

A Partial Characterization of the Solution of
the Multidimensional Screening Problem
with Nonlinear Preferences

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Abstract

In this paper I apply the Hamiltonian method to solve the relaxed multi-dimensional screening problem. I also illustrate by some examples that the Hamiltonian technique coupled with implementability criterion developed by Carlier [2002] sometimes allows us to arrive at a complete solution of a screening problem.

1 INTRODUCTION

In many industries the tariffs are not strictly proportional to the quantity purchased. Examples include railroad tariffs, electricity tariffs, and rental rates for durable goods and space. All these cases fall into a general category of *nonlinear tariffs*. The major justification for the nonlinear pricing is the existence of private information on the side of consumers. In the early papers on the subject, private information was captured either by assuming a finite number of types [e. g. Adams and Yellen, 1976] or by a one-dimensional continuum of types [Mussa and Rosen, 1978]. However, often nonlinear tariffs specify the payment as a function of a variety of characteristics. For example, railroad tariffs specify charges based on weight, volume, and distance of each shipment. Different customers may value each of these characteristics differently, hence the customer's type will not in general be captured by a one-dimensional characteristic and a problem of multi-dimensional screening arises.

The general formulation of the problem of multi-dimensional screening is due to Armstrong [1996] and Wilson [1993], and goes as follows. Consider a multi-product monopoly producing n goods with a convex cost function. The preferences of a consumer over these goods can be parameterized by

an m -dimensional vector. Types of consumers are distributed according to a density function $f(\cdot)$ defined over a convex open bounded set $\Omega \subset R^m$. Assume that $f(\cdot)$ is continuously differentiable on Ω and can be extended by continuity on its closure. The monopolist is interested in maximizing profits by choosing a tariff which is a function from the set of bundles of goods to the real line. The tariff determines how much a consumer will pay for a particular bundle of goods.

Armstrong [1996] formulated the problem for arbitrary m and n and derived a solution in some special cases. He assumed that the preferences of consumers are given by a utility function which is increasing in all arguments, and that is continuous, convex, and homogeneous of degree one in tastes. Under rather strong assumptions both on the utility function and the distribution over types, he showed that the optimal tariff is cost-based. Armstrong [1999] gave an approximate solution when the number of goods is large. He showed that in this case the optimal tariff can be approximated by a two-part tariff when taste parameters are distributed independently across products, and a menu of two-part tariffs when there is a correlation in the distribution of types.

Rochet and Chone [1998] developed a general technique for dealing with

the problem of multidimensional screening when $n = m$ and utilities are linear in types. Their technique was extended to the case $m \neq n$ by Basov [2001]. The last paper uses the Hamiltonian approach to solve the screening problem. In this paper I use the Hamiltonian approach for a partial characterization of the solution of multi-dimensional screening models with utilities nonlinear in types.

Following the strategy developed by Rochet and Chone [1998], I first deal with the relaxed problem, i.e. with the screening problem in which the implementability constraint is dropped. This is a standard optimal control problem and hence, it can be solved using the technique described in Basov [2001]. Recent results obtained by Carlier [2002] allow us to check whether the obtained solution is implementable. Their criterion is based on a generalized notion of convexity and will be described below. If the criterion is satisfied the derived allocation is implementable

The paper is organized as follows. In Section 2 I describe the general implementability result of Carlier and give a simpler and economically more transparent proof than the original paper. I also show that in the linear context the general implementability criterion is reduced to the requirement that the surplus is convex, while in the one-dimensional model satisfying

the single crossing property it is reduced to the demand that the allocation is increasing in type. Both these cases were studied in the literature, see Mussa and Rosen [1978] for the one-dimensional case and Rochet [1987] for the general linear case. In Section 3 I formulate the relaxed problem and obtain the first order characterization of the solution. I also show that if the solution to the relaxed problem is implementable and dimensionality of the types space is odd, under some weak assumptions on the types space there exist at least two types on the boundary of the participation region that are served efficiently. In Section 4 I apply the developed technique to solve some examples.

2 THE IMPLEMENTABILITY CRITERION

Assume that the preferences of a consumer of type α who purchases one unit of good with quality characteristics $x \in R_+^n$ and pays $t \in R$ are given by a utility function

$$u(\alpha, x, t) = v(\alpha, x) - t.$$

Here $\alpha \in \Omega$, where Ω is an open, bounded, simply connected¹ subset of R^m with a piece-wise smooth boundary, and v is a differentiable function of its arguments. The type is private information of the consumer, however, it is common knowledge that types are distributed according to an everywhere positive density function $f(\alpha)$.

The monopolist's problem is to derive a tariff $t^2: R_+^n \rightarrow R$ to maximize profits

$$\pi = \int [t(x(\alpha)) - c(x(\alpha))]f(\alpha)d\alpha \quad (1)$$

where $c(\cdot)$ is the cost of production and $x(\alpha)$ is the bundle purchased by all type- α consumers.

Given such a tariff, let

$$s(\alpha) = \max_{x \in R_+^n} (u(\alpha, x, t(x))). \quad (2)$$

Thus, $s(\alpha)$ is the surplus of a consumer of type α who chooses a bundle $x \in X$ that maximizes her utility. One can solve (1), (2) to get:

¹A set is called simply connected if all its homotopy groups are trivial. For example, any convex set is simply connected.

²Restricting the monopolist to devising a tariff rather than a more general allocation scheme is without loss of generality. See, for example, Rochet [1985].

$$t(\alpha) = v(\alpha, x) - s(\alpha). \quad (3)$$

It is important to note that the function $t(\cdot)$, defined by (3), depends on α only through x . More precisely, assume there exist two types α_1 and α_2 such that $x(\alpha_1) = x(\alpha_2)$, but $t_1 \neq t_2$, where $t_i \equiv t(\alpha_i)$. Without loss of generality, assume that $t_1 > t_2$. But then type α_1 would be better-off choosing $x(\alpha_2)$ and paying t_2 , which contradicts the utility maximization by type α_1 . This means that the function $t(\cdot)$ defined in (3) is indeed a tariff, since it maps bundles of goods into the real line. It is possible to show that $s(\cdot)$ is continuous, almost everywhere differentiable, and satisfies the envelope conditions:

$$\frac{\partial s}{\partial \alpha_i}(\alpha) = \frac{\partial v}{\partial \alpha_i}(\alpha, x(\alpha)), \quad i = \overline{1, m}. \quad (4)$$

For a proof see Carlier [2001]. This implies that the optimal tariff will be continuous, therefore the monopolist can without loss of generality restrict her attention to the continuous tariffs. Hence, I will call an allocation $x(\alpha)$ implementable if there exists a continuous tariff $t : R_+^n \rightarrow R$ such that

$$x(\alpha) \in \arg \max_{x \in R_+^n} (u(\alpha, x) - t(x)).$$

Consider an allocation $x(\alpha)$ and surplus function $s(\alpha)$ such that the envelope conditions

$$\nabla s(\alpha) = \nabla_{\alpha} u(\alpha, x(\alpha)).$$

are satisfied almost everywhere. Allocation rule $x(\alpha)$ is in general a correspondence. We assume that this correspondence is upper-hemicontinuous, which will always be the case if consumers face a continuous tariff. Let $X = x(\bar{\Omega})$, then since $x(\cdot)$ is u.h.c. and $\bar{\Omega}$ is compact, X is compact.

One might ask: Given a surplus function and an allocation satisfying the envelope conditions (4) does there exist a tariff that implements them? It turns out that the answer to this question is affirmative if and only if the consumer surplus satisfies a generalized convexity property³. This result was first proved by Carlier [2002]. For the sake of completeness I will give the basic definitions and the proof of the result. The form in which the result is formulated here and its proof differ from that given in Carlier [2002].

Definition Function $s^*(x)$ defined by

$$s^*(x) = \max_{\alpha \in \Omega} (v(\alpha, x) - s(\alpha))$$

³For a more detailed discussion of abstract convexity see, for example, Singer [1997]

is called v -conjugate of $s(\alpha)$.

Definition Function $s^{**}(\alpha)$ defined by

$$s^{**}(\alpha) = \max_{x \in X} (u(\alpha, x) - s^*(x))$$

is called v -biconjugate of $s(\alpha)$.

