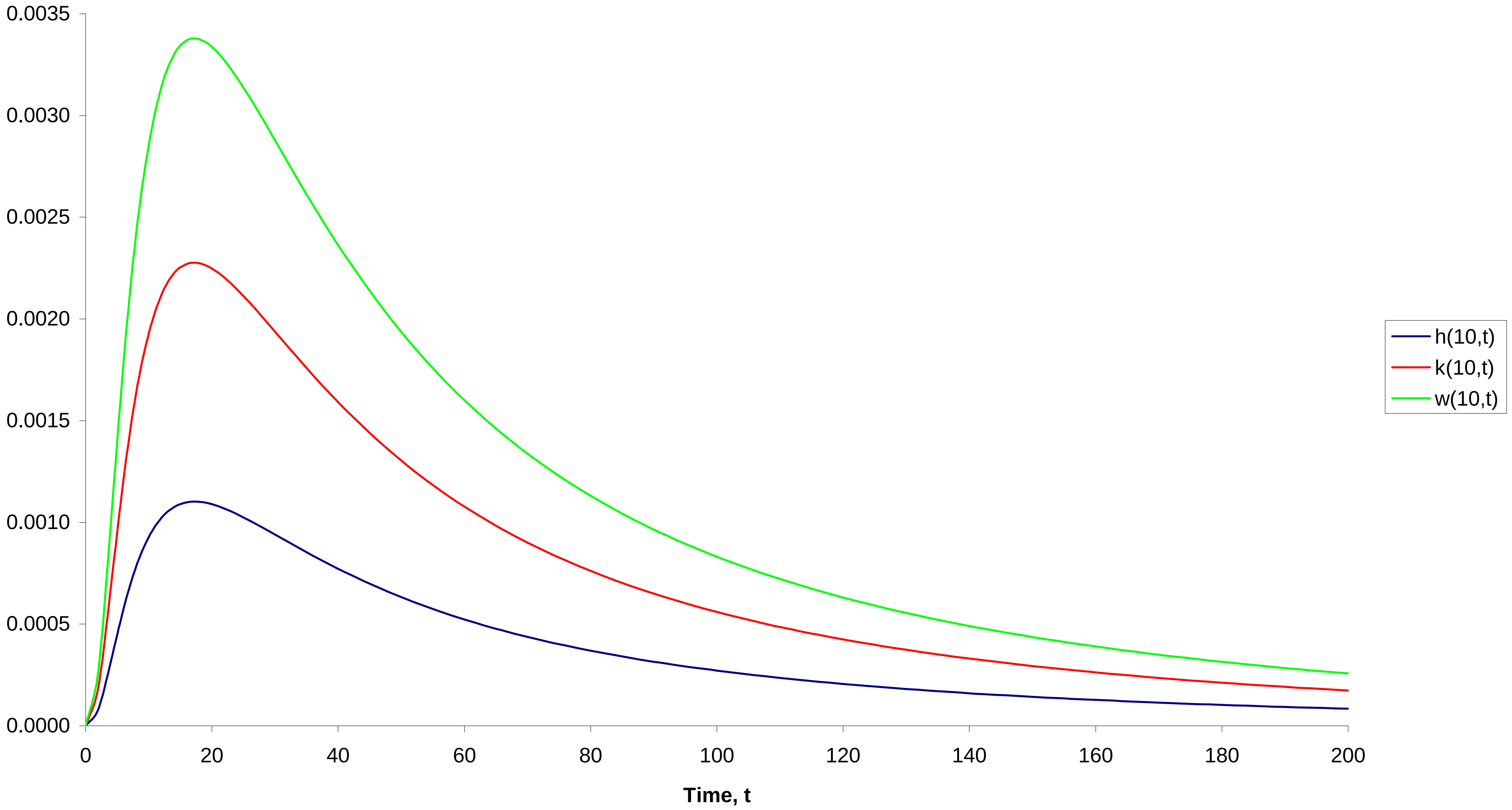


Figure 1: Erlang(2) claims, u = 0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
Time, t
h(0,t)
k(0,t)
w(0,t)

