FAST AND ACCURATE PRICING AND HEDGING OF LONG-DATED CMS SPREAD OPTIONS

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ABSTRACT. We present a fast method to price and hedge CMS spread options in the displaced-diffusion co-initial swap market model. Numerical tests demonstrate that we are able to obtain sufficiently accurate prices and Greeks with computational times measured in milliseconds. Further, we find that CMS spread options are weakly dependent on the at-the-money Black implied volatility skews.

1. INTRODUCTION

Constant maturity swap (CMS) spread options which are options written on the difference between two CMS rates have become increasingly popular in the fixed-income market since they give investors an opportunity to exercise their views on the shape of the yield curve. However, even if we use a simple model where the two underlying CMS rates are modelled by log-normal random variables, it is difficult to price such financial derivatives since the distribution of the sum of two log-normal random variables is not known and, in any case, convexity corrections distort the log-normality.

In practice, the standard model to price interest rate derivatives is the LIBOR market model (LMM) introduced in Brace et al (1997). Although it is computationally more expensive to implement such a high-dimensional model than the traditional short-rate models, the LMM has the benefits of automatic calibration to the market prices of European type derivatives. Furthermore, one-factor short rate models cannot price CMS spread options accurately since they are not able to capture the correlations between different swap-rates.

Tamminen (2008) carried out a numerical analysis on the pricing of CMS spread options in the LMM and found out that a LMM with at least 5 factors is required to produce accurate prices relative to a full-factor model. Thus, in order to obtain accurate prices of long-dated CMS spread options, we would need to run multi-step multi-factor LMM simulations, which would be very time consuming.

Date: December 8, 2009.

1991 *Mathematics Subject Classification.* 91B24, 60G40, 60G44. JEL Classification: G13.

Key words and phrases. Spread option, Gaussian quadrature rule, Delta, Vega, Market skew sensitivity.

In the literature, there are two main streams of research on fast and accurate pricing of spread options: approximation methods that seek analytical formula for the price, and numerical methods such as Monte Carlo simulation or numerical integration. Most attention has been paid to the equity and energy markets. Carmona and Durrleman (2003) provided a comprehensive review of the analytical approximation methods. Dempster and Hong (2000) used fast Fourier transform to approximate the price of spread options. Hurd and Zhou (2009) introduced a new formula based on Fourier transform and complex Gamma functions to price and hedge spread options.

Recently, two analytical formulae for CMS spread options were developed in Belomestny et al (2008) and Wu and Chen (2009), in which the authors used a drift-freezing LMM under the spot measure. Analytical methods have the virtue of simple and fast implementations. However, as we shall see in this paper, drift freezing can lead to large errors for long-dated options.

In this paper, we price CMS spread options numerically. We use the co-initial swap market model (ciSMM) developed in Galluccio and Hunter (2004). The underlying variables of the model are the co-initial swap-rates that have the same reset date but different ending dates. Hence, this model is a natural candidate to model CMS spread options, which only depend on the joint dynamics of the underlying rates at maturity. Leon (2006) carried out an analysis on pricing CMS spread options using a log-normal ciSMM. In that paper, a 'drift-freezing' Euler scheme is used to evolve the co-initial rates, and the prices of spread options are computed numerically via a two-dimensional integration.

From our numerical tests (in section 3), we find out that the one-step Euler scheme may give large errors for long-dated spread options. However, if a simple 2-factor predictor-corrector (PC) scheme that corrects the drift of the rates by averaging the values at both the start and the end of the step is used to evolve the co-initial rates, the pricing errors can be significantly reduced. In our numerical examples, the pricing error under the PC scheme is generally less than 1 basis point while the error under the Euler scheme can exceed 10 basis points.

Now, pricing using a 2-factor PC scheme Monte Carlo simulation is equivalent to computing a 2-dimensional integral numerically using random numbers. There exist faster and more efficient methods to perform the same task (see Press et al (1992)). The main purpose of this paper is to introduce a fast and accurate integration method to price and hedge CMS spread options with long maturities such as 20 years. We use a 2-dimensional Gaussian quadrature rule to numerically compute the 2-dimensional pricing integral. The main benefit of the Gaussian quadrature rules is that integrals can be computed by choosing only a small number of nodes, which results in substantial computational time reductions. Timing tests illustrate that we can price 1,000 CMS spread options in less than 0.5 seconds for certain sets of parameters! There appears to be no papers addressing the problem of computing the Greeks of a CMS spread option. We show how to compute model Greeks with the adjoint method. Numerical tests show that the 2-dimensional quadrature rule gives roughly the same prices and Greeks compared with the 2-factor PC scheme with 65,535 paths.

Furthermore, we use a displaced-diffusion version of the ciSMM (DD-ciSMM) since the model can produce implied volatility (IV) skews, which are prevalent in interest rate derivative markets. In the paper, we derive an analytical formula for IV skews, from which we are then able to compute market skew sensitivities using model Greeks. Numerical tests indicate that at-the-money skews have low impacts on the price of CMS spread options.

The paper is organized as follows. In section 2, we review the DDciSMM and various numerical schemes to implement the model. We show how to price a CMS spread option with a 2-factor PC scheme in section 3. In section 4, we illustrate how to use the two-dimensional Gaussian quadrature rule to price the spread option. We show how to estimate model sensitivities using quadrature rules in section 5. In section 6, we first derive an explicit formula for IV skews, we then analyze market skew sensitivities of CMS spread options. We conclude in section 7. An efficient implementation of the adjoint method is given in the appendix.

2. The Displaced-Diffusion Co-Initial Swap Market Model

2.1. **Notations.** The tenor structure is a finite set of dates

$$0 = T_{-1} < T_0 < T_1 < \cdots < T_{n-1} < T_n.$$

Let $\tau_{i-1} = T_i - T_{i-1}$, for all i, and P_i denote the price of the zero-coupon bond maturing at time T_i. SR_i is the co-initial swap-rates associated to times $T_0, \ldots, T_i$, for all i,

$$\mathrm{SR}_i = \frac{P_0 - P_i}{A_i}, \tag{2.1}$$

where $A_i = \sum_{k=1}^{i} \tau_{k-1} P_k$ is the value of the annuity of SR_i.

2.2. **Model set-up.** The n rates will be driven by an F-dimensional standard Brownian motion. We assume a piecewise constant volatility structure and therefore assign a pseudo-square root, $A = \{a_{ik}\}$, of the covariance matrix, C, for each step to determine the evolution. We can therefore write across each step

$$\mathrm{dSR}_i = \mu_i \mathrm{d}t + \big(\mathrm{SR}_i + \alpha_i\big) \sum_{k=1}^{F} a_{ik} \mathrm{d}Z_k \tag{2.2}$$

where μ_i is the drift of SR_i under the spot measure associated with the P_0 bond, $\{\alpha_i\}$ is a set of displaced-diffusion coefficients, and $\{Z_k\}$ is a vector of independent Brownian motions.

2.2.1. *The cross variation derivative.*

Definition 2.1. *The cross variation derivative for two Itô processes*

$$\mathrm{d}X_t = \mu_X\big(X_t, Y_t, t\big)\mathrm{d}t + \sigma_X\big(X_t, Y_t, t\big)\mathrm{d}W_t^X,$$
$$\mathrm{d}Y_t = \mu_Y\big(Y_t, Y_t, t\big)\mathrm{d}t + \sigma_Y\big(X_t, Y_t, t\big)\mathrm{d}W_t^Y$$

is defined to be the coefficient of $\mathrm{d}t$ *in* $\mathrm{d}X_t\mathrm{d}Y_t$*. If* $\mathrm{d}W_t^X\mathrm{d}W_t^Y = \rho\mathrm{d}t$ *then*

$$\big\langle X_t, Y_t\big\rangle = \rho\sigma_X\big(X_t, Y_t, t\big)\sigma_Y\big(X_t, Y_t, t\big). \tag{2.3}$$

We give a summary of the main properties of the cross variation derivative: for Itô processes X_t, Y_t and Z_t,

- (Linearity) $\big\langle X_t, Y_t + Z_t\big\rangle = \big\langle X_t, Y_t\big\rangle + \big\langle X_t, Z_t\big\rangle$;
- (Product rule) $\big\langle X_t, Y_tZ_t\big\rangle = Z_t\big\langle X_t, Y_t\big\rangle + Y_t\big\langle X_t, Z_t\big\rangle$;
- (Quotient rule) $\big\langle X_t, Y_t^{-1}\big\rangle = -Y_t^{-2}\big\langle X_t, Y_t\big\rangle$.

For detailed discussion of the cross variation derivative, we refer the reader to Joshi and Liesch (2007).

2.2.2. *General drift formulae.* For ease of notation, denote by $\overline{A}_i$ and $\overline{P}_i$ the annuity and bond prices deflated by P_0 respectively. Then (2.1) can be rewritten as

$$\mathrm{SR}_i = \frac{1 - \overline{P}_i}{\overline{A}_i}. \tag{2.4}$$

Joshi and Liesch (2007) derived the drifts in a log-normal ciSMM. In this paper, we derive drift formulae in a displaced-diffusion ciSMM.

Proposition 2.1. *The general drift formulae under the spot measure in the DD-ciSMM is*

$$\mu_i = -\frac{1}{\overline{A}_i}\big(\mathrm{SR}_i + \alpha_i\big)\sum_{k=1}^{F} a_{ik}\Big\langle Z_k, \overline{A}_i\Big\rangle, \tag{2.5}$$

where the cross variation derivatives are given by the recursive formula

$$\Big\langle Z_k, \overline{A}_i\Big\rangle = \frac{\Big\langle Z_k, \overline{A}_{i-1}\Big\rangle - a_{ik}\Big[\tau_{i-1}\big(\mathrm{SR}_i + \alpha_i\big)\overline{A}_i\Big]}{1 + \tau_{i-1}\mathrm{SR}_i}, \tag{2.6}$$

with $\big\langle Z_k, \overline{A}_0\big\rangle = 0$.