Definition $v(\alpha, x)$ is said to satisfy the generalized single-crossing (GSC) property if

$$[\nabla_{\alpha} v(\alpha, x_1) = \nabla_{\alpha} v(\alpha, x_2)] \Rightarrow (x_1 = x_2).$$

Theorem 1 Assume $v(\alpha, x)$ is continuous in both arguments, continuously differentiable in α and satisfies GSC. An allocation $x(\alpha)$ and surplus $s(\alpha)$ are implementable if and only if the following conditions hold:

- a. $s(\alpha)$ is continuous and almost everywhere differentiable;
- b. $x(\alpha)$ is upper hemicontinuous and the envelope condition holds almost everywhere;
- c. $s(\alpha) = s^{**}(\alpha)$.

Proof. Suppose conditions (a)-(c) hold. Consider a tariff

$$t(x) = \begin{cases} s^*(x), & \text{for } x \in X \\ \infty, & \text{for } x \in R_+^n/X. \end{cases}$$

Condition (c) implies that this tariff will implement the surplus $s(\alpha)$ and an allocation

$$h(\alpha) = \arg \max_{x \in X} (v(\alpha, x) - t(x)).$$

By the envelope theorem

$$\nabla s(\alpha) = \nabla_{\alpha} v(\alpha, h(\alpha)).$$

The GSC property now implies that $h(\alpha) = x(\alpha)$.

Now suppose that $s(\alpha)$ and $x(\alpha)$ are implementable. Then there exists continuous $t : R_+^n \rightarrow R$ such that

$$\begin{aligned} s(\alpha) &= \max_{x \in R_+^n} (v(\alpha, x) - t(x)) \\ x(\alpha) &= \arg \max_{x \in R_+^n} (v(\alpha, x) - t(x)). \end{aligned}$$

Condition (a) now follows from the Berge maximum theorem, while Condi-

tion (b) follows from the envelope theorem. To prove Condition (c) note that for any $(\alpha, x) \in \overline{\Omega} \times R_+^n$

$$s(\alpha) + t(x) \geq v(\alpha, x)$$

and

$$s(\alpha) + t(x(\alpha)) = v(\alpha, x(\alpha)).$$

Let $x \in X$, then $\exists \alpha^* \in \overline{\Omega}$ such that $x(\alpha^*) = x$.

$$t(x) = v(\alpha^*, x) - s(\alpha^*).$$

On the other hand, for any $\beta \in \overline{\Omega}$

$$t(x) \geq v(\beta, x) - s(\beta).$$

Hence, for any $x \in X$

$$t(x) = \max_{\alpha \in \overline{\Omega}} (v(\alpha, x) - s(\alpha)) = s^*(x).$$

The tariff is not determined uniquely outside X . It is sufficient to choose

any $t(x) > \max_{\alpha \in \bar{\Omega}} (v(\alpha, x) - s_0(\alpha))$ to ensure that bundles outside X are not chosen. But now

$$s(\alpha) = \max_{x \in X} (v(\alpha, x) - s^*(x)) = s^{**}(\alpha).$$

Q. E. D.

Note that the GSC property was used only in the proof of the sufficiency in the previous theorem. Hence, Conditions (a)-(c) are necessary even if GSC is not satisfied. Theorem 1 can be reformulated in a way that gets rid of GSC completely. Namely: An allocation $x(\alpha)$ and surplus $s(\alpha)$ are implementable if and only if they are implementable by a tariff:

$$t(x) = \begin{cases} s^*(x), & \text{for } x \in X \\ \infty, & \text{for } x \in R_+^n / X. \end{cases}$$

However, if GSC does not hold the tariff may implement allocation $x(\alpha)$ only weakly, namely the optimal choice of a consumer given the tariff need not be unique. Moreover, there may be several different tariffs implementing the same surplus function. To appreciate these points consider the following example.

Example 1. Let $m = 1$, $n = 2$. Assume that consumers' utilities are given by

$$v(\alpha, x_1, x_2) = \alpha(x_1 + x_2),$$

while the cost of the monopolist is given by:

$$c(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

Assume that $\Omega = [0, 1]$ and types are distributed uniformly on Ω . Note that GSC does not hold, since

$$\frac{\partial v}{\partial \alpha}(\alpha, x) = \frac{\partial v}{\partial \alpha}(\alpha, y) \Leftrightarrow x_1 + x_2 = y_1 + y_2 \not\Rightarrow x = y.$$

To find the optimal allocation, following Basov (2001), first solve

$$\begin{aligned} & \min c(x_1, x_2) \\ & \text{s.t. } x_1 + x_2 = y. \end{aligned}$$

The result is

$$\begin{aligned}x_1 &= x_2 = \frac{y}{2}. \\c(y) &= \frac{1}{4}y^2.\end{aligned}$$

Now the monopolist solves

$$\max \int_0^1 (\alpha y - s - \frac{1}{4}y^2) d\alpha = \int_0^1 (\alpha y - (1 - \alpha)y - \frac{1}{4}y^2) d\alpha.$$

The solution is

$$y(\alpha) = 4\alpha - 2$$

and

$$x_1(\alpha) = x_2(\alpha) = 2\alpha - 1.$$

Types with $\alpha < 1/2$ are excluded from the contract. Note that this allocation can be implemented by several tariffs. For example, a cost based tariff

$$t(y) = \frac{1}{2} \left(\left(\sqrt{\frac{x_1^2 + x_2^2}{2}} + 1 \right)^2 - 1 \right)$$

implements this allocation. It can also be implemented by a tariff

$$t(x) = \frac{1}{2} \left(\frac{(x_1 + x_2 + 2)^2}{4} - 1 \right).$$

The last tariff implements the above allocation only weakly, since any pair (x_1, x_2) satisfying

$$x_1 + x_2 = 4\alpha - 2$$

is an optimal choice for a consumer of type α given the tariff.

Note that in a special case, when

$$v(\alpha, x) = \sum_{i=1}^m \alpha_i x_i \tag{5}$$

Theorem 1 implies that a surplus is implementable if and only if it is convex.

This result was first obtained in Rochet (1987). Moreover, the optimal tariff is convex.

The practical use of Theorem 1 is that to check whether a surplus is implementable by *any* tariff it is sufficient to check whether it is implementable by *a particular* tariff, namely by the v -conjugate of the surplus. I demonstrate this point in the following two examples.

Example 2.

Assume $\Omega = (0, 1)$ and

$$v(\alpha, x) = \alpha x_1 + \sqrt{\alpha} x_2.$$

Is surplus $\psi(\alpha) = \alpha$ implementable? According to Theorem 1 it is sufficient to check whether it is implementable by a tariff

$$t(x) = \psi^*(x) = \begin{cases} \frac{x_2^2}{4(1-x_1)}, & x_1 < 1 \\ \infty, & x_1 \geq 1. \end{cases}$$

Solving a problem

$$\max_{x \in \mathbb{R}_+^2} (\alpha x_1 + \sqrt{\alpha} x_2 - t(x))$$

one obtains that the maximum is at points

$$\frac{x_2}{2(1-x_1)} = \sqrt{\alpha}$$

and the value of the maximum is α . Hence the surplus is implementable.

Example 3.

Assume $\Omega = (0, 1)$ and

$$v(\alpha, x) = \alpha x_1 + \alpha^{3/2} x_2.$$

Is surplus $\psi(\alpha) = \alpha$ implementable? According to Theorem 1 it is sufficient to check whether it is implementable by a tariff

$$t(x) = \psi^*(x) = \begin{cases} x_2, & \text{if } 2x_1 + 3x_2 \geq 2 \\ 0, & \text{if } 2x_1 + 3x_2 < 2. \end{cases}$$

But

$$\max_{x \in R_+^2} (\alpha x_1 + \alpha^{3/2} x_2 - t(x)) = \infty.$$

Hence, the surplus is not implementable.

To get another look on the implementability criterion consider two problems

$$s^*(x) = \max_{\alpha \in \Omega} (v(\alpha, x) - s(\alpha))$$

and

$$s^{**}(\alpha) = \max_{x \in X} (v(\alpha, x) - s^*(x)).$$

Note that these problems are dual in the sense that the first order conditions for one of them are the envelope conditions for the other. Assume that functions $v(\cdot, \cdot)$ and $s(\cdot)$ and set Ω are such that the first order conditions are necessary and sufficient for each maximization problem (e.g. $v(\cdot, \cdot)$ is concave in α and linear in x , $s(\cdot)$ is convex and set Ω is convex). Then $s^{**} = s$ and $s(\cdot)$ is implementable.

