Log-linearization of perturbed dynamical systems, with applications to optimal growth

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Abstract

It is common to study the asymptotic properties of log-linear stochastic systems by analyzing the behavior of their linear counterparts. In this paper a formal justification for analysis by log-linearization is given. As an application, a new existence, uniqueness and stability condition is derived for equilibria in a standard class of multisector growth models with stochastic production. The condition can be seen as a generalization of existing equilibrium conditions for this class of models. *Journal of Economic Literature* Classification Numbers: C62, E13, O41.

Key words: Log-linearization, stochastic equilibria, multisector growth.

1 Introduction

The investigation of equilibria and stability in a dynamic, stochastic setting begins with the contributions of, among others, Brock and Mirman (1972), Mirman (1972; 1973) and Green and Majumdar (1975). Standard techniques for establishing the existence and uniqueness of stochastic equilibria in the

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sense of Brock and Mirman were outlined in Razin and Yahav (1979) and Futia (1982). Since then, the methodology has been extended in several directions (see, e.g., Hopenhayn and Prescott (1992), Duffie et al. (1994), Bhattacharya and Majumdar (2001), Stachurski (2001)).

In economics, given that the state space often corresponds to bundles of goods, prices or physical quantities, it is common to analyse stochastic systems that evolve on the positive cone of finite or infinite dimensional Euclidean space. A large class of these models have the property of log-linearity, and the evolution of such systems is usually studied in linearized form, a transformation which considerably simplifies analysis. (Typical examples include Long and Plosser (1983), or Stokey et al (1989, Section 2.2).) However, it is not immediately clear that the original system retains all of the dynamic properties that are found in the linearized system.

Here a formal justification for log-linearization of stochastic systems is provided. In particular, it is shown that parallel existence, uniqueness and stability results hold for stochastic steady states in the original and linearized models. The approach is to establish topological conjugacy between the systems by showing that they commute with a homeomorphism.

Topological conjugacy arguments relate to deterministic dynamical systems. On the other hand, a stochastic system on state space X can be reformulated as a deterministic system generated by a linear operator on the space of all probability measures on X. The operators that characterize the evolution of perturbed dynamical systems in this manner are called Foias operators. The main technical innovation of this paper is application of the theory of topological conjugacy to Foias operators.

As well as using this approach to formalize a standard methodology, the techniques provide new insights into stability in log-linear systems. As an illustration, a new existence, uniqueness and stability condition is given for equilibria in a well-known multisector growth model with stochastic production. The condition is easy to verify in applications.

Finally, it should be noted that specifications exist where the formal methods used here are unnecessary for asymptotic analysis, as may be the case when the distribution of the disturbance term is characterized by a finite number of parameters that can be tracked through time, thereby obtaining an estimate of the limiting distribution. This paper generalizes to the case where the shock is arbitrary, and satisfies only a mild regularity condition which limits its expected influence on the system to finite quantities.

Section 2 frames the standard definitions of equilibria and stability in terms of Foias operators. Section 3 focuses on log-linear systems and their linear counterparts, and proves that equivalent results hold for stochastic equilibria. Section 4 applies the preceding analysis to a class of multisector growth models.

2 Equilibria and Stability

It is advantageous to first state the standard definition of stochastic equilibria and stability in terms of Foias operators. [1]

[1] For an exposition of Foias operators, see Lasota and Mackey (1994, Section 12.4).

In what follows, $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, X is an arbitrary metric space, $\mathcal{B}(X)$ is the Borel subsets of X, and $\mathcal{M}(X)$ is the set of finite signed measures on X. In other words, $\mathcal{M}(X)$ is the class of countably additive set functions $\mu\colon \mathcal{B}(X) \to \mathbb{R}$. The subset $\mathcal{P}(X) \subset \mathcal{M}(X)$ denotes the probabilistic measures on X, that is,

$$\mathcal{P}(X) = \{\mu \in \mathcal{M}(X) \,:\, \mu \geq 0 \text{ and } \mu(X) = 1\}.$$

Let a discrete-time stochastic dynamical system be given:

$$x_{t+1} = T(x_t, \varepsilon_t), \quad t \geq 0. \tag{1}$$

The state vectors x_t and the uncorrelated and identically distributed shocks ε_t take values in X, and $T\colon X \times X \to X$. In particular, $\varepsilon\colon \Omega \to X$ is a fixed $\mathcal{F}, \mathcal{B}(X)$-measurable function, and ε_t denotes the value of the function at the time t draw $\omega_t \in \Omega$. Each ω_t is drawn independently by $\mathbf{P}$. The distribution $\psi \in \mathcal{P}(X)$ of ε is defined in the usual way, i.e., $\psi(B) = \mathbf{P}(\varepsilon^{-1}(B))$.