Proof. The drifts of the co-initial swap-rates are determined by no-arbitrage considerations to ensure that the ratio of every bond price to the *numeraire* bond P_0 is a martingale. Since SR_iA_i, A_i are tradables, then $\mathrm{SR}_i\overline{A}_i$ and $\overline{A}_i$ are martingales under the spot measure. Therefore the following stochastic differential equation

$$\begin{aligned}\mathrm{d}\Big(\mathrm{SR}_i\overline{A}\Big) &= \overline{A}_i\mathrm{dSR}_i + \mathrm{SR}_i\mathrm{d}\overline{A}_i + \Big\langle\mathrm{SR}_i, \overline{A}_i\Big\rangle\mathrm{d}t\\ &= \Big[\mu_i\overline{A}_i + \Big\langle\mathrm{SR}_i, \overline{A}_i\Big\rangle\Big]\mathrm{d}t + \overline{A}_i\big(\mathrm{SR}_i + \alpha_i\big)\sum_{k=1}^{F} a_{ik}\mathrm{d}Z_k + \mathrm{SR}_i\mathrm{d}\overline{A}_i\end{aligned}$$

has no drift term, then

$$\mu_i = -\frac{1}{\overline{A}_i}\Big\langle \mathrm{SR}_i, \overline{A}_i \Big\rangle$$
$$= -\frac{1}{\overline{A}_i}\big(\mathrm{SR}_i + \alpha_i\big)\sum_{k=1}^{F} a_{ik}\Big\langle Z_k, \overline{A}_i \Big\rangle.$$

From the definition of the annuities

$$\overline{A}_i = \overline{A}_{i-1} + \tau_{i-1}\overline{P}_i, \tag{2.7}$$

we have the following equality

$$\Big\langle Z_k, \overline{A}_i \Big\rangle = \Big\langle Z_k, \overline{A}_{i-1} \Big\rangle + \tau_{i-1}\Big\langle Z_k, \overline{P}_i \Big\rangle. \tag{2.8}$$

Joshi and Liesch (2007) derived the analytical formula for the latter cross variation derivative in (2.8), we use a different method to derive the recursive formula (2.6). Using (2.4) and the product rule, the latter cross variation term is given by

$$\Big\langle Z_k, \overline{P}_i \Big\rangle = -\Big\langle Z_k, \mathrm{SR}_i\overline{A}_i \Big\rangle$$
$$= -\Big[\big\langle Z_k, \mathrm{SR}_i\big\rangle\overline{A}_i + \Big\langle Z_k, \overline{A}_i\Big\rangle\mathrm{SR}_i\Big]$$
$$= -\Big[a_{ik}\big(\mathrm{SR}_i + \alpha_i\big)\overline{A}_i + \mathrm{SR}_i\Big\langle Z_k, \overline{A}_i\Big\rangle\Big]. \tag{2.9}$$

Substitute (2.9) into (2.8), we have the recursive formula (2.6). □

2.3. **Efficient computation of annuity ratios and drifts.** We define $\overline{A}_0 = 0$. Joshi and Liesch (2007) used the following algorithms to compute bond and annuity ratios:

$$\overline{P}_i = \frac{1 - \mathrm{SR}_i\overline{A}_{i-1}}{1 + \tau_{i-1}\mathrm{SR}_i}, \quad i = 1, 2, \ldots, n, \tag{2.10}$$

and the annuity ratios are given by (2.7).

Galluccio and Hunter (2004) proposed an algorithm of order $\mathcal{O}(n^2)$ to compute the drifts of the co-initial rates, and an algorithm involving solving a forward linear system to compute the bond ratios. The algorithms in Joshi and Liesch (2007) give an order of $\mathcal{O}(nF)$ per step.

2.4. **Numerical schemes.** We introduce two popular numerical schemes to evolve the co-initial swap-rates.

2.4.1. *'Drift-freezing' Euler scheme.* We first compute drifts $\widehat{\mu}_i(T_r)$ using the values of the annuity ratios and co-initial rates at time T_{r-1}, we then use the following scheme to evolve the set of discretized co-initial swap-rates $\widehat{\mathrm{SR}}_i$:

$$\widehat{\mathrm{SR}}_i(T_r) = \Big(\widehat{\mathrm{SR}}_i(T_{r-1}) + \alpha_i\Big)\exp\left[\widehat{\mu}_i(T_r) + \sum_{k=1}^{F}\left(a_{ik}Z_k - \frac{a_{ik}^2}{2}\right)\right] - \alpha_i, \tag{2.11}$$

where $\{Z_k\}$ is a set of independent normal variates, for $i = 1, \ldots, n$.

2.4.2. *Predictor-corrector scheme.* The PC scheme to approximate stochastic differential equations is due to Kloeden and Platen (2000). The scheme corrects the stochastic drift by averaging the values at both the start and the end of the step. Hunter et al (2001) applied the PC scheme in the LMM. In the DDciSMM: we first evolve the predictor swap-rates $\widehat{\mathrm{SR}}_i(T_r)$ using the Euler scheme (2.11). Next, we compute the predictor annuity ratios using (2.7) and (2.10) with $\widehat{\mathrm{SR}}_i(T_r)$. Then we compute the predictor drifts $\widetilde{\mu}_i(T_r)$ using the predictor swap-rates and predictor annuity ratios. Finally, we use the following scheme to evolve the discretized swap-rates $\widetilde{\mathrm{SR}}_i$:

$$\begin{aligned}\widetilde{\mathrm{SR}}_i(T_r) &= \Big(\widetilde{\mathrm{SR}}_i(T_{r-1}) + \alpha_i\Big) \exp\left[\frac{\widetilde{\mu}_i(T_r) + \widehat{\mu}_i(T_r)}{2} + \sum_{k=1}^{F}\Big(a_{ik}Z_k - \frac{a_{ik}^2}{2}\Big)\right] \\ &\quad - \alpha_i \\ &= \Big(\widehat{\mathrm{SR}}_i(T_r) + \alpha_i\Big) \exp\left[\frac{\widetilde{\mu}_i(T_r) - \widehat{\mu}_i(T_r)}{2}\right] - \alpha_i, \end{aligned} \tag{2.12}$$

where $\{Z_k\}$ is a set of independent normal variates, for $i = 1, \ldots, n$.

3. Pricing CMS Spread Option Using a 2-Factor Model

3.1. **Pricing under the spot measure.** Published research papers concentrate on using the P_0 bond as the numeraire. The price of a CMS spread option under the spot measure is given by

$$V(0) = P_0(0)\widetilde{\mathbb{E}}^{(T_0)}\left[\tau\big(\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K\big)^+\right], \tag{3.1}$$

where τ is an accrual factor. We use Monte Carlo simulation to compute (3.1) using Euler and PC schemes introduced in section 2.4.

3.2. **Pricing under the annuity measure.** Alternatively, we can price spread options with an annuity numeraire. The price of a CMS spread option under the annuity measure associated with A_l is given by

$$V(0) = A_l(0)\widetilde{\mathbb{E}}^{(A_l)}\left[\frac{\tau\big(\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K\big)^+}{A_l(T_0)}\right]. \tag{3.2}$$

We normally choose an annuity A_l with $l = \min\{i, j\}$ as A_l depends on fewer swap-rates, therefore we expect A_l to be exposed to less discretization errors in the Monte Carlo simulation. We use the annuity numeraire to make one of the underlying swap-rates in (3.2) driftless, which will simplify the numerical integration in section 4.

The dynamics of the co-initial swap-rates under the annuity measure is given in the following proposition.

Proposition 3.1. *Under the annuity measure $\widetilde{\mathbb{P}}^{(A_l)}$, one co-initial swap-rate is driftless and the others have state-dependent drifts*

$$\mu_i^{(A_l)} = -\frac{1}{\overline{A}_i}\big(\mathrm{SR}_i + \alpha_i\big)\sum_{k=1}^{F} a_{ik}\Big\langle Z_k, \overline{A}_i\Big\rangle^{A_l}, \tag{3.3}$$

where

$$\Big\langle Z_k, \overline{A}_i\Big\rangle^{A_l} = \Big\langle Z_k, \overline{A}_i\Big\rangle - \frac{\overline{A}_i}{\overline{A}_l}\Big\langle Z_k, \overline{A}_l\Big\rangle$$

with $\overline{A}_i = A_i/P_0$. In particular, $\mu_l^{(A_l)}$ is equal to zero.