Now let us consider the case when $m = n = 1$. Assume that $\Omega = (0, 1)$, function $v(\cdot, \cdot)$ is twice differentiable with $v_1 \geq 0$, $v_2 \geq 0$ and $v_{12} > 0$, where v_i denotes the partial derivative with respect to the i^{th} argument. Moreover, $v(0, 0) = v_2(0, 0) = 0$. Inequality $v_{12} > 0$ is known as the single crossing property. The following result holds:

Corollary 1. *Assume a consumer has a twice differentiable utility function satisfying the single crossing property. Given a differentiable function $s(\cdot)$ define $x = h(\alpha)$ as the unique solution to*

$$s'(\alpha) = v_1(\alpha, x). \tag{6}$$

Then $s(\cdot) = s^{**}(\cdot)$ (and therefore implementable) if and only if $h(\alpha)$ is weakly increasing.

This result was first obtained by Mussa and Rosen (1978). Here I want to prove it as a corollary of the general implementability theorem. Note that even in the case $m = n = 1$ Theorem 1 is stronger than Corollary 1 since it does not require the single-crossing property to hold.

Proof. First, suppose that $h(\alpha)$ is weakly increasing. We have to prove that $s(\cdot)$ is v -convex. Define a correspondence $h^{-1}(x)$ by

$$\alpha \in h^{-1}(x) \Leftrightarrow h(\alpha) = x. \quad (7)$$

This correspondence is increasing, namely $x_1 > x_2$, $\alpha_1 \in h(x_1)$, $\alpha_2 \in h(x_2)$ implies $\alpha_1 > \alpha_2$. I will write $\alpha < h^{-1}(x)$ ($\alpha > h^{-1}(x)$) if for $\forall \beta \in h^{-1}(x)$ $\alpha < \beta$ ($\alpha > \beta$). It is easy to see that $h^{-1}(x)$ is either singleton or a closed

interval for any x . Define $s^*(x)$ by

$$s^*(x) = \max_{\alpha \in \Omega} (v(\alpha, x) - s(\alpha)). \quad (8)$$

Let

$$\psi(\alpha) = v(\alpha, x) - s(\alpha). \quad (9)$$

Then

$$\psi'(\alpha) = v_1(\alpha, x) - s'(\alpha) = v_1(\alpha, x) - v_1(\alpha, h(\alpha)). \quad (10)$$

The last equality follows from the definition of $h(\alpha)$. Let $\alpha < h^{-1}(x)$. Then $x > h(\alpha)$ and by the single crossing property $v_1(\alpha, x) > v_1(\alpha, h(\alpha))$ and $\psi'(\alpha) > 0$. Similar, if $\alpha > h^{-1}(x)$ then $\psi'(\alpha) < 0$. Finally, if $\alpha \in h(x)$ then (9) implies $\psi'(\alpha) = 0$. Hence function $\psi(\alpha)$ is maximized at $\alpha \in h(x)$ (all such α will deliver the same value for $\psi(\alpha)$) and

$$s^*(x) = v(h^{-1}(x), x) - s(h^{-1}(x)). \quad (11)$$

According to (9), the value of $s^*(x)$ does not depend on the selection from the correspondence $h^{-1}(x)$, hence $s^*(x)$ is well defined. Note also that by the

envelope theorem

$$s^{*'}(x) = v_2(h^{-1}(x), x). \quad (12)$$

If $h^{-1}(x)$ is a singleton then $s^{*'}(x)$ is well defined. If it is an interval, $s^{*'}(x)$ will be multivalued (and correspond to the sub-differential rather than to a derivative). In this case $s^{*'}(x)$ will have a kink.

Now define

$$s^{**}(x) = \sup_{x \in R_+} (v(\alpha, x) - s^*(x)). \quad (13)$$

Let

$$\varphi(x) = v(\alpha, x) - s^*(x). \quad (14)$$

The derivative (subdifferential) of φ is given by

$$\varphi'(x) = v_2(\alpha, x) - v_2(h^{-1}(x), x). \quad (15)$$

Using the single crossing property in the same way as above, it is easy to see that $\varphi(x)$ is increasing for $x < h(\alpha)$ and decreasing afterwards. Hence, a maximum is achieved at $x = h(\alpha)$, and using (14) and (16)

$$s^{**}(x) = v(\alpha, h(\alpha)) - v(h^{-1}(h(\alpha)), h(\alpha)) - s(h^{-1}(h(\alpha))) = s(\alpha). \quad (16)$$

Now let us assume that $s(\cdot)$ is v -convex.

$$s(\alpha) = \max_{x \in R_+} (v(\alpha, x) - s^*(x)). \quad (17)$$

Let

$$x^*(\alpha) = \arg \max_{x \in R_+} (v(\alpha, x) - s^*(x)) \quad (18)$$

The generalized envelope theorem (Milgrom and Segal, 2002) implies that $s'(\alpha) = v_1(\alpha, x^*)$. Hence by the single crossing property $x^* = h(\alpha)$.

According to the Revelation Principle (see, for example, Mas-Colel, Whinston, and Green, 1995) the same allocation can be implemented by a direct revelation mechanism. That is, the consumer is asked to reveal her type. If the consumer announces α' she is allocated her $h(\alpha')$ and pays $t(\alpha')$, where

$$t(\beta) = v(\beta, h(\beta)) - \int_0^\beta v_1(\theta, h(\theta)) d\theta. \quad (19)$$

The last expression follows from the definition of the surplus and envelope conditions (4). It should be optimal for the consumer to reveal her type truthfully.

If a consumer announces type α' when her true type is α she will earn

utility

$$V(\alpha, \alpha') = v(\alpha, h(\alpha')) - t(\alpha'). \quad (20)$$

The truth-telling conditions imply

$$\frac{\partial V}{\partial \alpha'}(\alpha, \alpha) = 0, \quad \frac{\partial^2 V}{\partial \alpha'^2}(\alpha, \alpha) \leq 0. \quad (21)$$

Differentiating the first order condition with respect to α' along the line $\alpha' = \alpha$ and taking into account the second order condition implies

$$\frac{\partial^2 V}{\partial \alpha \partial \alpha'}(\alpha, \alpha) \geq 0. \quad (22)$$

On the other hand, the definition of $V(\cdot, \cdot)$ implies

$$\frac{\partial^2 V}{\partial \alpha \partial \alpha'}(\alpha, \alpha) = v_{12}(\alpha, h(\alpha))h'(\alpha). \quad (23)$$

Hence, taking into account the single-crossing property, $h'(\alpha) \geq 0$.

In this Section I described Carlier's general implementability Theorem and showed that previously known results on implementability can be obtained as its corollaries. In the next Section I will drop the implementability

constraint and formulate the relaxed problem.

3 THE RELAXED PROBLEM

Recall that the monopolist's problem is to derive a tariff $t : R_+^n \rightarrow R$ to maximize profits (1) subject to

$$x(\alpha) = \arg \max_{x \in R_+^n} (u(\alpha, x, t(x))). \quad (24)$$

Using (3) to exclude the tariff from the monopolist's objective the monopolist's problem can be written as:

$$\begin{aligned} & \max_x \int [v(\alpha, x) - s(\alpha) - c(x)] f(\alpha) d\alpha \quad (25) \\ s.t. \frac{\partial s}{\partial \alpha_i}(\alpha) &= \frac{\partial v}{\partial \alpha_i}(\alpha, x), \quad i = \overline{1, m}, \quad s^{**}(\alpha) = s(\alpha), \quad s(\alpha) \geq s_0(\alpha). \quad (26) \end{aligned}$$

The constraints include envelope conditions (4), a participation constraint, and the implementability condition provided by Theorem 1. Function $s_0(\alpha)$ gives the value of the outside option for a consumer of type α . One might one to assume that $s_0^{**}(\alpha) = s_0(\alpha)$. The economic meaning of this assumption is that $s_0(\alpha)$ itself results from participation in some economic mechanism.