Equation (1) implies that state variable x evolves according to a Markovian stochastic process denoted here by (x_t). In other words, (x_t) is a random element of the space of all X-valued sequences $\prod_{t=0}^{\infty} X$. The distribution ν^{∞} of the process (x_t) is a probability measure from the Borel sets of $\prod_{t=0}^{\infty} X$ into $[0, 1]$ that can be constructed from knowledge of the distribution of the shock ε, the function T and an initial state value x_0.[2] By definition, the

[2] Formal construction of discrete stochastic process in this manner is usually based on a well-known theorem of Ionescu Tulcea. See, e.g., Shiryaev (1996, Theorem 2, p. 249).

marginal distribution of x_t (call it ν_t) gives the probability that $x_t \in B$ for any $B \in \mathcal{B}(X)$, that is, $\nu_t(B) = \nu^\infty(\prod_{\tau=0}^{t-1} X \times B \times \prod_{\tau=t+1}^{\infty} X)$.

It follows from (1) that if the current state is $x \in X$, then the probability that the next-period state is in $B \in \mathcal{B}(X)$ is

$$\int \mathbf{1}_B[T(x,z)]\psi(dz), \tag{2}$$

where $\mathbf{1}_B$ is the characteristic function of B and the integral is over state space X. Suppose that the marginal distribution ν_t of x_t is known. The marginal distribution ν_{t+1} of x_{t+1} can be obtained via the recursion

$$\nu_{t+1}(B) = \int\int \mathbf{1}_B[T(x,z)]\psi(dz)\nu_t(dx), \quad B \in \mathcal{B}(X). \tag{3}$$

The intuition is that (3) sums the probability (2) of the state moving to B from x in one step over all possible values of x, weighted by the probability $\nu_t(dx)$ of x occurring as the current state.

From (3) it is possible to define an operator P from $\mathcal{P}(X)$ into itself by

$$(P\nu)(B) = \int\int \mathbf{1}_B[T(x,z)]\psi(dz)\nu(dx), \quad B \in \mathcal{B}(X).$$

In this context, P is called the *Foias operator* on $\mathcal{P}(X)$ corresponding to dynamical system (1). Using P allows (3) to be rewritten as $\nu_{t+1} = P\nu_t$. If x_0 is the initial state for the system (1), then $P^t\delta_{x_0}$ is the distribution for the state at time t, where δ_{x_0} is the probability measure concentrated at x_0 and P^t is defined recursively by $P^t = P \circ P^{t-1}$, $P^1 = P$.

A *steady state* or *equilibrium* for the system is a distribution ν^* such that $P\nu^* = \nu^*$, i.e. a fixed point of P on $\mathcal{P}(X)$. This definition is consistent with the standard definition used in economics (Green and Majumdar (1975, Eq. (4.8), p. 652)). The intuition is that P updates the probabilistic laws that govern the system; an equilibrium is a distribution that is invariant under this operation.

2.2 *Stability*

In order to discuss stability of equilibria, a notion of convergence is needed. For $\mu \in \mathcal{M}(X)$, the map $\mu \mapsto \sup \sum_{n=1}^{N} |\mu(X_n)|$, where the supremum is over the class of all measurable partitions of X, is called the total variation norm on $\mathcal{M}(X)$. In what follows, convergence in $\mathcal{P}(X)$ is in terms of this metric.

Let P be any Foias operator on $\mathcal{P}(X)$, and let μ^* be an equilibrium for P. Distribution μ^* will be called globally stable for P if $P^t\mu \to \mu^*$ as $t \to \infty$ for any $\mu \in \mathcal{P}(X)$.[3] For such a system, the current probabilistic state converges to the unique equilibrium regardless of initial conditions.