Proof. Dynamics of the swap-rates under the annuity measure $\widetilde{\mathbb{P}}^{(A_l)}$ are characterized by the stochastic differential equation

$$\mathrm{dSR}_i = \mu_i^{(A_l)}\mathrm{d}t + \big(\mathrm{SR}_i + \alpha_i\big)\sum_{k=1}^{F} a_{ik}\mathrm{d}Z_k.$$

To ensure ratios of tradables to the numeraire annuity are martingales, the following stochastic differential equation

$$\mathrm{d}\Big(\frac{\mathrm{SR}_i A_i}{A_l}\Big) = \mathrm{dSR}_i\Big(\frac{A_i}{A_l}\Big) + \mathrm{SR}_i\mathrm{d}\Big(\frac{A_i}{A_l}\Big) + \Big\langle \mathrm{SR}_i, \frac{A_i}{A_l}\Big\rangle\mathrm{d}t$$

has no drift term. Hence the drift of SR_i under the annuity measure is

$$\begin{aligned}\mu_i^{(A_l)} &= -\frac{A_l}{A_i}\Big\langle \mathrm{SR}_i, \frac{A_i}{A_l}\Big\rangle \\ &= -\frac{1}{\overline{A}_i}\Big\langle \mathrm{SR}_i, \overline{A}_i\Big\rangle - \overline{A}_l\Big\langle \mathrm{SR}_i, \frac{1}{\overline{A}_l}\Big\rangle. \end{aligned}\tag{3.4}$$

The first term in (3.4) is the drift under the spot measure. Using the quotient rule for cross variation derivatives (see section 2.2.1), the second term in (3.4) can be rewritten as

$$\overline{A}_l\Big\langle \mathrm{SR}_i, \frac{1}{\overline{A}_l}\Big\rangle = -\frac{1}{\overline{A}_l}\Big\langle \mathrm{SR}_i, \overline{A}_l\Big\rangle.$$

The result then follows. □

3.3. **Decomposition of the correlation matrix.** Given that a CMS spread option depends on the co-movement of two swap-rates, it is natural to use a 2-factor Monte Carlo simulation. However, we know that a CMS spread option is highly correlation-dependent so that any factor-reduced correlation matrix that does not preserve the original correlation between the two underlying rates may lead to price distortions. Therefore we introduce an algorithm that can preserve the correlation between the two underlying CMS swap-rates.

3.3.1. *Rank-2 algorithm.* Suppose that the correlation matrix is an $n \times n$ full-rank symmetric matrix $\rho = [\rho_{ij}]$. If we want to preserve the kth row of ρ, we construct an $n \times 2$ matrix B using the following algorithm:

$$\begin{cases} b_{i1} = \rho_{ik}, \\ b_{i2} = \sqrt{1 - b_{i1}^2}, \end{cases} \tag{3.5}$$

for $i = 1, 2, \ldots, n$. Then we have $\widehat{\rho}_{ki} = \rho_{ki}$, for all i, where $\widehat{\rho} = BB^\top$.

3.3.2. *Spectral decomposition.* Another common approach to factor-reduce the correlation matrix is the spectral decomposition method, see Joshi (2003). Leon (2006) used this method to factor-reduce ρ to a rank-2 matrix. However, numerical tests in section 3.4 indicate that mispricing of CMS spread options written on SR_{10} and SR_2 arises if we factor reduce naïvely. One possible reason might be that the method does not preserve the correlation between the two underlying CMS swap-rates, which is crucial in pricing spread options.

3.4. **Numerical testing.** According to Brigo and Mercurio (2006): the first CMS rate of a CMS spread option is typically a long-maturity swap-rate (from ten years onwards), the second a short maturity one (one or two years). We consider CMS spread options with the following pay-off at maturity

$$\tau\big(\mathrm{SR}_{10}(T_0) - \mathrm{SR}_2(T_0) - 0.5\%\big)^+,$$

where $T_0 = 20$ and the accrual factor τ is set to 1.

3.4.1. *Market data.* As a market scenario, we consider the tenor structure $0 < T_0 < \cdots < T_{10}$ with $T_{j+1} - T_j = 1$. The T_0 bond has value equal to $e^{-0.05\cdot 20}$. The co-initial swap-rates all equal to 5.127%, with displacements of 2% for each swap rate. The displaced-diffusion volatilities are given by the approximation formula in Rebonato (2002)

$$\sigma_i^\alpha = \sigma_i^B \frac{\mathrm{SR}_i}{\mathrm{SR}_i + \alpha_i},$$

where σ_i^B is the at-the-money Black IV of a European swaption written on SR_i. The instantaneous correlations are given by

$$\rho_{ij} = e^{-0.05\cdot|T_i - T_j|}. \tag{3.6}$$

We use Sobol quasi-random number sequence with Brownian Bridging (for details of Brownian bridging, see Jäckel (2002)).

3.4.2. *True price.* Given that there is no analytical solution for the spread option, we first approximate the true price using a multi-step full-factor PC scheme with 20 steps and $1,048,575$ paths under the spot measure. For different one-step schemes, we run $65,535$ paths to compute the price.

Scheme	Number of Factors	Measure	Time	Price given Black IV 10% IV	20% IV	30% IV
Euler	2	Spot	0.265	21.96	61.68	118.75
Euler	2	Annuity	0.328	21.95	61.56	118.84
PC	2	Spot	0.407	**22.25**	**66.62**	**150.60**
PC	2	Annuity	0.515	**22.25**	**66.64**	**150.51**
PC	10	Spot	166.8	22.28	66.82	155.80

TABLE 3.1. Prices (in basis points) for a 20 year CMS spread option corresponding to various at-the-money Black IVs using one-step full-factor schemes. The true prices on the last row are given by a full-factor 20-step PC scheme.

3.4.3. *Comparison of one-step full-factor schemes.* The pricing results for different full-factor one-step numerical schemes are given in table 3.1. We can see that the one-step PC schemes give very accurate prices in all market scenarios. The errors of the Euler schemes increase as Black IV becomes larger. It seems that the pricing errors are indifferent to the pricing measure.

3.4.4. *The one-step two-factor PC schemes.* We now test the effectiveness of the two-factor PC schemes. The pricing results are given in table 3.2. Firstly, we can see that the rank-2 algorithm gives better results than the spectral decomposition method. Secondly, fixing ρ_{ij} has large impacts on pricing errors and we find that fixing $\rho_{10,j}$ gives very accurate prices while fixing $\rho_{2,j}$ can give worse results than the spectral decomposition method. Similar to the full-factor schemes, the errors seem to be indifferent to the pricing measure.

3.4.5. *Comparison of one-step 2-factor schemes.* The pricing results for different 2-factor schemes are given in table 3.3. The Euler schemes are fast, but they give inaccurate prices when volatility is high. The rank-2 algorithm gives similar prices compared with spectral decomposition method under the Euler schemes. Again, the errors seem to be indifferent to the pricing measure.

3.4.6. *Sensitivity of error to Black IV.* We also test the stability of the schemes relative to different values of Black IVs. We choose three schemes: two-factor Euler under spot measure, two-factor PC under annuity measure, and full-factor PC under spot measure. We show the graphs in figure 1. We can see that the error for the Euler scheme grows exponentially while the increase in Black IV has far less impacts on the errors for the PC schemes.

3.4.7. *Test conclusion.* The numerical tests indicate that the PC scheme under the annuity measure with $\rho_{10,j}$ fixed is the best scheme to price spread options in terms of speed and accuracy. This finding motivates the use of a fast two-dimensional integration method to price CMS spread options.

Measure	Correlation matrix Decomposition	Correlation Preservation	Time	Price given Black IV 10% IV	20% IV	30% IV
Spot	Rank-2	ρ_{10j}	0.188	22.20	66.09	149.37
Spot	Rank-2	ρ_{2j}	0.172	22.79	70.30	164.49
Annuity	Rank-2	ρ_{10j}	0.219	**22.20**	**66.18**	**150.74**
Annuity	Rank-2	ρ_{2j}	0.203	22.79	70.30	164.18
Spot	Spectral	n/a	0.172	21.78	67.34	167.72
Annuity	Spectral	n/a	0.203	21.78	67.36	168.33
PC	n/a	n/a	166.8	22.28	66.82	155.80

TABLE 3.2. Prices (in basis points) for a 20 year CMS spread option corresponding to various at-the-money Black IVs using one-step 2-factor PC schemes. The true prices on the last row are given by a full-factor 20-step PC scheme.

Scheme	Correlation matrix Decomposition	Measure	Time	Price given Black IV 10% IV	20% IV	30% IV
Euler	Rank-2	Spot	0.125	21.96	61.67	119.05
Euler	Rank-2	Annuity	0.156	21.96	61.61	120.52
PC	Rank-2	Annuity	0.203	**22.20**	**66.18**	**150.74**
Euler	Spectral	Spot	0.125	21.43	61.55	120.92
Euler	Spectral	Annuity	0.156	21.43	61.43	121.75
PC	n/a	Spot	166.8	22.28	66.82	155.80

TABLE 3.3. Prices (in basis points) for a 20 year CMS spread option corresponding to various at-the-money Black IVs using one-step 2-factor schemes. The true prices on the last row are given by a full-factor 20-step PC scheme.