This assumption, however, is not important for our purposes. Carlier (2001) proved that the problem (25)-(26) has a solution.

Introduce new variables z by formulae

$$z_i = \frac{\partial v}{\partial \alpha_i}(\alpha, x(\alpha)), \quad i = \overline{1, m-n} \quad (27)$$

$$z_i = x_i, \quad i = \overline{m-n+1, n}. \quad (28)$$

Define the relaxed problem by

$$\max_z \int [v(\alpha, z) - s(\alpha) - c(z)] f(\alpha) d\alpha \quad (29)$$

$$s.t. \frac{\partial s}{\partial \alpha_i}(\alpha) = \frac{\partial v}{\partial \alpha_i}(\alpha, z), \quad i = \overline{1, m}, \quad s(\alpha) \geq s_0(\alpha) \quad (30)$$

$$z_i = \frac{\partial v}{\partial \alpha_i}(\alpha, z), \quad i = \overline{1, m-n} \quad (31)$$

The relaxed problem (29)-(31) is obtained from the problem (25)-(26) by dropping the implementability constraint and using (27)-(28) to exclude x .

It is an optimal control problem with a state variable s and a vector of control variables z . Note that there are two types of constraints that govern this problem: constraints (30) link controls with the partial derivatives of the state variable. They are similar to the capital accumulation equation in the

optimal growth theory. Constraints (31), on the other hand, restrict the set of possible controls and do not contain the state variable or its derivatives. Note that if $m \leq n$ constraints (31) vanish.

Suppose function v is linear in type and $\partial v / \partial \alpha_i = x_i$, $i = \overline{m - n + 1, n}$. In that case components of vector z can be viewed as artificial goods, utils, which are produced from physical goods, x using production technology (27)-(28). If $m > n$ not all util combinations can be produced, i. e. there are production constraints. Hence, a multidimensional screening problem with $m > n$ is equivalent to a screening problem with $m = n$ with production constraints. A similar interpretation can be useful in the nonlinear case. In that case, however, both utils and production constraints become type specific.

Theorem 2 *Suppose there exists $\gamma > 0$ such that $f(\alpha) \geq \gamma$ for any $\alpha \in \Omega$. Let function $\partial v / \partial \alpha_i(\alpha, \cdot) : R_+^n \rightarrow R$ be twice continuously differentiable and concave and $c(\cdot)$ be twice continuously differentiable and the matrix of its second derivatives $B = D^2c$ has uniformly bounded eigenvalues, i. e. $\exists \varepsilon > 0$*

and $\exists M > \varepsilon$ such that $\forall x \in R_+^n$ and $\forall h \in R^n$

$$\varepsilon \|h\|^2 \leq \langle h, Bh \rangle \leq M \|h\|^2.$$

Then problem (29)-(32) has a unique solution.

Proof. See Appendix.

Note that, as we have seen in Example 1, the uniqueness of surplus guaranteed by Theorem 2 does not imply the uniqueness of the optimal tariff, unless the GSC property holds.

This Theorem generalizes similar results in Rochet and Chone [1998] and Basov [2001] for a nonlinear case. Note that this result does not follow from the results obtained by Carlier [2001], since he does not deal with the relaxed problem. Our next step is to provide the first order characterization of the solution of the problem (29)-(31).

3.1 A RECOURSE IN THE OPTIMAL CONTROL THEORY

Consider a problem:

$$\max \int_{\Omega} F(\varphi, y, t) dt \quad (32)$$

$$s.t. A\varphi = b(y, t), g(y, t) = 0, \varphi(t) \geq \varphi_0(t) \quad (33)$$

where $t \in \Omega \subset R^m$ is an open, bounded, one-connected set with a smooth boundary, $\varphi : \Omega \rightarrow R$ and each component of $y : \Omega \rightarrow R_+^n$ belongs to $H^1(\Omega)$, $A : H^1(\Omega) \rightarrow \times_{i=1}^m H^0(\Omega) \equiv H_m^0(\Omega)$ is a linear differential operator, $b : R^{n+m} \rightarrow R^m$ and $g : R^{n+m} \rightarrow R^\ell$ are continuously differentiable functions, and $\varphi_0 : \Omega \rightarrow R$ is a continuous function. Here $H^p(\Omega)$ is the p^{th} Sobolev space, i. e. the space of functions possessing square integrable derivatives up to the order p , $H_k^p(\Omega)$ is the space of k -dimensional vector functions, such that each component belongs to $H^p(\Omega)$. For any two functions $\zeta, \xi \in H_m^0(\Omega)$ define their inner product by

$$(\zeta, \xi) = \sum_{i=1}^m \int_{\Omega} \zeta_i(t) \xi_i(t) dt. \quad (34)$$

Let $\nu(\alpha)$ denote the unit vector normal to the boundary $\partial\Omega$ of the set Ω and pointing in the outward direction.

Definition 5. Operator $A^* : H_m^1(\Omega) \rightarrow H^0(\Omega)$ is called adjoint for the operator A if for any $\zeta \in H^1(\Omega)$ and $\xi \in H_m^1(\Omega)$ such that $\langle \xi, \nu \rangle = 0$

$$(\xi, A\zeta) = (A^*\xi, \zeta). \quad (35)$$

To formulate the first order conditions for the problem (35)-(36) form a Hamiltonian:

$$H(\varphi, y, t; \lambda, \mu) = F(\varphi, y, t) + \langle \lambda, \nabla_\alpha v(\alpha, z) \rangle + \langle \mu, g(y, t) \rangle + \eta(\varphi(t) - \varphi_0(t)). \quad (36)$$

where $\lambda : \Omega \rightarrow R^m$ and $\mu : \Omega \rightarrow R^\ell$ are continuously differentiable vector functions. Then the following result holds.⁵

Theorem 3 Suppose function $\varphi(\cdot)$ solves the problem (32)-(33). Let Γ be the

⁵For a discussion, see Eberhard [1984] or Funk [1962].

set $\{t \in \Omega : \varphi(t) \geq \varphi_0(t)\}$. Then there exist continuously differentiable vector functions $\lambda : \Omega \rightarrow R^m$, $\mu : \Omega \rightarrow R^\ell$, and a continuous function $\eta : \Omega \rightarrow R_+$ such that

$$A^* \lambda = \frac{\partial H}{\partial \varphi} \text{ a.e. on } \Gamma. \quad (37)$$

$$\langle \lambda, \nu \rangle = 0 \text{ a.e. on } \partial\Gamma \cap \partial\Omega. \quad (38)$$

$$\eta(\varphi(t) - \varphi_0(t)) = 0 \quad (39)$$

$$y \in \arg \max H(\varphi, y, t; \lambda, \mu). \quad (40)$$

Equation (37) governs the evolution of the costate vector λ . It is easy to see that if $m = 1$ and $A = d/dt$ then (37) is reduced to⁶

$$\dot{\lambda} = -\frac{\partial H}{\partial \varphi}. \quad (41)$$

This condition is well known to economists from optimal growth theory. Equation (38) is a straightforward generalization of the transversality condition, (39) is the complementary slackness condition. Finally, equation (40) is Pontryagin's maximum principle.

⁶In this case $\Omega = [0, T]$ and integrating and taking into account $\xi(T) = 0$ one obtains $\int_0^T \zeta(t) \dot{\zeta}(t) dt = -\int_0^T \dot{\zeta}(t) \xi(t) dt$. Hence $(d/dt)^* = -(d/dt)$.

3.2 THE FIRST ORDER CHARACTERIZATION OF THE SOLUTION TO THE RELAXED PROBLEM LEM

After a short recourse in the optimal control theory we are ready to provide the first order characterization of the solution to the relaxed problem.

Define the Hamiltonian by

$$\begin{aligned}
 H(s, z, \alpha; \lambda, \mu) = & [v(\alpha, z) - s(\alpha) - c(z)]f(\alpha) + \langle \lambda, \nabla_{\alpha} v(\alpha, z) \rangle + \\
 & \sum_{i=1}^{m-n} \mu_i (z_i - \frac{\partial v}{\partial \alpha_i}(\alpha, z)) + \eta(s(\alpha) - s_0(\alpha)), \quad (42)
 \end{aligned}$$

where

$$[\nabla_{\alpha} v(\alpha, z)]_i = \frac{\partial v}{\partial \alpha_i}(\alpha, z). \quad (43)$$

Recall that for a continuously differentiable vector function $q : R^m \rightarrow R^m$ divergence of q is defined by

$$div q = \sum_{i=1}^n \frac{\partial q_i}{\partial \alpha_i}. \quad (44)$$

The following result holds:

Theorem 4 Suppose the surplus function $s^*(\cdot)$ solves the problem (29)-(31).