3 Log-Linearization

Let $\mathbb{R}_{++}$ denote the positive real numbers. Consider a system on $\mathbb{R}^n_{++} = \times_{i=1}^{n} \mathbb{R}_{++}$ that is linear (affine) under the taking of logarithms:

$$x_{i,t+1} = m_i \left(\prod_{j=1}^{n} x_{jt}^{a_{ij}} \right) \varepsilon_{it}, \quad i = 1, \ldots, n. \tag{4}$$

Here $m_i > 0$ and ε_{it} is an $\mathbb{R}_{++}$-valued shock, $i = 1, \ldots, n$. The vector of shocks $\varepsilon_t = (\varepsilon_{it})_{i=1}^{n} \in \mathbb{R}^n_{++}$ is serially independent with common distribution $\psi \in \mathcal{P}(\mathbb{R}^n_{++})$, $\psi(B) = \mathbf{P}(\varepsilon^{-1}(B))$.

[3] The majority of research in economics has used weak or weak* convergence to define stability for stochastic equilibria. Convergence in total variation norm implies convergence in both of these topologies.

Given any n-dimensional vector x, it is convenient to use the abbreviations $\log x$ for $(\log x_i)_{i=1}^n$ and $\exp x$ for $(\exp x_i)_{i=1}^n$. In this notation, (4) can be expressed more compactly as

$$x_{t+1} = \exp(\hat{m} + A \log x_t + \log \varepsilon_t), \quad x \in \mathbb{R}_{++}^n,$$

where $\hat{m} = \log m = (\log m_i)_{i=1}^n$ and A is the $n \times n$ matrix (a_{ij}).

Denote by P the Foias operator associated with (4).

In practice it is common to study the process generated by (4) using log-linearization, which corresponds to analysis of the system

$$x_{t+1} = \hat{m} + Ax_t + \hat{\varepsilon}_t, \quad x \in \mathbb{R}^n, \tag{5}$$

where $\hat{\varepsilon} = \log \varepsilon$.

Denote by $\hat{P}$ the Foias operator associated with (5).

It is shown below that (i) the system (5) has a stochastic equilibrium if and only if (4) has a stochastic equilibrium, (ii) the system (5) has at most one equilibrium if and only if (4) has at most one equilibrium, and (iii) the system (5) has a globally stable equilibrium if and only if (4) has a globally stable equilibrium.

For this purpose, define $E\colon \mathcal{P}(\mathbb{R}^n) \ni \mu \mapsto E\mu \in \mathcal{P}(\mathbb{R}_{++}^n)$ by $(E\mu)(B) = \mu(\log B)$, $B \in \mathcal{B}(\mathbb{R}_{++}^n)$, $\log B = \{\log x : x \in B\}$. The image $E\mu$ is non-negative, assigns 1 to the whole space $\mathbb{R}_{++}^n$ and is countably additive. Hence $E\mu \in \mathcal{P}(\mathbb{R}_{++}^n)$ as claimed. In fact E defines a one-to-one correspondence from $\mathcal{P}(\mathbb{R}^n)$ onto $\mathcal{P}(\mathbb{R}_{++}^n)$, where if $\nu \in \mathcal{P}(\mathbb{R}_{++}^n)$, then $E^{-1}\nu = \nu(\exp(\cdot))$.

The question arises as to how to integrate with respect to these measures.

Let $f\colon \mathbb{R}^n_{++} \to \mathbb{R}$ be any $\mathcal{B}(\mathbb{R}^n_{++})$-measurable function that is summable with respect to $\nu \in \mathcal{P}(\mathbb{R}^n_{++})$. By a change of variables result (Berezansky et al. (1996, Chapter 5, Theorems 4.1 and 4.2)), $\mathbb{R}^n \ni x \mapsto f(\exp x) \in \mathbb{R}$ is a $\mathcal{B}(\mathbb{R}^n)$-measurable function and

$$\int_{\mathbb{R}^n_{++}} f(x)(E\mu)(dx) = \int_{\mathbb{R}^n} f(\exp x)\mu(dx). \tag{6}$$

Conversely, if $g\colon \mathbb{R}^n \to \mathbb{R}$ is any $\mathcal{B}(\mathbb{R}^n)$-measurable function that is summable with respect to $\mu \in \mathcal{P}(\mathbb{R}^n)$, then $\mathbb{R}^n_{++} \ni x \mapsto g(\log x) \in \mathbb{R}$ is a $\mathcal{B}(\mathbb{R}^n_{++})$-measurable function and

$$\int_{\mathbb{R}^n} g(x)(E^{-1}\nu)(dx) = \int_{\mathbb{R}^n_{++}} g(\log x)\nu(dx). \tag{7}$$

As stated above, E is a bijection. The next lemma shows that E and E^{-1} are both continuous. Bijections with this property are called *homeomorphisms*, or, more suggestively, *topological isomorphisms*.