4. APPLICATION OF THE GAUSSIAN QUADRATURE RULES

4.1. **Integral representation of CMS spread option price.** When we implement the one-step two-factor PC scheme under the annuity measure, we are computing the following two-dimensional integral

$$\int_{(0,1)^2} P\Big[\Phi^{-1}(u_0), \Phi^{-1}(u_1)\Big] \mathrm{d}\mathbf{u}, \tag{4.1}$$

where Φ^{-1} is the inverse cumulative normal distribution function and P is an $\mathbb{R}^2$ to $\mathbb{R}$ function that computes the price of a CMS spread option. Instead of simulating (quasi-) random numbers u_0 and u_1, we can replace the uniform variables on a two-dimensional unit hypercube by a two-dimensional standard normal random variable on $\mathbb{R}^2$. Thus (4.1) can be rewritten as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{c(x)} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} f(x,y) \mathrm{d}y \mathrm{d}x, \tag{4.2}$$

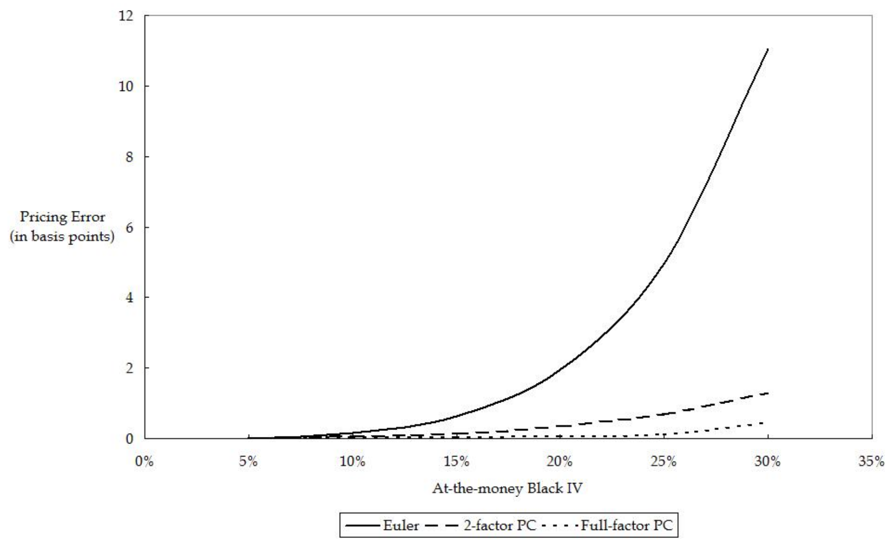

FIGURE 1. Pricing errors of a 20-year CMS spread option corresponding to different at-the-money Black IVs for three numerical schemes.

where $c(x)$ is the critical value of the y-axis given x such that the function f is zero and

$$f(x, y) = A_2(0) \frac{\mathrm{SR}_{10}(T_0) - \mathrm{SR}_2(T_0) - K}{A_2(T_0)}. \tag{4.3}$$

The swap-rates in (4.3) are computed using the 1-step 2-factor PC scheme with $\rho_{10,j}$ being preserved[1]

$$\mathrm{SR}_{10}(T_0) = \big(\mathrm{SR}_{10}(0) + \alpha_{10}\big) e^{\mu_{10} - \frac{1}{2}(\sigma_{10}^{\alpha})^2 T_0 + a_{10,1} x} - \alpha_{10}, \tag{4.4}$$

$$\mathrm{SR}_2(T_0) = \big(\mathrm{SR}_2(0) + \alpha_2\big) e^{-\frac{1}{2}(\sigma_2^{\alpha})^2 T_0 + a_{2,1} x + a_{2,2} y} - \alpha_2, \tag{4.5}$$

where $a_{ik} = b_{ik} \sqrt{T_0} \sigma_i^{\alpha}$ and the PC drift μ_{10} is the average of the Euler drifts calculated using the time zero and time T_0 swap-rates and annuity ratios. The reason why we integrate from $-\infty$ to $c(x)$ is that $\partial f / \partial y < 0$.

We cannot reduce the inner integral in (4.2) to a Margrabe formula since μ_{10} depends on y. Consequently, we need to compute a two-dimensional integral. We use Brent's method to locate the root $c(x)$. Note that y is not present in (4.4), we are able to obtain an accurate initial guess by replacing the PC drift with the Euler drift $\widehat{\mu}_{10}$ in (4.4).

4.2. **Gaussian quadrature rules.** The integrand of the two-dimensional integral (4.2) is smooth, and the boundary of integration is simple. Thus, we

[1]A consequence of preserving $\rho_{10,j}$ is that $b_{10,1} = 1$ and $b_{10,2} = 0$.

can evaluate the integral with a two-dimensional Gaussian quadrature rule (see Press et al. (1992)).

Proposition 4.1. *The price of a CMS spread option can be approximated by the following two-dimensional Gaussian quadrature rule*

$$V(0) \simeq \sum_{i=1}^{N}\sum_{j=1}^{M} w_i v_j \frac{c(\sqrt{2}x_i)+L}{2\sqrt{2}\pi} g\left(\sqrt{2}x_i, \frac{c(\sqrt{2}x_i)+L}{2}y_j + \frac{c(\sqrt{2}x_i)-L}{2}\right), \tag{4.6}$$

where (w_i, x_i) are weights and abscissas under the Gauss-Hermite rule, (v_j, y_j) are weights and abscissas under the Gauss-Legendre rule, L is some constant, $c(x)$ is the critical value of the integral (4.2) and

$$g(x,y) = e^{-\frac{y^2}{2}} f(x,y) \tag{4.7}$$

with f specified in (4.3).

Proof. The inner integral of (4.2) is a function of x, we write

$$h(x) = \int_{-\infty}^{c(x)} e^{-\frac{y^2}{2}} f(x,y)\mathrm{d}y \simeq \int_{-L}^{c(x)} g(x,y)\mathrm{d}y.$$

where L is some real number such that the integrand $g(x,L)$ is small. Rescaling the variable y, we can apply the Gauss-Legendre quadrature rule:

$$\begin{aligned} h(x) &= \frac{c(x)+L}{2}\int_{-1}^{1} g\left(x, \frac{c(x)+L}{2}y' + \frac{c(x)-L}{2}\right)\mathrm{d}y' \\ &\simeq \frac{c(x)+L}{2}\sum_{j=1}^{M} v_j g\left(x, \frac{c(x)+L}{2}y_j + \frac{c(x)-L}{2}\right). \end{aligned} \tag{4.8}$$

The outer integral of (4.2)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} h(x)\mathrm{d}x$$

can be approximated by the Gauss-Hermite quadrature rule:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} h(x)\mathrm{d}x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{2}e^{-(x')^2} h\left(\sqrt{2}x'\right)\mathrm{d}x' \\ &\simeq \frac{1}{\sqrt{2}\pi}\sum_{i=1}^{N} w_i h\left(\sqrt{2}x_i\right). \end{aligned} \tag{4.9}$$

Substituting (4.8) into (4.9), we have the double sum formula (4.6). □

4.3. **Numerical results.** To test the accuracy of the Gaussian quadrature rule, we compute the prices using the double sum formula (4.6) with market data as in section 3.4. We set L = 10 in (4.8) and use an 8×12 grid to compute CMS spread options numerically. We list the prices for the 1-step 2-factor Monte Carlo simulation and the double sum formula in table 4.1.

	IV = 10%		IV = 20%		IV = 30%	
Method	Time	Price	Time	Price	Time	Price
2-factor PC scheme	0.2186	22.20	0.2186	66.18	0.2186	150.74
Gaussian quadrature	0.0003	22.20	0.0043	66.07	0.0085	148.80

TABLE 4.1. Prices (in basis points) between the 8×12 Gaussian quadrature formula (4.6) and the two-factor PC scheme under annuity measure with 65,535 paths.

The Gaussian quadrature rule gives prices almost identical to those produced by the simulation. When the IV increases, it takes the non-linear solver longer to locate the critical value $c(x)$. One major notable improvement is that the quadrature rule approach can price CMS spread options within high degree of accuracy extremely fast.

5. GREEKS COMPUTATION AND THE ADJOINT METHOD

An algorithm that can compute prices extremely accurate but fails to produce Greeks will not be implemented by banks in practice. We showed how to compute accurate prices using a fast algorithm in section 4, we now use a similar algorithm to compute accurate Greeks of CMS spread options.

5.1. **Model Greeks.** The model Greeks are the partial derivatives of the price with respect to a model parameter θ

$$\frac{\partial V(0)}{\partial \theta} = \frac{\partial}{\partial \theta}\left\{\widetilde{\mathbb{E}}^{(N)}\left[N(0)\frac{\left(\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K\right)^+}{N(T_0)}\right]\right\}. \tag{5.1}$$

Common model Greeks are Deltas and Vegas, which are sensitivities with respect to initial swap-rates and volatility parameters respectively. In this paper, we also consider the sensitivity with respect to displacements.

Since the pay-off of the spread option is Lipschitz-continuous, we can interchange the differentiation operator and the expectation operator in (5.1)

$$\frac{\partial V(0)}{\partial \theta} = \widetilde{\mathbb{E}}^{(N)}\left\{\frac{\partial}{\partial \theta}\left[N(0)\frac{\left(\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K\right)^+}{N(T_0)}\right]\right\}. \tag{5.2}$$

The partial derivative in the expectation operator in (5.2) can be computed by the pathwise method in a Monte Carlo simulation (see Glasserman (2004)).

5.2. **Computing Greeks using Gaussian quadrature rules.**

Proposition 5.1. *The model Greeks of a CMS spread option can be approximated by the following two-dimensional Gaussian quadrature rule*

$$\frac{\partial V(0)}{\partial \theta} \simeq \sum_{i=1}^{N}\sum_{j=1}^{M} w_i v_j \frac{c(\sqrt{2}x_i)+K}{2\sqrt{2}\pi} g_\theta\left(\sqrt{2}x_i, \frac{c(\sqrt{2}x_i)+K}{2}y_j + \frac{c(\sqrt{2}x_i)-K}{2}\right), \tag{5.3}$$

where (w_i, x_i) *are weights and abscissas under the Gauss-Hermite rule,* (v_j, y_j) *are weights and abscissas under the Gauss-Legendre rule, ,* $c(x)$ *is the critical value of the integral (4.2) and*

$$g_\theta(x, y) = e^{-\frac{y^2}{2}} \frac{\partial f}{\partial \theta}(x, y) \tag{5.4}$$

with f *specified in (4.3). The parameter* θ *can be any model input such as initial swap-rates, displacements and pseudo-root elements.*

Proof. The partial derivative of the two-dimensional integral (4.2) with respect to θ is given by

$$\begin{aligned}
&\frac{\partial}{\partial \theta}\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{c(x)} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} f(x, y)\mathrm{d}y\mathrm{d}x\Big] \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi}\Big[e^{-\frac{x^2+c(x)^2}{2}} f\big(x, c(x)\big)\frac{\partial c(x)}{\partial \theta} + \int_{-\infty}^{c(x)} e^{-\frac{x^2+y^2}{2}} \frac{\partial f}{\partial \theta}(x, y)\mathrm{d}y\Big]\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{c(x)} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \frac{\partial f}{\partial \theta}(x, y)\mathrm{d}y\mathrm{d}x. \qquad (5.5)
\end{aligned}$$

This two-dimensional integral is the same as (4.2) except that f is replaced by one of its partial derivatives. Similar arguments of the proof of **Proposition 4.1** leads to (5.3). □

5.3. **The adjoint method.** Using quadrature methods to compute model Greeks will be faster than estimating the expectation (5.2) using a Monte Carlo simulation. The key to use formula (5.3) is to efficiently compute the (pathwise) partial derivatives of the discounted CMS spread option price (4.3) with respect to different model input paratmeters, $\partial f/\partial\theta$, given the random normal variates of one path, (x, y).