Then there exist continuously differentiable vector functions $\lambda : \Omega \rightarrow R^m$,

$\mu : \Omega \rightarrow R^{m-n}$, and a continuous function $\eta : \Omega \rightarrow R_+$ and continuous almost

everywhere differentiable function $s(\cdot)$ such that $s^*(\alpha) = \max(s(\alpha), s_0(\alpha))$

and the following first order conditions hold:

$$\operatorname{div} \lambda = -\frac{\partial H}{\partial s} \text{ a.e. on } \Omega. \quad (45)$$

$$\langle \lambda, \nu \rangle = 0 \text{ a.e. } \partial\Omega. \quad (46)$$

$$\eta \geq 0 \quad (47)$$

$$\eta(s(\alpha) - s_0(t)) = 0 \quad (48)$$

$$z \in \arg \max H(s, z, \alpha; \lambda, \mu). \quad (49)$$

Proof. Constraints (30) can be written as

$$As = \frac{\partial v}{\partial \alpha_i}(\alpha, x), \quad (50)$$

where $A = \nabla$ is the gradient operator. To find its adjoint take arbitrary

continuously differentiable function $\xi : \Omega \rightarrow \Omega$ satisfying

$$\langle \xi, \nu \rangle = 0 \quad (51)$$

and any continuously differentiable $\zeta : \Omega \rightarrow R^m$ and consider

$$\sum_{i=1}^m \int_{\Omega} \xi_i(\alpha) \frac{\partial \zeta}{\partial \alpha_i} d\alpha. \quad (52)$$

Note that

$$\text{div}(\zeta \xi) = \xi \nabla \zeta + \zeta \text{div} \xi. \quad (53)$$

Using the divergence theorem (54) can be transformed to

$$\int_{\Omega} \langle \xi_i(\alpha), \nabla \zeta \rangle d\alpha = \oint_{\partial \Omega} \zeta \langle \xi, \nu \rangle d\alpha - \int_{\Omega} \zeta \text{div} \xi d\alpha. \quad (54)$$

Due to (51), the first term on the right hand side of (55) vanishes and we obtain $\nabla^* = -\text{div}$. Now Theorem 4 follows from Theorem 3.

Q.E.D.

Consider a particular case, when $n > m$ and the utility is linear in types,

namely

$$v(\alpha, x) = \sum_{i=1}^m \alpha_i v_i(x).$$

Assume that functions $v_i(\cdot)$ are twice continuously differentiable and strictly quasiconcave, the Jacoby matrix $Dv(x)$ has a full rank for all $x \in R_+^n$, $v_i(0) = 0$, and $\lim_{x_k \rightarrow \infty} v_i(x) = \infty$. More specifically, assume

$$\det \frac{\partial [v_1, \dots, v_n]}{\partial [x_1, \dots, x_n]} \neq 0$$

for $\forall x \in R_+^n$. Let the cost function be increasing, convex, twice continuously differentiable, and the eigenvalues of the matrix of its second derivatives are uniformly bounded from above and are uniformly bounded away from zero from below.

First, consider the problem:

$$\min c(x) \tag{55}$$

$$s.t. v_i(x) \geq v_i, \quad i = 1, \dots, n. \tag{56}$$

Given the assumptions, the solution exists and is unique. Denote this

solution by $x^*(v)$ and define $\theta(v) = c(x^*(v))$. Then $\theta(v)$ will be convex, twice continuously differentiable, and the eigenvalues of the matrix of its second derivatives will be uniformly bounded from above and uniformly bounded away from zero from below. Now, the monopolist's problem (25)-(31) can be rewritten in the form:

$$\max_s \int [\sum_{i=1}^m \alpha_i v_i - \theta(v) - s(\alpha)] f(\alpha) d\alpha \quad (57)$$

$$s.t.. \frac{\partial s}{\partial \alpha_i}(\alpha) = \frac{\partial v}{\partial \alpha_i}(\alpha, z), \quad i = \overline{1, m}, \quad s \geq s_0(\alpha). \quad (58)$$

It is straightforward to check that the first order conditions to problem (57)-(60) are equivalent to (48)-(51). Characterization (57)-(60) for the linear case was found earlier by Basov (2001).

3.3 SOME PROPERTIES OF THE SOLUTION WHEN THE IMPLEMENTABILITY CONSTRAINT DOES NOT BIND

Assume that the implementability constraint does not bind, that is the solution to the relaxed problem found in the previous subsection is imple-

mentable. In this case we can establish some interesting properties of the solution. Let Γ be the set $\{\alpha \in \Omega : s(\alpha) > s_0(\alpha)\}$ and $\Omega = \Omega_0 \cup \Gamma$. Since

$$-\frac{\partial H}{\partial s} = f(\alpha) - \eta(\alpha)$$

equations (45), (46) and the divergence theorem imply

$$\int_{\Omega} \eta(\alpha) d\alpha = 1.$$

Condition (47) implies that $\eta(\cdot)$ is a probability density function, call the probability measure induced by this density μ . The complimentary slackness condition (48) implies that the support of μ is Ω_0 . Moreover, μ is absolutely continuous with respect to the Lebesgue measure on Ω_0 and $\mu(\Omega_0) = 1$. This result generalizes a similar finding by Rochet and Chone [1998]. In the linear context Rochet and Chone were able to show that this result still holds for the solution of the complete problem. Whether this is still true in the nonlinear case is unknown. Another interesting result is that if $m > 1$ is odd, there exist at least two types on the boundary of the participation region that are served efficiently. The result is summarized in the following lemma.

Lemma 1 *Assume that Ω is convex, $v(\alpha, x)$ is convex in types, m is odd, and $s_0(\alpha) = 0$. Moreover, assume that the solution to the relaxed problem is implementable. Then there are at least two types on the boundary of the participation region that are served efficiently under the optimal mechanism.*

Proof. First note that it is sufficient to prove that there are at least two points $\alpha_1, \alpha_2 \in \partial\bar{\Gamma}$ such that $\lambda(\alpha_1) = \lambda(\alpha_2) = 0$. Indeed, at these points the Pontryagin maximum principle (49) implies that the social surplus is maximized. To show that such points exist, note that our assumptions on Ω imply that $\partial\Omega$ is topologically equivalent to a sphere of dimension $(m - 1)$. Our assumptions on utility imply that $s(\alpha)$ is convex, hence the exclusion region is convex. Define mapping $\varphi : \partial\Omega \rightarrow \partial\bar{\Gamma}$ as follows. Let $\gamma = \partial\Omega \cap \partial\bar{\Gamma}$, select $O \notin \Omega$ to be the vertex of the cone with base γ and draw a straight line through O and a point $X \in \partial\Omega$. Assume, $X \notin \gamma$. Since Ω is convex, there exists unique $Y \neq X$ such that $Y \in \partial\Omega \cap (OX)$. Since, Ω/Γ is convex, there exists unique $Z \neq Y$ such that $Z \in \partial\Gamma \cap (OX)$. Define $\varphi(X) = X$ if either $X \in \gamma$ or Z lies between X and Y . Otherwise, define $\varphi(X) = Z$. It is straightforward to check that φ is a homeomorphism. Therefore, $\partial\bar{\Gamma}$ will be topologically equivalent to a sphere of dimension $(m - 1)$. If m is odd, $m - 1$

is even and the Euler characteristic of the sphere will be equal to 2. Hence, there exist at least two points on $\partial\bar{\Gamma}$ at which a continuous vector field λ will become zero, and the corresponding types will be served efficiently.

Q. E. D.

4 APPLICATION OF THE DEVELOPED TECHNIQUE

In this Section I am going to apply the developed technique to solve a particular example. I will consider a market for a good with one quality characteristic with two-dimensional private information. The example is similar to Laffont, Maskin, and Rochet [1987], where both the demand slope and demand intercept were private information. Unlike the latter case, however, in my example the market collapses completely, that is nobody is served in equilibrium.