Lemma 1 *The map $E\colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n_{++})$ is a homeomorphism.*

PROOF. Let $\mu_n \to \mu$ in $\mathcal{P}(\mathbb{R}^n)$. It is shown in Stokey et al. (1989, Theorem 11.6) that for any metric space X, sequence (μ_n) in $\mathcal{P}(X)$ and $\mu \in \mathcal{P}(X)$, the statement $\mu_n \to \mu$ in total variation norm is equivalent to $|\mu_n(B) - \mu(B)| \to 0$ uniformly on $\mathcal{B}(X)$.

Fix $\varepsilon > 0$. By hypothesis, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\mu_n(A) - \mu(A)| < \varepsilon, \quad \forall A \in \mathcal{B}(\mathbb{R}^n),$$

where N is independent of A. But then

$$n \geq N \implies |(E\mu_n)(B) - (E\mu)(B)| = |\mu_n(\log B) - \mu(\log B)| < \varepsilon$$

for any $B \in \mathcal{B}(\mathbb{R}^n_{++})$. Hence $E\mu_n \to E\mu$ in $\mathcal{P}(\mathbb{R}^n_{++})$. This proves continuity of E. The proof of continuity of E^{-1} is similar.

The relationship between P and $\hat{P}$ can be expressed in terms of E.

Proposition 2 *If P is the Foias operator on $\mathcal{P}(\mathbb{R}^n_{++})$ associated with (4) and $\hat{P}$ is the Foias operator on $\mathcal{P}(\mathbb{R}^n)$ associated with the linearized system (5), then*

$$P = E\hat{P}E^{-1} \quad \text{on} \quad \mathcal{P}(\mathbb{R}^n_{++}). \tag{8}$$

PROOF. We prove the equivalent result

$$(PE\mu)(B) = (E\hat{P}\mu)(B), \quad \forall \mu \in \mathcal{P}(\mathbb{R}^n), \quad \forall B \in \mathcal{B}(\mathbb{R}^n_{++}). \tag{9}$$

Note first that since $\varepsilon \sim \psi \in \mathcal{P}(\mathbb{R}^n_{++})$, it follows that $\hat{\varepsilon} \sim E^{-1}\psi \in \mathcal{P}(\mathbb{R}^n)$, because

$$\mathbf{P}[\hat{\varepsilon}^{-1}(B)] = \mathbf{P}[\varepsilon^{-1}(\exp B)] = \psi(\exp B) = (E^{-1}\psi)(B), \quad \forall B \in \mathcal{P}(\mathbb{R}^n).$$

Hence the Foias operator $\hat{P}$ corresponding to (5) is given by

$$(\hat{P}\mu)(B) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_B(\hat{m} + Ax + z)(E^{-1}\psi)(dz)\mu(dx), \quad B \in \mathcal{B}(\mathbb{R}),$$

where $\mu \in \mathcal{P}(\mathbb{R}^n)$.

On the other hand,

$$(PE\mu)(B) = \int_{\mathbb{R}^n_{++}}\int_{\mathbb{R}^n_{++}} \mathbf{1}_B(\exp(\hat{m} + A\log x + \log z)\psi(dz)(E\mu)(dx).$$

But then

$$
\begin{aligned}
(E\hat{P}\mu)(B) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{\log B}(\hat{m} + Ax + z)(E^{-1}\psi)(dz)\mu(dx) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_B[\exp(\hat{m} + Ax + z)](E^{-1}\psi)(dz)\mu(dx) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n_{++}} \mathbf{1}_B[\exp(\hat{m} + Ax + \log z)]\psi(dz)\mu(dx) \\
&= \int_{\mathbb{R}^n_{++}}\int_{\mathbb{R}^n_{++}} \mathbf{1}_B[\exp(\hat{m} + A\log x + \log z)]\psi(dz)(E\mu)(dx)
\end{aligned}
$$

where we have used (6) and (7) to change variables. This proves (9).

A pair of dynamical systems is called topologically conjugate if they commute with a homeomorphism in the manner of (8). Qualitatively, the two systems exhibit the same dynamics. In particular, the following statement is true.