Giles and Glasserman (2006) developed the adjoint method that speeds up computation of the pathwise derivatives in the LMM context. The existence of the complex PC drift in the non-linear function f makes straightforward partial differentiation a difficult task. Thus we adopt the main idea used in Joshi and Yang (2009): we divide all the computations of f into a sequence of simple vector operations, each of which is easily differentiated and of low computational order.

We show how to compute the model Greeks of a CMS spread option in appendix A.

5.4. **Numerical tests.** We compute the sensitivities of a CMS spread option with respect to initial swap-rates, displaced-diffusion volatilities and displacements using market data in section 3.4. We list the results in tables 5.1, 5.2 and 5.3.

We can see that the differences between all three methods are negligible and the time reduction in computing all these model Greeks using a Gaussian quadrature rule is, however, substantial.

Swap-rate	Quadrature	2-factor PC	Full-factor PC
SR_1	-0.19%	-0.18%	-0.19%
SR_2	-16.57%	-15.92%	-16.47%
SR_3	0.08%	0.08%	0.08%
SR_4	0.13%	0.13%	0.13%
SR_5	0.19%	0.19%	0.19%
SR_6	0.26%	0.26%	0.27%
SR_7	0.37%	0.37%	0.38%
SR_8	0.50%	0.50%	0.53%
SR_9	0.68%	0.68%	0.71%
SR_{10}	27.63%	27.63%	27.70%
Time	0.00067		

TABLE 5.1. Deltas of a 20-year CMS spread option computed using three methods and a Black IV of 20%.

Swap-rate	Quadrature	2-factor PC	Full-factor PC
SR_1	-0.11%	-0.11%	-0.10%
SR_2	0.76%	0.75%	0.73%
SR_3	0.07%	0.07%	0.07%
SR_4	0.10%	0.10%	0.11%
SR_5	0.14%	0.14%	0.15%
SR_6	0.19%	0.19%	0.21%
SR_7	0.26%	0.26%	0.28%
SR_8	0.36%	0.36%	0.39%
SR_9	0.49%	0.49%	0.53%
SR_{10}	5.66%	5.67%	5.69%
Time	0.00075		

TABLE 5.2. Sensitivities of a 20-year CMS spread option with respect to displaced-diffusion volatilities computed using three methods and a Black IV of 20%.

6. MARKET SKEW SENSITIVITY

Definition 6.1. *IV skews at strike level K are defined to be the partial derivatives*

$$\mathrm{Skew}_i(K) = \frac{\partial \widehat{\sigma}_i(K)}{\partial K},$$

where $\widehat{\sigma}_i(K)$ is the IV generated using the algorithm in section (6.2.1). The (market) at-the-money skew sensitivities of CMS spread options are defined as

$$\left.\frac{\partial V(0)}{\partial \mathrm{Skew}_i(K)}\right|_{K=\mathrm{SR}_i}. \tag{6.1}$$

Swap-rate	Quadrature	2-factor PC	Full-factor PC
SR_1	-0.20%	-0.20%	-0.19%
SR_2	1.52%	1.53%	1.59%
SR_3	0.13%	0.13%	0.13%
SR_4	0.19%	0.19%	0.20%
SR_5	0.26%	0.26%	0.27%
SR_6	0.35%	0.35%	0.36%
SR_7	0.46%	0.46%	0.47%
SR_8	0.59%	0.59%	0.62%
SR_9	0.77%	0.77%	0.80%
SR_{10}	9.61%	9.64%	9.68%
Time	0.00066		

TABLE 5.3. Sensitivities of a 20-year CMS spread option with respect to displacements computed using three methods and a Black IV of 20%.

6.1. **Black's formulae.** In order to derive formulae for skew and skew sensitivities, we need to use the following Black's formula in a DDciSMM:

$$\begin{aligned}\mathrm{Swaption}\big(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\big) &= \mathrm{BLACK}\Big(\mathrm{SR}_i + \alpha_i, K + \alpha_i, T_0; \sigma_i^\alpha\Big) \\ &= A_i\Big[\big(\mathrm{SR}_i + \alpha_i\big)\Phi\big(d_1^\alpha\big) - \big(K + \alpha_i\big)\Phi\big(d_2^\alpha\big)\Big], \qquad (6.2)\end{aligned}$$

where σ_i^α is the flat displaced-diffusion volatility and

$$d_1^\alpha = \frac{\log\big(\frac{\mathrm{SR}_i + \alpha_i}{K + \alpha_i}\big) + \frac{1}{2}(\sigma_i^\alpha)^2 T_0}{\sigma_i^\alpha \sqrt{T_0}}, \qquad (6.3)$$

$$d_2^\alpha = d_1^\alpha - \sigma_i^\alpha \sqrt{T_0}. \qquad (6.4)$$

For notational purposes, we rewrite d_i^α in (6.3) as d_i if we use a log-normal ciSMM.

6.2. **Analytical formula of IV skews.** We first show how to generate a set of skewed IVs, we then derive an analytical formula for the IV skews.

6.2.1. *Generation of skewed IV.* We use the following algorithm to generate skewed IV:

1. Calibrate the DDciSMM to market prices of swaptions using (6.2).
2. Invert Black's formula (6.2) and solve for $\widehat{\sigma}_i(K)$ such that

$$\mathrm{Swaption}\big(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\big) = \mathrm{BLACK}\big(\mathrm{SR}_i, K, T_0; \widehat{\sigma}_i(K)\big). \qquad (6.5)$$

3. Repeat step 2 for different values of K.

6.2.2. *IV skew formula.* Having generated $\widehat{\sigma}_i(K)$, we now derive an explicit formula for skews. Differentiating (6.5)

$$\begin{aligned}\frac{\partial \mathrm{Swaption}}{\partial K}\big(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\big) = \frac{\partial \mathrm{BLACK}}{\partial K}\big(\mathrm{SR}_i, K, T_0; \widehat{\sigma}_i(K)\big) \\ + \frac{\partial \mathrm{BLACK}}{\partial \widehat{\sigma}_i(K)}\big(\mathrm{SR}_i, K, T_0; \widehat{\sigma}_i(K)\big)\frac{\partial \widehat{\sigma}_i(K)}{\partial K},\end{aligned}$$

we have

$$\frac{\partial \widehat{\sigma}_i(K)}{\partial K} = \left(\frac{\partial \mathrm{Swaption}}{\partial K} - \frac{\partial \mathrm{BLACK}}{\partial K}\right)\Bigg/\frac{\partial \mathrm{BLACK}}{\partial \widehat{\sigma}_i(K)}. \tag{6.6}$$

The partial derivatives on the right-hand-side of equation (6.6) are

$$\begin{cases} \frac{\partial \mathrm{Swaption}}{\partial K} = -A_i \Phi\big(d_2^\alpha\big), \\ \frac{\partial \mathrm{BLACK}}{\partial K} = -A_i \Phi(d_2), \\ \frac{\partial \mathrm{BLACK}}{\partial \widehat{\sigma}_i(K)} = A_i \mathrm{SR}_i \sqrt{T_0}\varphi\big(d_1\big). \end{cases}$$

Therefore, the analytical expression for the IV skew in a displaced-diffusion SMM is given by

$$\frac{\partial \widehat{\sigma}_i(K)}{\partial K} = \frac{\Phi(d_2) - \Phi\big(d_2^\alpha\big)}{\mathrm{SR}_i \sqrt{T_0}\varphi(d_1)}. \tag{6.7}$$

6.3. **Skew sensitivities.** In section 5, we computed the following model Greeks

$$\left(\frac{\partial V(0)}{\partial \alpha_i}, \frac{\partial V(0)}{\partial \sigma_i^\alpha}\right).$$

Now, if we are able to compute the partial derivatives

$$\left(\frac{\partial \alpha_i}{\partial \mathrm{Skew}_i(K)}, \frac{\partial \sigma_i^\alpha}{\partial \mathrm{Skew}_i(K)}\right)^\top, \tag{6.8}$$

then the market skew sensitivity is given by

$$\left(\frac{\partial V(0)}{\partial \alpha_i}, \frac{\partial V(0)}{\partial \sigma_i^\alpha}\right)\left(\frac{\partial \alpha_i}{\partial \mathrm{Skew}_i(K)}, \frac{\partial \sigma_i^\alpha}{\partial \mathrm{Skew}_i(K)}\right)^\top.$$

We show how to compute the partial derivatives in (6.8). Consider the following mapping

$$\mathbf{I} : \begin{pmatrix} \alpha_i \\ \sigma_i^\alpha \end{pmatrix} \mapsto \begin{pmatrix} \widehat{\sigma}_i(K) \\ \mathrm{Skew}_i(K) \end{pmatrix}. \tag{6.9}$$