Example 4. Assume that consumers are interested in buying a single unit of a good. If a consumer of type $\alpha \in (0, 1) \times (0, 1)$ pays t for a good of quality

x she obtains utility

$$u(\alpha, x, t) = \alpha_1(x - \alpha_2)^2 - t. \quad (59)$$

Assume that the types are distributed uniformly on the unit square and the monopolist's cost of production is

$$c(x) = \frac{x^2}{2}. \quad (60)$$

Note that under perfect information all types with $\alpha_1 > 1/2$ should be served and obtain an infinite quality. That is potential gains to trade are infinite. However, as we will see, due to the private information the market fails completely.

Introducing $z_1 = (x - \alpha_2)^2$ and $z_2 = x$ the monopolist's problem can be written as

$$\max \int_0^1 (\alpha_1 z_1 - s - \frac{z_2^2}{2}) d\alpha. \quad (61)$$

$$s.t. s_1 = z_1, s_2 = 2\alpha_1(\alpha_2 - z_2), z_1 - (z_2 - \alpha_2)^2 = 0, s \geq 0. \quad (62)$$

Here I assume that the outside option is the same for all consumers and is

normalized to be zero. The Hamiltonian has a form

$$H(z, s; \lambda, \mu) = \alpha_1 z_1 - s - \frac{z_2^2}{2} + \lambda_1 z_1 + 2\lambda_2 \alpha_1 (\alpha_2 - z_2) + \mu(z_1 - (z_2 - \alpha_2)^2). \quad (63)$$

Conditions (48)-(51) are reduced to the boundary value problem for the system of partial differential equations

$$-\alpha_1 \frac{\partial \mu}{\partial \alpha_1} + \frac{s_2}{2\alpha_1} \frac{\partial \mu}{\partial \alpha_2} = 2\alpha_1 - \mu - \left(\frac{1}{2} + \mu\right) \left(1 - \frac{s_{22}}{2\alpha_1}\right) \quad (64)$$

$$4\alpha_1 s_1 - s_2^2 = 0 \quad (65)$$

$$\mu(1, \alpha_2) = -1. \quad (66)$$

The solution to (66)-(68) is given by

$$s(\alpha) = \alpha_1 \alpha_2^2 \quad (67)$$

$$\mu(\alpha) = -\alpha_1 - 2 \ln \alpha_1. \quad (68)$$

This implies that

$$x(\alpha) = z_2(\alpha) = \alpha_2 - \frac{s_2}{2\alpha_1} = 0. \quad (69)$$

Define tariff $t(\cdot)$ by

$$t(x) = \max_{\alpha \in [0, 1] \times [0, 1]} (\alpha_1(x - \alpha_2)^2 - \alpha_1\alpha_2^2) = x^2. \quad (70)$$

It is straightforward to check that tariff (72) implements allocation (71) (in doing so one should take into account the non-negativity constraint $x \geq 0$).

Hence, the solution to the relaxed problem is also a solution for the original problem. Nobody is served in equilibrium and the market fails completely.

Note also that this tariff is not unique. For example, $t(x) = 2x^2$ will do the job. As we saw in Section 2 the tariff is determined uniquely only on set X , which in this case is $\{0\}$.

Example 5. Let the individual's utility be given by:

$$u(\alpha, x, t) = \alpha_1 x_1 + \alpha_2 x_2 + \sqrt{\alpha_1 \alpha_2} x_3 - t$$

and the cost of production is

$$c(x) = \frac{1}{2}(x_1^2 + x_2^2 + \beta x_3^2).$$

The set $\Omega = \{\alpha \in R_+^2 : \alpha_1 + \alpha_2 < b\}$. Distribution of types is given by

$$f(\alpha_1, \alpha_2) = \frac{\exp(-\alpha_1 - \alpha_2)}{1 - (b + 1) \exp(-b)}.$$

The value of the outside option is type independent and normalized to be zero.

I will show that in the case $\beta = 1/2$ the optimal tariff is cost based. In this particular case the solution can be found in three different ways: reducing the problem to the one-dimensional one, using integration by rays technique (Armstrong, 1996) and using the Hamiltonian approach. We will see that the solution to the relaxed problem is implementable in this case. Of course, the results of all three approaches agree. If $\beta \neq 1/2$ the optimal tariff is no longer cost based, hence the problem is no longer reducible to a one-dimensional one. Moreover, the candidate solution obtained using integration by rays technique is no longer implementable. One can still write the first order conditions for the Hamiltonian approach. They can be reduced to a von-Neumann boundary value problem for an elliptic partial differential equation on consumer surplus. Moreover, for β sufficiently close $1/2$ the solution to

this problem is implementable. However, it cannot be found analytically in a closed form.

To substantiate the claims made in the previous paragraph, let us first define

$$\begin{aligned} v(\alpha, y) &= \max_{x \in R_+^3} (\alpha_1 x_1 + \alpha_2 x_2 + \sqrt{\alpha_1 \alpha_2} x_3) \\ \text{s.t. } \frac{1}{2}(x_1^2 + x_2^2 + \beta x_3^2) &= y. \end{aligned}$$

The solution to this problem is

$$\begin{aligned} x_1 &= \frac{\alpha_1 \sqrt{2y}}{\alpha_1 + \alpha_2} \\ x_2 &= \frac{\alpha_2 \sqrt{2y}}{\alpha_1 + \alpha_2} \\ x_3 &= x_1 = \frac{2\sqrt{\alpha_1 \alpha_2} \sqrt{2y}}{\alpha_1 + \alpha_2} \\ v(\alpha, y) &= (\alpha_1 + \alpha_2) \sqrt{2y}. \end{aligned}$$

where

$$\frac{1}{\lambda(\alpha, y)} = \frac{\sqrt{2y}}{\sqrt{\alpha_1^2 + \alpha_2^2 + \frac{\alpha_1 \alpha_2}{\beta}}}.$$

Following Armstrong (1996) we conclude that the optimal tariff is cost based if and only if $u(\alpha, y) = v(y)\varphi(\alpha_1 + \alpha_2)$ which happens for $\beta = 1/2$. Let us

concentrate on case $\beta = 1/2$ for a while.

1. *Reduction of the Problem to a One-Dimensional One Using Cost-Based Tariffs.*

The optimal tariff is cost based and for a given tariff $t(y)$ the consumer surplus is

$$s(\alpha) = \max((\alpha_1 + \alpha_2)\sqrt{2y} - t(y)).$$

It is easy to see that it depends only on $(\alpha_1 + \alpha_2)$ and the monopolist's problem becomes

$$\max \iint_{\Omega} ((\alpha_1 + \alpha_2)\sqrt{2y} - s(\alpha_1 + \alpha_2) - y) \exp(-\alpha_1 - \alpha_2) d\alpha_1 d\alpha_2,$$

where integration is over the triangle $\Omega = \{\alpha \in R_+^2 : \alpha_1 + \alpha_2 < 1\}$. Make a change of variables

$$\begin{cases} \alpha_1 = \frac{1}{2}(\gamma_1 + \gamma_2) \\ \alpha_2 = \frac{1}{2}(\gamma_1 - \gamma_2) \end{cases}.$$

Then the monopolist's objective is transformed into

$$\frac{1}{2} \int_0^1 \left[\int_{-\gamma_1}^{\gamma_1} d\gamma_2 (\gamma_1 \sqrt{2y} - s(\gamma_1) - y) \right] \exp(-\gamma_1) d\gamma_1 = \int_0^1 (\gamma \sqrt{2y} - s(\gamma) - y) \gamma \exp(-\gamma) d\gamma,$$

where I dropped subscript 1 to simplify notation. Using standard integration by parts technique (Mussa and Rosen, 1978) it can be shown that the monopolist should solve the following pointwise maximization problem:

$$\max_y \left(\sqrt{2y} \left(\gamma - \frac{1 - F(\gamma)}{h(\gamma)} \right) - y \right),$$

where

$$h(\gamma) = \frac{\gamma \exp(-\gamma)}{1 - (b+1) \exp(-b)}$$

and $H(\gamma)$ is the corresponding cumulative distribution function. The solution is

$$\sqrt{2y} = \gamma - \frac{\gamma + 1}{\gamma} + \frac{b + 1}{\gamma} \exp(\gamma - b).$$