Figure 2: Erlang(2) claims, u = 10



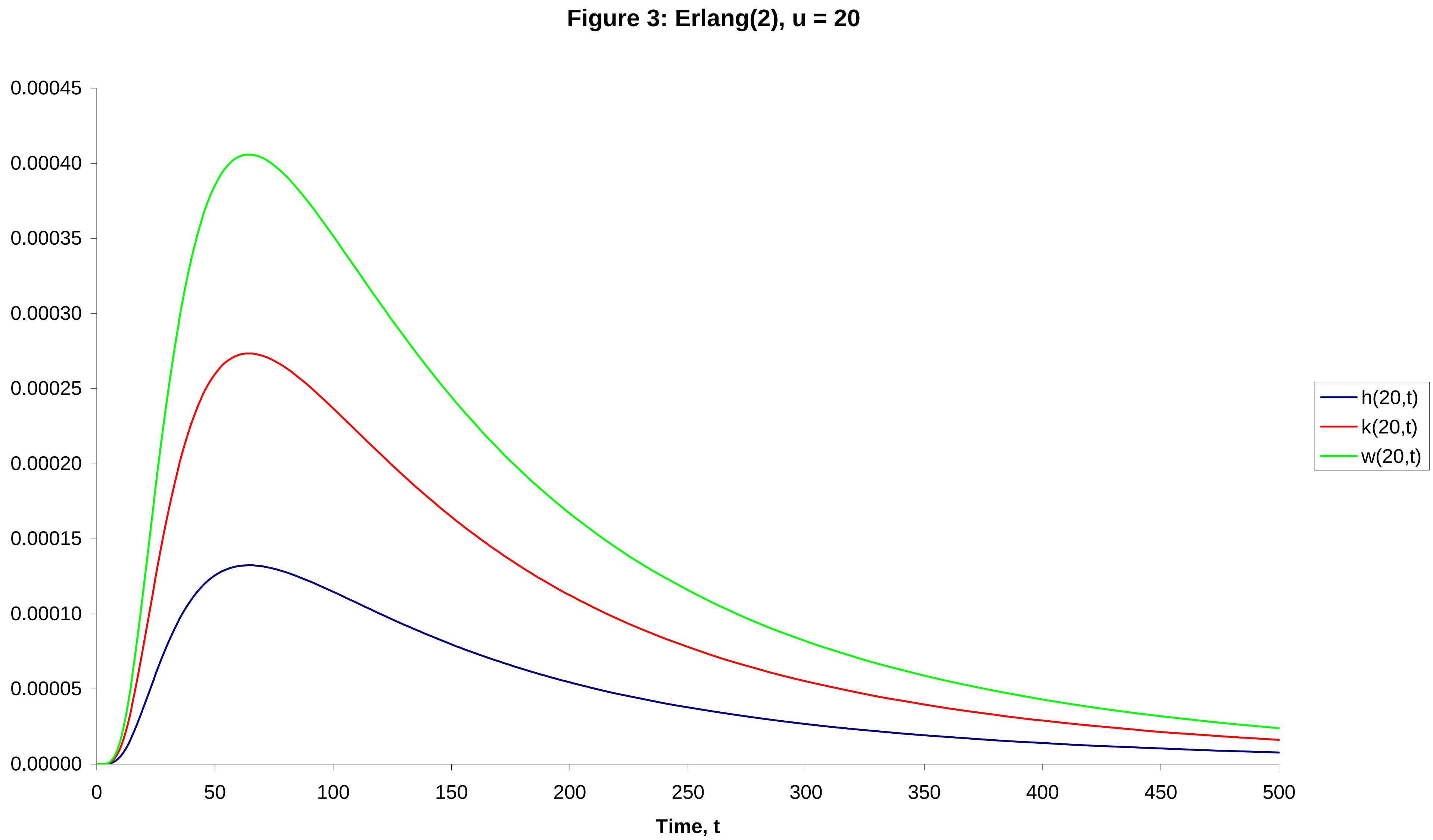


Figure 3: Erlang(2), u = 20
0.00045
0.00040
0.00035
0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0
50
100
150
200
250
300
350
400
450
500
Time, t
h(20,t)
k(20,t)
w(20,t)