We know that the Jacobian matrix $\big(\mathbf{I}^{-1}\big)' = \big(\mathbf{I}'\big)^{-1}$ is a 2×2 matrix of the form

$$\begin{pmatrix} \frac{\partial \alpha_i}{\partial \widehat{\sigma}_i(K)} & \frac{\partial \alpha_i}{\partial \mathrm{Skew}_i(K)} \\ \frac{\partial \sigma_i^\alpha}{\partial \widehat{\sigma}_i(K)} & \frac{\partial \sigma_i^\alpha}{\partial \mathrm{Skew}_i(K)} \end{pmatrix}.$$

It is difficult to compute the entries of the Jacobian matrix $\mathbf{I}'$ directly, therefore we consider the following sub-mappings:

$$\mathbf{I}_0 : \begin{pmatrix} \alpha_i \\ \sigma_i^\alpha \end{pmatrix} \mapsto \begin{pmatrix} \alpha_i \\ \sigma_i^\alpha \\ \widehat{\sigma}_i(K) \end{pmatrix},$$
$$\mathbf{I}_1 : \begin{pmatrix} \alpha_i \\ \sigma_i^\alpha \\ \widehat{\sigma}_i(K) \end{pmatrix} \mapsto \begin{pmatrix} \widehat{\sigma}_i(K) \\ \mathrm{Skew}_i(K) \end{pmatrix}.$$

Thus, $\mathbf{I} = \mathbf{I}_1 \circ \mathbf{I}_0$ and computing $\left(\mathbf{I}^{-1}\right)'$ is equivalent to computing $\left(\mathbf{I}_1'\mathbf{I}_0'\right)^{-1}$.

6.3.1. *The Jacobian matrix* $\mathbf{I}_0'$. The Jacobian matrix $\mathbf{I}_0'$ is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial \widehat{\sigma}_i(K)}{\partial \alpha_i} & \frac{\partial \widehat{\sigma}_i(K)}{\partial \sigma_i^\alpha} \end{pmatrix}$$

Differentiating (6.5), we have

$$\begin{cases} \frac{\partial \mathrm{Swaption}}{\partial \alpha_i}\left(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\right) = \frac{\partial \mathrm{BLACK}}{\partial \widehat{\sigma}_i(K)}\left(\mathrm{SR}_i, K, T_0; \widehat{\sigma}_i(K)\right)\frac{\partial \widehat{\sigma}_i(K)}{\partial \alpha_i}, \\ \frac{\partial \mathrm{Swaption}}{\partial \sigma_i^\alpha}\left(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\right) = \frac{\partial \mathrm{BLACK}}{\partial \widehat{\sigma}_i(K)}\left(\mathrm{SR}_i, K, T_0; \widehat{\sigma}_i(K)\right)\frac{\partial \widehat{\sigma}_i(K)}{\partial \sigma_i^\alpha}. \end{cases}$$

From (6.2), we have

$$\begin{cases} \frac{\partial \mathrm{Swaption}}{\partial \alpha_i}\left(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\right) = A_i\left[\Phi\left(d_1^\alpha\right) - \Phi\left(d_2^\alpha\right)\right], \\ \frac{\partial \mathrm{Swaption}}{\partial \sigma_i^\alpha}\left(\mathrm{SR}_i, \sigma_i^\alpha, \alpha_i\right) = A_i\left(\mathrm{SR}_i + \alpha_i\right)\sqrt{T_0}\varphi\left(d_1^\alpha\right). \end{cases}$$

The non-trivial entries of $\mathbf{I}_0'$ are given by

$$\begin{cases} \frac{\partial \widehat{\sigma}_i(K)}{\partial \alpha_i} = \frac{\Phi\left(d_1^\alpha\right) - \Phi\left(d_2^\alpha\right)}{\mathrm{SR}_i\sqrt{T_0}\varphi(d_1)}, \\ \frac{\partial \widehat{\sigma}_i(K)}{\partial \sigma_i^\alpha} = \frac{(\mathrm{SR}_i + \alpha_i)\varphi\left(d_1^\alpha\right)}{\mathrm{SR}_i\varphi(d_1)}. \end{cases} \tag{6.10}$$

6.3.2. *The Jacobian matrix* $\mathbf{I}_1'$. The Jacobian matrix $\mathbf{I}_1'$ is of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{\partial \mathrm{Skew}_i(K)}{\partial \alpha_i} & \frac{\partial \mathrm{Skew}_i(K)}{\partial \sigma_i^\alpha} & \frac{\partial \mathrm{Skew}_i(K)}{\partial \widehat{\sigma}_i(K)} \end{pmatrix}.$$

Straightforward differentiation of (6.7) gives

$$\begin{cases} \frac{\partial \mathrm{Skew}_i(K)}{\partial \alpha_i} = -\frac{\varphi\left(d_2^\alpha\right)}{\varphi(d_1)\mathrm{SR}_i\sigma_i^\alpha T_0}\left(\frac{1}{\mathrm{SR}_i + \alpha_i} - \frac{1}{K + \alpha_i}\right), \\ \frac{\partial \mathrm{Skew}_i(K)}{\partial \sigma_i^\alpha} = \frac{\varphi\left(d_2^\alpha\right)d_1^\alpha}{\varphi(d_1)\mathrm{SR}_i\sigma_i^\alpha\sqrt{T_0}}, \\ \frac{\partial \mathrm{Skew}_i(K)}{\partial \widehat{\sigma}_i(K)} = \frac{1}{\varphi(d_1)\mathrm{SR}_i\widehat{\sigma}_i(K)\sqrt{T_0}}\left[\frac{\Phi(d_2)}{\varphi(d_1)}d_2 - \varphi(d_2)d_1\right]. \end{cases} \tag{6.11}$$

	IV $= 10\%$	IV $= 20\%$	IV $= 30\%$
$\partial V(0)/\partial \mathrm{Skew}_2$	0.0012%	0.0032%	0.0823%
$\partial V(0)/\partial \mathrm{Skew}_{10}$	0.0156%	0.1037%	0.5177%

TABLE 6.1. Market skew sensitivities of a 20-year CMS spread option corresponding to different at-the-money Black IV.

6.3.3. *Numerical tests.* We use model Greeks in tables 5.2 and 5.3, and the Jacobian matrices in sections 6.3.1 and 6.3.2 to compute market skew sensitivities. The numerical values for skew sensitivities are given in table 6.1.

Although market skew sensitivities increase when Black IV increases, the relatively small magnitude of the market skew sensitivities implies that the price of a CMS spread option will not vary much if we vary the displacements given that the at-the-money Black IV is fixed. To confirm the hypothesis, we vary α_i, $i = 2, 10$, from 0 to 4% one at a time and compute the corresponding prices fixing Black IV at 20%. The graphs are shown in figure 2. We can see that changing α_{10} from 0 to 4% causes price to decrease by approximately 4 basis points while changing α_2 from 0 to 4% virtually keeps the price unchanged. However, these changes in prices are small relative to changes in initial swap-rates or volatilities. Hence, we conclude that CMS spread options are weakly dependent on at-the-money IV skews.

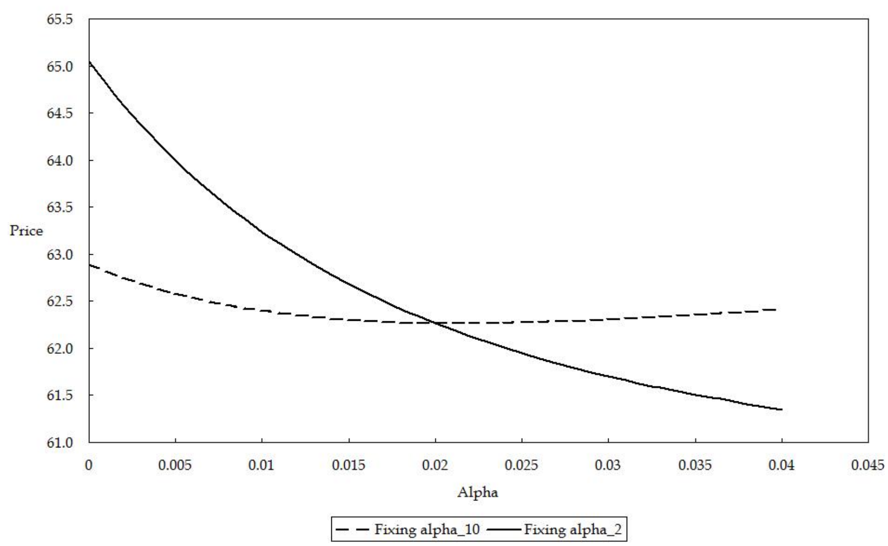

FIGURE 2. The Prices of a 20-year CMS spread option corresponding to different values of one displacement while fixing the other. The at-the-money Black IV is fixed at 20%. The at-the-money skews are equal to 0, -0.554 and -0.871 corresponding to α_i equal to 0, 0.02 and 0.04 respectively.

7. Conclusion

We have described and implemented a fast and accurate method to price CMS spread options. The evolution of the swap-rates is based on an efficient two-factor PC scheme under an annuity measure. Pricing options using the new method takes extremely low computational times compared with traditional Monte Carlo simulations. The additional time for computing model Greeks using the adjoint method can be measured in milliseconds. Mostly importantly, the enormous reduction in computational times does not come at a cost of poorer accuracy. In fact, the prices and Greeks produced by the new method are sufficiently close to those produced by a one-step two-factor PC scheme with 65,535 paths.

In addition, we also derive a general analytical formula for IV skews in a displaced-diffusion model which is applicable to other markets. Numerical tests show that CMS spread options are not highly sensitive to the changes in at-the-money IV skews.