One can check that the function is increasing. Hence, the allocation is implementable. Returning to the original variables

$$\begin{aligned} x_1 &= \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right) \\ x_2 &= \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right) \\ x_3 &= 2\sqrt{\alpha_1 \alpha_2} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right). \end{aligned}$$

The exclusion region is given by

$$\Omega_0 = \{\alpha \in R_+^2 : 1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \leq 0\}.$$

It is easy to see that the exclusion region is non-empty, since for small values of $\alpha_1 + \alpha_2$ the first term can be neglected, and the second negative term dominates the third positive term, since $e^b > b + 1$ for any $b \in R$. It is also interesting to note that all types on the boundary $\alpha_1 + \alpha_2 = b$ are served efficiently. As $b \rightarrow \infty$ the solution becomes

$$\begin{aligned} x_1 &= \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_2 &= \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_3 &= 2\sqrt{\alpha_1 \alpha_2} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right). \end{aligned}$$

The exclusion region in this case is

$$\Omega_0 = \{\alpha \in R_+^2 : \alpha_1 + \alpha_2 \leq \frac{1 + \sqrt{5}}{2}\}.$$

2. *Integration Along Rays*

Note that utility is homogeneous of degree one in types and assume that $b = \infty$, so $\Omega = R_+^2$. Then we can, following Armstrong's integration along rays approach, define a candidate solution as a solution to a pointwise maximization of the following expression

$$\left(1 - \frac{g(\alpha)}{f(\alpha)}\right)u(\alpha, x) - c(x),$$

where

$$g(\alpha) = \int_1^{\infty} t f(t\alpha) dt.$$

The first order conditions are

$$x_1 = \alpha_1 \left(1 - \frac{g(\alpha)}{f(\alpha)}\right) \tag{71}$$

$$x_2 = \alpha_2 \left(1 - \frac{g(\alpha)}{f(\alpha)}\right) \tag{72}$$

$$x_3 = 2\sqrt{\alpha_1 \alpha_2} \left(1 - \frac{g(\alpha)}{f(\alpha)}\right). \tag{73}$$

Evaluating $g(\cdot)$ as

$$g(\alpha) = \int_1^{\infty} t \exp(-t(\alpha_1 + \alpha_2)) dt = \frac{\exp(-\alpha_1 - \alpha_2)}{(\alpha_1 + \alpha_2)^2} (\alpha_1 + \alpha_2 + 1).$$

we obtain

$$\frac{g(\alpha)}{f(\alpha)} = \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}.$$

Plugging it into (71)-(73) one obtains

$$\begin{aligned} x_1 &= \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_2 &= \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \\ x_3 &= 2\sqrt{\alpha_1\alpha_2} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right). \end{aligned}$$

This coincides with the solution we obtained by the previous method in the limit $b \rightarrow \infty$. Since we proved that previous method is implementable, this is a solution to our problem. Note that using this procedure for $\beta \neq 1/2$

$$x_1 = \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \tag{74}$$

$$x_2 = \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \tag{75}$$

$$x_3 = \frac{\sqrt{\alpha_1\alpha_2}}{\beta} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right). \tag{76}$$

and calculating $\nabla_{\alpha} v(\alpha, x(\alpha))$

$$\frac{\partial u}{\partial \alpha_1}(\alpha, x(\alpha)) = \left(\alpha_1 + \frac{\alpha_2}{2\beta}\right) \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right) \quad (77)$$

$$\frac{\partial u}{\partial \alpha_2}(\alpha, x(\alpha)) = \left(\alpha_2 + \frac{\alpha_1}{2\beta}\right) \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2}\right). \quad (78)$$

It is easy to check that right hand sides of (77)-(78) cannot be components of the gradient of any function. Hence, allocation (74)-(76) is not implementable.

3. *Hamiltonian Approach*

In this approach I will try to go as far as possible assuming an arbitrary value of β . The envelope conditions for the consumer surplus are

$$\begin{aligned} s_1 &= x_1 + \frac{1}{2} \sqrt{\frac{\alpha_2}{\alpha_1}} x_3 \\ s_2 &= x_2 + \frac{1}{2} \sqrt{\frac{\alpha_1}{\alpha_2}} x_3 \end{aligned}$$

Form a Hamiltonian

$$\begin{aligned}
H(\alpha, s, x; \lambda) = & [\alpha_1 x_1 + \alpha_2 x_2 + \sqrt{\alpha_1 \alpha_2} x_3 - \frac{1}{2}(x_1^2 + x_2^2 + \frac{\beta}{2} x_3^2)] \exp(-\alpha_1 - \alpha_2) - \\
& s \exp(-\alpha_1 - \alpha_2) + \lambda_1(x_1 + \frac{1}{2}\sqrt{\frac{\alpha_2}{\alpha_1}} x_3) + \lambda_2(x_2 + \frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}} x_3). \quad (79)
\end{aligned}$$

The first order conditions have the form

$$\lambda_1 = (x_1 - \alpha_1) \exp(-\alpha_1 - \alpha_2) \quad (80)$$

$$\lambda_2 = (x_2 - \alpha_2) \exp(-\alpha_1 - \alpha_2) \quad (81)$$

$$x_3 = \frac{1}{\beta}(\sqrt{\alpha_1 \alpha_2} + \frac{1}{2}(\lambda_1 \sqrt{\frac{\alpha_2}{\alpha_1}} + \lambda_2 \sqrt{\frac{\alpha_1}{\alpha_2}})) \exp(\alpha_1 + \alpha_2) \quad (82)$$

$$\text{div} \lambda = \exp(-\alpha_1 - \alpha_2) - \eta \quad (83)$$

$$\lambda_1 + \lambda_2 = 0 \text{ at } \alpha_1 + \alpha_2 = b \quad (84)$$

$$\lambda_1 = 0 \text{ at } \alpha_1 = 0 \quad (85)$$

$$\lambda_2 = 0 \text{ at } \alpha_2 = 0. \quad (86)$$

Note, that since the Hamiltonian is concave in x first order conditions are necessary and sufficient for maximum. Hence conditions (80)-(82) capture the Pontryagin maximum principle. Equation (83) is the evolution equation for the costate variable. To understand the boundary conditions note that

the boundary consists of three segments: a segment $\alpha_1 + \alpha_2 = b$ with normal vector $n = (1, 1)$, a segment $\alpha_1 = 0$ with a normal vector $(1, 0)$, and a segment $\alpha_2 = 0$ with a normal vector $(0, 1)$. Equations (84)-(86) are therefore the transversality conditions.

Within the participation region $\eta = 0$, let us look for a candidate solution in a form:

$$\lambda_i = \alpha_i \exp(-\alpha_1 - \alpha_2) \varphi(\alpha_1 + \alpha_2),$$

where φ is some continuously differentiable function. Denote $z = \alpha_1 + \alpha_2$.

Then

$$\text{div} \lambda = \exp(-\alpha_1 - \alpha_2) \Rightarrow z\varphi' + (2 - z)\varphi = 1.$$

The condition on the boundary $\alpha_1 + \alpha_2 = b$ implies $\varphi(b) = 0$, while two other boundary conditions are always satisfied. It is straightforward to check that

$$\varphi = -\frac{1}{z} - \frac{1}{z^2}$$

is a particular solution of this equation. The general solution is a sum of a

particular solution and the general solution of the uniform equation

$$z\varphi' + (2 - z)\varphi = 0.$$

To find the general solution to the uniform equation write it in the form

$$\frac{d\varphi}{\varphi} = \frac{z - 2}{z} dz \Leftrightarrow \varphi'(z) = \frac{d\varphi}{dz} = \frac{z - 2}{z} \varphi.$$

Now, integrating,

$$\ln \varphi - \ln C = z - 2 \ln z$$

or

$$\varphi = C \frac{\exp z}{z^2}.$$

Hence, the general solution to the original problem is given by

$$\varphi(z) = -\frac{1}{z} - \frac{1}{z^2} + C \frac{\exp z}{z^2}.$$

The boundary condition $\varphi(b) = 0$ implies

$$C = (b + 1)e^{-b}.$$

Hence

$$\lambda_i = -\alpha_i \exp(-\alpha_1 - \alpha_2) \left(\frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} - \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right).$$

Now the first two conditions imply that

$$\begin{aligned} x_1 &= \alpha_1 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right) \\ x_2 &= \alpha_2 \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right), \end{aligned}$$

while the third equation implies

$$x_3 = \frac{\sqrt{\alpha_1 \alpha_2}}{\beta} \left(1 - \frac{\alpha_1 + \alpha_2 + 1}{(\alpha_1 + \alpha_2)^2} + \frac{b + 1}{(\alpha_1 + \alpha_2)^2} \exp(\alpha_1 + \alpha_2 - b) \right).$$

But this is exactly allocation (74)-(76), which is implementable if and only if $\beta = 1/2$.