Proposition 3 *The operator $\hat{P}$ has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n)$ if and only if P has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n_{++})$.*

PROOF. Suppose first that μ^* is a fixed point of $\hat{P}$. Then

$$PE\mu^* = (E\hat{P}E^{-1})E\mu^* = E\hat{P}\mu^* = E\mu^*.$$

Hence $E\mu^*$ is a fixed point of P. Conversely, let ν^* be a fixed point of P. Then

$$\hat{P}E^{-1}\nu^* = (E^{-1}PE)E^{-1}\nu^* = E^{-1}P\nu^* = E^{-1}\nu^*.$$

Hence $E^{-1}\nu^*$ is a fixed point of $\hat{P}$.

Uniqueness is immediate from the above argument, given that E is a bijection.

Regarding stability, suppose that $\hat{P}$ has globally stable equilibrium μ^*. Define $\nu^* = E\mu^*$. Let ν be any distribution in $\mathcal{P}(\mathbb{R}_{++})$, and set $\mu = E^{-1}\nu$. By hypothesis, $\hat{P}^t\mu \to \mu^*$ as $t \to \infty$. But then $E\hat{P}^t\mu \to E\mu^*$ by Lemma 1. Given that $E\mu^* = \nu^*$ and

$$E\hat{P}^t\mu = E(E^{-1}P^tE)(E^{-1}\nu) = P^t\nu,$$

this is the same statement as $P^t\nu \to \nu^*$. This proves sufficiency. The proof of necessity is similar.

4 Example

As an illustration of the utility of these results, a new stability condition is derived for the multisector unit-elastic growth model. Asymptotic stability for nonlinear multisector growth models was studied by Brock and Majumdar (1978) using the weak* topology. Their analysis depends on assumed compactness of the underlying state space X, in which case $\mathcal{P}(X)$ is compact in the weak* topology (but not, in general, in the strong topology used here).[4] In turn, compactness of the state space requires that the disturbance term ε have compact support. Here a weaker condition is used, which can be viewed as a generalization of the compact support condition.

4.1 The Model

The following n-sector growth model, which has become something of a workhorse in intertemporal macroeconomics, is due to Long and Plosser (1983).[5]

[4] See, e.g., Aliprantis and Border (1999, Section 14.1).

[5] For a more recent discussion of stochastic optimal growth and introduction to the literature see, e.g., Amir (1997).

Let $C_t = (C_{it})_{i=1}^n$ be time t consumption. Utility is given by $u(C_t) = \sum_{i=1}^n \theta_i \log C_{it}$, $\theta_i > 0$. Production is according to the Cobb-Douglas technology

$$Y_{i,t+1} = L_{it}^{b_i} \Big(\prod_{j=1}^{n} X_{ijt}^{a_{ij}} \Big) \varepsilon_{it}, \quad i = 1, \ldots n, \tag{10}$$

where Y_i is output of commodity i, L_i is labor allocated to sector i, X_{ij} is the amount of commodity j used in the production of good i, and ε_i is a sector-specific shock. The vector of shocks is uncorrelated and identically distributed over time. [6]

The parameters a_{ij} and b_i are all positive. Production is assumed to be constant returns to scale, i.e.,

$$b_i + \sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, \ldots, n. \tag{11}$$

The economy faces resource constraints

$$C_{jt} + \sum_{i=1}^{n} X_{ijt} = Y_{jt}, \quad j = 1, \ldots, n, \tag{12a}$$

$$\sum_{i=1}^{n} L_i = L, \quad i = 1, \ldots, n. \tag{12b}$$

The representative agent seeks a solution to $\max E[\sum_{t=0}^{\infty} \beta^t u(C_t)]$ subject to (10) and (12), where $\beta \in (0,1)$ is a discount factor. The optimal controls are

$$X_{ijt} = a_{ij} \beta \frac{\gamma_i}{\gamma_j} Y_{jt}, \tag{13a}$$

$$L_{it} = \gamma_i b_i \Big(\sum_{j=1}^{n} \gamma_j b_j \Big)^{-1} L, \tag{13b}$$

where $\gamma' = \theta'(1 - \beta A)$, A being the matrix (a_{ij}) of output elasticities with

[6] The objective of Long and Plosser was to generate fluctuations in time series consistent with the business cycle from a general equilibrium framework and without serial dependence in external noise.

respect to commodity inputs. Substitution of (13) into (10) gives

$$Y_{i,t+1} = m_i \Big(\prod_{j=1}^{n} Y_{jt}^{a_{ij}} \Big) \varepsilon_{it}, \quad i = 1, \ldots n, \tag{14}$$

with m_i a positive constant.