Appendix A. Implementing the adjoint methods

We briefly discuss the efficient estimation of model Greeks with the PC scheme in the DDciSMM. The algorithms for the PC scheme are similar to those for the Euler scheme, which were discussed in detail in sections 5 and 6 in Joshi and Yang (2009). For interested reader, we refer them to that paper for detailed implementations.

A.1. **Adjoint Deltas.** We consider the following vector operations:

$$
\begin{aligned}
\mathbf{SR}(0) &\xrightarrow{\mathbf{F}_0} \begin{bmatrix} \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \end{bmatrix} \xrightarrow{\mathbf{F}_1} \begin{bmatrix} \mathbf{SR}(0) \\ \widehat{\mu}(0) \end{bmatrix} \xrightarrow{\mathbf{F}_2} \begin{bmatrix} \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \end{bmatrix} \\
&\xrightarrow{\mathbf{F}_3} \begin{bmatrix} \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widehat{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{F}_4} \begin{bmatrix} \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widetilde{\mu}(0) \end{bmatrix} \xrightarrow{\mathbf{F}_5} \mathbf{SR}(T_0) \\
&\xrightarrow{\mathbf{F}_6} \begin{bmatrix} \mathbf{SR}(T_0) \\ \overline{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{F}_7} \frac{\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K}{A_l(T_0)}.
\end{aligned} \tag{A.1}
$$

The price of a CMS spread option (A_l is the numeraire) is given by

$$V(0) = A_l(0)\mathbf{F}\big(\mathbf{SR}(0)\big),$$

where $\mathbf{F} = \mathbf{F}_7 \circ \mathbf{F}_6 \circ \cdots \circ \mathbf{F}_0$. The Deltas of a CMS spread option are given by

$$\Delta = \frac{\partial A_l(0)}{\partial \mathbf{SR}(0)} + A_l(0)\mathbf{F}', \tag{A.2}$$

where $\mathbf{F}' = \mathbf{F}_7'\mathbf{F}_6' \cdots \mathbf{F}_0'$. The computation of the gradient vector $\mathbf{F}_7'$ is trivial, we show how to compute the other Jacobian matrices below.

A.1.1. *Annuity ratio adjoints.* The Jacobian matrix $\mathbf{F}_0'$ have exactly the same non-zero elements as the Jacobian matrix $\mathbf{F}_{0,0}'$ in section 5.1 in Joshi and Yang (2009). The only difference between $\mathbf{F}_0'$ and $\mathbf{F}_3'$, $\mathbf{F}_6'$ is that the dimensions of $\mathbf{F}_3'$ and $\mathbf{F}_6'$ are larger than that of $\mathbf{F}_0'$ while the non-trivial entries are the same.

A.1.2. *Drift adjoints.* We consider the following sub-mappings of $\mathbf{F}_1$:

$$\begin{bmatrix}\mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\end{bmatrix} \xrightarrow{\mathbf{F}_{1,0}} \begin{bmatrix}\mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \left\{\langle Z_k, \overline{A}_i\rangle\right\}_{i=1}^{1}\end{bmatrix} \xrightarrow{\mathbf{F}_{1,1}} \begin{bmatrix}\mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \left\{\langle Z_k, \overline{A}_i\rangle\right\}_{i=1}^{2}\end{bmatrix}$$

$$\cdots \xrightarrow{\mathbf{F}_{1,n-1}} \begin{bmatrix}\mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \left\{\langle Z_k, \overline{A}_i\rangle\right\}_{i=1}^{n}\end{bmatrix} \xrightarrow{\mathbf{F}_{1,n}} \begin{bmatrix}\mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \left\{\langle Z_k, \overline{A}_i\rangle^{A_l}\right\}_{i=1}^{n}\end{bmatrix} \xrightarrow{\mathbf{F}_{1,n+1}} \begin{bmatrix}\mathbf{SR}(0)\\ \widehat{\mu}(0)\end{bmatrix}$$

The mappings $\mathbf{F}_{1,i}$, $i = 0, \dots, n-1$, update the cross variation derivatives, $\mathbf{F}_{1,n}$ changes the pricing measure from the spot measure to the annuity measure, and $\mathbf{F}_{1,n+1}$ computes all the drifts under the annuity measure.

- The Jacobian matrices $\mathbf{F}_{1,i}'$, $i = 0, \dots, n-1$, have similar the same non-zero elements as the Jacobian matrix $\mathbf{H}_{0,j}'$ in section 5.2.1 in Joshi and Yang (2009). The differences are due to the displaced-diffusion model we use in this paper: the partial derivatives in section 5.2.1 are changed to

$$\begin{cases} \frac{\partial}{\partial \mathrm{SR}_j}\left\langle Z_k, \overline{A}_j\right\rangle = -\tau_{j-1}\frac{\langle Z_k,\overline{A}_j\rangle + a_{jk}\overline{A}_j}{1+\tau_{j-1}\mathrm{SR}_j}, \\ \frac{\partial}{\partial \overline{A}_j}\left\langle Z_k, \overline{A}_j\right\rangle = -\frac{\tau_{j-1}(\mathrm{SR}_j+\alpha_j)}{1+\tau_{j-1}\mathrm{SR}_j}a_{jk}, \\ \frac{\partial}{\partial\left\langle Z_k,\overline{A}_{j-1}\right\rangle}\left\langle Z_k, \overline{A}_j\right\rangle = \frac{1}{1+\tau_{j-1}\mathrm{SR}_j}, \end{cases}$$

  for $k = 1, \dots, F$.
- The Jacobian matrix $\mathbf{F}_{1,n}'$ has the following non-trivial partial derivatives:

$$\begin{cases} \frac{\partial}{\partial\left\langle Z_k,\overline{A}_j\right\rangle}\left\langle Z_k, \overline{A}_j\right\rangle^{A_l} = 1, \quad (j \neq l), \\ \frac{\partial}{\partial \overline{A}_j}\left\langle Z_k, \overline{A}_j\right\rangle^{A_l} = -\frac{1}{\overline{A}_l}\left\langle Z_k, \overline{A}_l\right\rangle, \\ \frac{\partial}{\partial \overline{A}_l}\left\langle Z_k, \overline{A}_j\right\rangle^{A_l} = \frac{\overline{A}_j}{(\overline{A}_l)^2}\left\langle Z_k, \overline{A}_l\right\rangle, \\ \frac{\partial}{\partial\left\langle Z_k,\overline{A}_l\right\rangle}\left\langle Z_k, \overline{A}_j\right\rangle^{A_l} = -\frac{\overline{A}_j}{\overline{A}_l}, \end{cases}$$

  for $k = 1, \dots, F$.
- The Jacobian matrix $\mathbf{F}_{1,n+1}'$ has exactly the same entries at the Jacobian matrix $\mathbf{H}_0'$ in section 5.2.2 in Joshi and Yang (2009).

The only difference between $\mathbf{F}_4'$ and $\mathbf{F}_1'$ is that the dimension of $\mathbf{F}_4'$ is larger than that of $\mathbf{F}_1'$ while the non-trivial entries are the same.

A.1.3. *Evolution adjoints.* Based on the Euler discretization scheme (2.11), the non-trivial partial derivatives of the Jacobian matrix $\mathbf{F}_2'$ are

$$\begin{cases} \frac{\partial \widehat{\mathrm{SR}}_j(T_0)}{\partial \mathrm{SR}_j(0)} = \frac{\widehat{\mathrm{SR}}_j(T_0)+\alpha_i}{\mathrm{SR}_j(0)+\alpha_i}, \\ \frac{\partial \widehat{\mathrm{SR}}_j(T_0)}{\partial \widehat{\mu}_j} = \widehat{\mathrm{SR}}_j(T_0)+\alpha_i, \end{cases}$$

for $j = 1, \ldots, n$.

Based on the PC scheme (2.12), the non-trivial partial derivatives of the Jacobian matrix $\mathbf{F}_5'$ are

$$\begin{cases} \frac{\partial \mathrm{SR}_j(T_0)}{\partial \widehat{\mathrm{SR}}_j(T_0)} = \frac{\mathrm{SR}_j(T_0)+\alpha_i}{\widehat{\mathrm{SR}}_j(T_0)+\alpha_i}, \\ \frac{\partial \mathrm{SR}_j(T_0)}{\partial \widehat{\mu}_j} = -\frac{\mathrm{SR}_j(T_0)+\alpha_i}{2}, \\ \frac{\partial \mathrm{SR}_j(T_0)}{\partial \widetilde{\mu}_j} = \frac{\mathrm{SR}_j(T_0)+\alpha_i}{2}, \end{cases}$$

for $j = 1, \ldots, n$.

A.1.4. *Computational order.* As the computations of Jacobian matrices are of $\mathcal{O}(nF)$, we can compute Deltas of a CMS spread option with an order $\mathcal{O}(nF)$.

A.2. **Adjoint Vegas.** The model elementary Vegas are defined as

$$\frac{\partial V(0)}{\partial a_{ik}},$$

for all i and k. The sensitivity with respect to a displaced-diffusion volatility parameter is given by the linear combination of the elementary Vegas

$$\frac{\partial V(0)}{\partial \sigma_i^\alpha} = \frac{a_{i1}}{\sigma_i^\alpha}\frac{\partial V(0)}{\partial a_{i1}} + \frac{a_{i2}}{\sigma_i^\alpha}\frac{\partial V(0)}{\partial a_{i2}}, \tag{A.3}$$

since we are using a 2-factor DDciSMM.

We consider the following vector operations in estimating elementary Vegas:

$$\begin{bmatrix} \{a_{ik}\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \end{bmatrix} \xrightarrow{\mathbf{G}_1} \begin{bmatrix} \{a_{ik}\} \\ \mathbf{SR}(0) \\ \widehat{\mu}(0) \end{bmatrix} \xrightarrow{\mathbf{G}_2} \begin{bmatrix} \{a_{ik}\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{G}_3} \begin{bmatrix} \{a_{ik}\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widehat{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{G}_4} \begin{bmatrix} \{a_{ik}\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widetilde{\mu}(0) \end{bmatrix}$$

$$\xrightarrow{\mathbf{G}_5} \mathbf{SR}(T_0) \xrightarrow{\mathbf{F}_6} \begin{bmatrix} \mathbf{SR}(T_0) \\ \overline{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{F}_7} \frac{\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K}{A_l(T_0)}. \tag{A.4}$$

The nF model elementary Vegas will be given by the first nF entries of the following gradient vector

$$\mathbf{F}_7'\mathbf{F}_6'\mathbf{G}_5'\mathbf{G}_4' \cdots \mathbf{G}_1'.$$

We show how to compute the Jacobian matrices $\mathbf{G}_i'$, $i = 1, \ldots, 5$ below.

A.2.1. *Drift adjoints.* We consider the following sub-mappings of $\mathbf{G}_1$:

$$\begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\end{bmatrix} \xrightarrow{\mathbf{G}_{1,0}} \begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{1}\end{bmatrix} \xrightarrow{\mathbf{G}_{1,1}} \begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{2}\end{bmatrix}$$

$$\cdots \xrightarrow{\mathbf{G}_{1,n-1}} \begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{n}\end{bmatrix} \xrightarrow{\mathbf{G}_{1,n}} \begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \overline{\mathbf{A}}(0)\\ \{\langle Z_k, \overline{A}_i\rangle^{A_l}\}_{i=1}^{n}\end{bmatrix} \xrightarrow{\mathbf{G}_{1,n+1}} \begin{bmatrix}\{a_{ik}\}\\ \mathbf{SR}(0)\\ \widehat{\mu}(0)\end{bmatrix}$$

The mappings $\mathbf{G}_{1,i}$, $i = 0, \ldots, n-1$, update the cross variation derivatives, $\mathbf{G}_{1,n}$ changes the pricing measure from the spot measure to the annuity measure, and $\mathbf{G}_{1,n+1}$ computes all the drifts under the annuity measure.

- The Jacobian matrices $\mathbf{G}'_{1,i}$, $i = 0, \ldots, n-1$, have similar non-zero elements as the Jacobian matrix $\mathbf{F}_{0,j}$ in section A.1.2. In addition to the three partial derivatives, we have an extra partial derivative with respect to pseudo-root elements:

$$\frac{\partial}{\partial a_{jk}}\big\langle Z_k, \overline{A}_j\big\rangle = -\frac{\tau_{j-1}(\mathrm{SR}_j + \alpha_j)\overline{A}_j}{1+\tau_{j-1}\mathrm{SR}_j},$$

  for $k = 1, \ldots, F$.
- The Jacobian matrix $\mathbf{G}'_{1,n}$ is the same as the Jacobian matrix $\mathbf{F}'_{1,n}$ in section A.1.2.
- The Jacobian matrix $\mathbf{G}'_{1,n+1}$ has similar non-zero elements as the Jacobian matrix $\mathbf{F}'_{0,n+1}$ in section A.1.2. In addition to the two partial derivatives, we have an extra partial derivative with respect to pseudo-root elements:

$$\frac{\partial\widehat{\mu}_j}{\partial a_{jk}} = -\frac{\big\langle Z_k, \overline{A}_j\big\rangle}{\overline{A}_j},$$

  for $k = 1, \ldots, F$.

The only difference between $\mathbf{G}'_4$ and $\mathbf{G}'_1$ is that the dimension of $\mathbf{G}'_4$ is larger than that of $\mathbf{G}'_1$ while the non-trivial entries are the same.

A.2.2. *Evolution adjoints.* The Jacobian matrix $\mathbf{G}'_2$ have one more non-trivial partial derivative (with respect to pseudo-root elements) than the Jacobian matrix $\mathbf{F}'_2$ in section A.1.3:

$$\frac{\partial\widehat{\mathrm{SR}}_j(T_0)}{\partial a_{jk}} = \big(\widehat{\mathrm{SR}}_j(T_0) + \alpha_i\big)\big(Z_k - a_{jk}\big),$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, F$.

Since the pseudo-root elements do not appear in the PC scheme (2.12), the non-trivial partial derivatives of the Jacobian matrix $\mathbf{G}'_5$ are exactly the same as those of the Jacobian matrix $\mathbf{F}'_5$ in section A.1.3.

A.2.3. *Computational order.* Similar to Delta computations, we can compute elementary Vegas of a CMS spread option with an order $\mathcal{O}(nF)$.

A.3. **Adjoint Displacement sensitivities.** The computation of displacement sensitivities is similar to that of elementary Vegas. We consider the following vector operations:

$$\begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \end{bmatrix} \xrightarrow{\mathbf{H}_1} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \widehat{\mu}(0) \end{bmatrix} \xrightarrow{\mathbf{H}_2} \begin{bmatrix} \{\alpha_i\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{H}_3} \begin{bmatrix} \{\alpha_i\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widehat{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{H}_4} \begin{bmatrix} \{\alpha_i\} \\ \widehat{\mu}(0) \\ \widehat{\mathbf{SR}}(T_0) \\ \widetilde{\mu}(0) \end{bmatrix}$$
$$\xrightarrow{\mathbf{H}_5} \mathbf{SR}(T_0) \xrightarrow{\mathbf{F}_6} \begin{bmatrix} \mathbf{SR}(T_0) \\ \overline{\mathbf{A}}(T_0) \end{bmatrix} \xrightarrow{\mathbf{F}_7} \frac{\mathrm{SR}_i(T_0) - \mathrm{SR}_j(T_0) - K}{A_l(T_0)}. \tag{A.5}$$

The n displacement sensitivities will be given by the first n entries of the following gradient vector

$$\mathbf{F}_7'\mathbf{F}_6'\mathbf{H}_5'\mathbf{H}_4' \cdots \mathbf{H}_1'.$$

We show how to compute the Jacobian matrices $\mathbf{H}_i'$, $i = 1, \ldots, 5$ below.

A.3.1. *Drift adjoints.* We consider the following sub-mappings of $\mathbf{H}_1$:

$$\begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \end{bmatrix} \xrightarrow{\mathbf{H}_{1,0}} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{1} \end{bmatrix} \xrightarrow{\mathbf{H}_{1,1}} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{2} \end{bmatrix}$$
$$\cdots \xrightarrow{\mathbf{H}_{1,n-1}} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \\ \{\langle Z_k, \overline{A}_i\rangle\}_{i=1}^{n} \end{bmatrix} \xrightarrow{\mathbf{H}_{1,n}} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \overline{\mathbf{A}}(0) \\ \{\langle Z_k, \overline{A}_i\rangle^{A_l}\}_{i=1}^{n} \end{bmatrix} \xrightarrow{\mathbf{H}_{1,n+1}} \begin{bmatrix} \{\alpha_i\} \\ \mathbf{SR}(0) \\ \widehat{\mu}(0) \end{bmatrix}$$

The mappings $\mathbf{H}_{1,i}$, $i = 0, \ldots, n-1$, update the cross variation derivatives, $\mathbf{H}_{1,n}$ changes the pricing measure from the spot measure to the annuity measure, and $\mathbf{H}_{1,n+1}$ computes all the drifts under the annuity measure.

- The Jacobian matrices $\mathbf{H}_{1,i}'$, $i = 0, \ldots, n-1$, have similar non-zero elements as the Jacobian matrix $\mathbf{F}_{0,j}$ in section A.1.2. In addition to the three partial derivatives, we have an extra partial derivative with respect to displacements:
$$\frac{\partial}{\partial \alpha_j}\langle Z_k, \overline{A}_j\rangle = -\frac{\tau_{j-1}\overline{A}_j}{1 + \tau_{j-1}\mathrm{SR}_j} a_{jk},$$
for $k = 1, \ldots, F$.
- The Jacobian matrix $\mathbf{H}_{1,n}'$ is the same as the Jacobian matrix $\mathbf{F}_{1,n}'$ in section A.1.2.
- The Jacobian matrix $\mathbf{H}_{1,n+1}'$ is the same as the Jacobian matrix $\mathbf{F}_{0,n+1}'$ in section A.1.2.

The only difference between $\mathbf{H}_4'$ and $\mathbf{H}_1'$ is that the dimension of $\mathbf{H}_4'$ is larger than that of $\mathbf{H}_1'$ while the non-trivial entries are the same.

A.3.2. *Evolution adjoints.* The Jacobian matrix $\mathbf{H}_2'$ have one more non-trivial partial derivative (with respect to pseudo-root elements) than the Jacobian matrix $\mathbf{F}_2'$ in section A.1.3:

$$\frac{\partial \widehat{\mathrm{SR}}_j(T_0)}{\partial \alpha_j} = \frac{\widehat{\mathrm{SR}}_j(T_0) + \alpha_i}{\mathrm{SR}_j(0) + \alpha_i} - 1.$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, F$.

The Jacobian matrix $\mathbf{H}_5'$ have one more non-trivial partial derivative (with respect to pseudo-root elements) than the Jacobian matrix $\mathbf{F}_5'$ in section A.1.3:

$$\frac{\partial \mathrm{SR}_j(T_0)}{\partial \alpha_j} = \frac{\mathrm{SR}_j(T_0) + \alpha_i}{\widehat{\mathrm{SR}}_j(T_0) + \alpha_i} - 1.$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, F$.

A.3.3. *Computational order.* Similar to Delta and Vega computations, we can compute displacement sensitivities of a CMS spread option with an order $\mathcal{O}(nF)$.

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