The following proposition is the main result of this section. The norm used in the statement refers to the standard Euclidean vector norm. As before, the log of a vector is to be interpreted as the vector of the logs of the individual components.

Proposition 4 *Let an initial commodity vector $Y_0 \in \mathbb{R}^n_{++}$ be given. If the expectation $E\|\log \varepsilon\|$ is finite, then the economy has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n_{++})$.*

PROOF. Consider again the system (1). It has been shown by Lasota and Mackey (1994, Proposition 12.7.1, Theorem 12.7.2) that if $X = \mathbb{R}^n$, $T(x_t, \varepsilon_t) = T(x_t) + \varepsilon_t$, where $T \colon X \to X$ satisfies

$$\|T(x) - T(x')\| \leq \rho \|x - x'\|, \quad \forall x, x' \in \mathbb{R}^n, \tag{15}$$

for some $\rho < 1$, an initial condition $x_0 \in X$ is given and the sequence (ε_t) of independent and identically distributed shocks satisfies $E\|\varepsilon\| = \int \|x\| \psi(dx) < \infty$, then the system has a unique, globally stable equilibrium in $\mathcal{P}(X)$.

The linearized version of (14) can be written as

$$y_{t+1} = \log m + A y_t + \log \varepsilon_t, \tag{16}$$

where $A = (a_{ij})$ is as above. We now show that (16) satisfies all of the conditions of Lasota and Mackey. Given the hypotheses of the proposition, only verification of (15) is nontrivial. Let ρ denote the spectral radius of the matrix A. Then, by a standard result from linear operator theory,

$$\|Ay - Ay'\| \leq \rho\|y - y'\|, \quad \forall y, y' \in \mathbb{R}^n.$$

Moreover, $\rho < 1$, because the spectral radius of any matrix $A = (a_{ij})$, $a_{ij} \geq 0$, is less than or equal to the maximum of the row sums.[7] These sums are all strictly less than one by (11). Hence (15) holds. It follows that (16) has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n)$.

But then (14) has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n_{++})$ by Proposition 3.

Remark 5 *By Jensen's inequality,*

$$E\|\log\varepsilon\| \leq \sqrt{\sum_{i=1}^{n} E[(\log\varepsilon_i)^2]}.$$

Thus the finiteness condition on the vector of sector-specific shocks ε *is satisfied if* $\log\varepsilon_i$ *has finite second moment for each* i.

Suppose that the distribution of ε_i *can be represented by density function* ψ_i *on* $\mathbb{R}_{++}$. *By the Chebyshev inequality,*

$$\int_0^{e^{-a}} \psi_i(x)dx + \int_{e^a}^{\infty} \psi_i(x)dx \leq \frac{E[(\log\varepsilon_i)^2]}{a^2}$$

For any positive number a. *Hence finiteness of the second moment of* $\log\varepsilon_i$ *can be interpreted as a restriction on the left- and right-hand tails of the distribution of* ε_i. *This can be viewed as a generalization of the assumption of compact support used in Brock and Majumdar (1978).*

Note that for a positive shock ε_i, *a bound such as* $E(\varepsilon_i) < \infty$ *or* $E(\varepsilon_i^2) < \infty$

[7] For details, see, e.g., Lancaster (1969, Theorem 7.2.1).

would be insufficient, because it would restrict only the right-hand tail of the distribution. Without a restriction on the left-hand tail, many small shocks may inhibit stable growth.

Remark 6 *The result of Lasota and Mackey is independent of the choice of norm on* $\mathbb{R}^n$*. Different norms imply different conditions on the shock* ε *in Proposition 4. For example, it is also sufficient that* $E|\log \varepsilon_i| < \infty$ *for* $i = 1, \ldots, n$.

5 Conclusion

The paper examined a new technique for studying dynamic economies on the positive cone of real Euclidean space that are subject to the influence of serially uncorrelated noise. The technique led to a new stability condition for a standard multisector growth model. The condition was obtained by exploiting log-linearity in a formal dynamical systems framework.

The techniques may also lead to insights into dynamic behavior of perturbed systems studied in other areas of economics. Further, it may be possible to extend the analysis by considering other types of topological conjugacies, or by combining the conjugacy relation used here with different stability conditions.

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