If $\beta \neq 1/2$ one can solve for x_1, x_2 in terms of the components of the surplus gradient s_1, s_2 to get

$$x_1(\alpha, \nabla s) = \frac{\alpha_1 [(4\beta\alpha_2 + \alpha_1)s_1 - \alpha_2 s_2]}{\alpha_1^2 + \alpha_2^2 + 4\beta\alpha_1\alpha_2} \quad (87)$$

$$x_2(\alpha, \nabla s) = \frac{\alpha_2 [(4\beta\alpha_1 + \alpha_2)s_2 - \alpha_1 s_1]}{\alpha_1^2 + \alpha_2^2 + 4\beta\alpha_1\alpha_2}. \quad (88)$$

Then plug (87)-(88) into (80)-(81) to express components of λ through the components of surplus gradient. Note that conditions (85)-(86) hold automatically and plugging this expression for λ into (83)-(84) results in

$$\sum_{i=1}^2 \left(\frac{\partial(x_i(\alpha, \nabla s) - \alpha_i)}{\partial \alpha_i} - x_i(\alpha, \nabla s) + \alpha_i \right) = 1 \quad (89)$$

$$\sum_{i=1}^2 x_i(\alpha, \nabla s) = b \text{ for } \alpha_1 + \alpha_2 = b. \quad (90)$$

This system (89)-(90) holds in the participation region. This is the von-Neumann boundary value problem for an elliptic partial differential equation for the surplus. The solution to this problem is unique up to an additive constant, where the constant will determine the participation region. For $\beta = \frac{1}{2}$ the solution is

$$s(\alpha_1, \alpha_2) = \varphi(\alpha_1 + \alpha_2), \quad (91)$$

where

$$\varphi(z) = \frac{z^2}{2} - z - \ln z + \int_1^z \frac{\exp(t-b)}{t} dt + C \quad (92)$$

where C is found from a system

$$C + \varphi(z^*) = 0 \tag{93}$$

$$\varphi'(z^*) = 0. \tag{94}$$

Using (87)-(88) one can find the optimal allocation and show that it is the same as one derived using two other approaches. Both other approaches break, however, for $\beta \neq 1/2$. The Hamiltonian approach, on the other hand, is still applicable. In general, system (89)-(90) should be solved numerically. For β sufficiently close to $1/2$ its solution is implementable.

5 DISCUSSION AND CONCLUSIONS

In this paper I applied the Hamiltonian method to solve the relaxed problem for the multidimensional screening problem. The technique was previously developed for the case of the utilities linear in types (Basov, 2001). Together with the implementability results developed by Carlier [2002] this technique sometimes allows us to arrive at a complete solution of a screening problem. In the linear case the Hamiltonian approach can be thought of as a two-step procedure, first introduce new quality dimensions, utils, which

are produced using physical quality dimensions. The number of utils always equals the dimensionality of the type vector. If dimension of type is bigger than the number of physical quality characteristics, then this leads to restrictions on the utils production. Similar intuition is useful in the nonlinear case. Utils in this case are, however, type specific.

The technique developed in this paper was used to solve some examples. This paper leaves open the question: What is the solution to the screening problem if the solution to the relaxed problem is not implementable? The solution to this problem is known only in the linear case and in the case $m = n = 1$. For a discussion, see Rochet and Chone [1998]. I hypothesize that in the general case it should be a monotone rearrangement of the solution to the relaxed problem with respect to an appropriately chosen measure.

APPENDIX

Proof of Theorem 3: Let $z' \in R^n$ be a vector defined by

$$z'_i = z_{m-n+i} \tag{A.1}$$

if $m > n$ and $z'_i = z_i$ otherwise. Define a normed space:

$$H^1(\Omega) = \{\phi : \phi \in L^2(\Omega), \nabla\phi \in L^2(\Omega)\} \quad (\text{A.2})$$

$$|\phi|_{H^1} = \int_{\Omega} (\phi^2 + \|\nabla\phi\|^2) d\alpha. \quad (\text{A.3})$$

Define a functional π by a formula:

$$\pi(s) = \int_{\Omega} [v(a, z) - c(z) - s(\alpha)] f(\alpha) d\alpha. \quad (\text{A.4})$$

Let

$$\begin{aligned} K &= \{s \in H^1(\Omega) : \frac{\partial s}{\partial \alpha_i}(\alpha) = \frac{\partial v}{\partial \alpha_i}(\alpha, z'), \ i = \overline{1, m}, s(\alpha) \geq s_0(\alpha), \\ z_i &\leq \frac{\partial v}{\partial \alpha_i}(\alpha, z'), \ i = \overline{1, m-n}\}. \end{aligned} \quad (\text{A.5})$$

I will prove that there exists a unique surplus function $s^* \in K$ and allocation $z^*(\cdot)$ such that $\pi(s^*) \geq \pi(s)$ for any $s \in K$ and $z_i^* = \partial v / \partial \alpha_i(\alpha, z^*)$. Since under assumptions on functions $\partial v / \partial \alpha_i$ the set K is a convex closed set, to prove the first assertion it is sufficient to prove that the functional π is coercive on K (see, e.g., Kinderlehrer and Stampacchia [1980]), i. e. that $\pi(s)$ tends to $-\infty$ when $|s|_{H^1}$ tends to $+\infty$. For all $s \in H^1(\Omega)$, denote by \underline{s}

the mean value of s over Ω :

$$\underline{s} = \frac{1}{|\Omega|} \int_{\Omega} s(\alpha) d\alpha. \quad (\text{A.6})$$

By the Poincare inequality (see, e.g., Kinderlehrer and Stampacchia [1980]) there exists a constant $M(\Omega)$ such that for all $s \in H^1(\Omega)$, $|s - \underline{s}|_{L^2} \leq M(\Omega) |\nabla s|_{L^2}$. This implies that

$$|s|_{H^1} \rightarrow +\infty \Leftrightarrow \underline{s} \rightarrow +\infty \text{ or } |\nabla s|_{L^2} \rightarrow +\infty. \quad (\text{A.7})$$

Note that since each $v_j(\cdot)$ is concave and $v_j(0) = 0$ one obtains $z \leq \langle \nabla v(0), z' \rangle$ and hence $|\nabla s|_{L^2} \leq N(\Omega) |z'|_{L^2}$. Under the assumptions on the cost and the distribution of types:

$$\pi(s) \leq -\gamma\varepsilon |z|_{L^2}^2 + M(\Omega)N(\Omega) |z|_{L^2} - \underline{s}. \quad (\text{A.8})$$

For the details of derivation of (A.8) see Rochet and Chone [1998]. Note that since $z_j \leq v_j(z')$ for $\overline{1, m-n}$ the condition $|\nabla s|_{L^2} \rightarrow +\infty$ implies $z' \rightarrow +\infty$. Coerciveness of π then follows from (A.7) and (A.8).

Hence, I have proven that π achieves maximum for some $s^* \in K$. To see that $z_i^* = \partial v / \partial \alpha_i(\alpha, z^*)$ should hold, assume that there exists k such that $z_k < \partial v / \partial \alpha_k(\alpha, z^*)$. Consider a function $s'(\alpha) = s^*(\alpha) + \delta \alpha_i - \varepsilon$ for each α such that $s^*(\alpha) > s_0(\alpha)$ and $i = \overline{1, m - n}$. Note that $s' \in K$ for sufficiently small $\delta > 0$ and $\varepsilon > 0$. Since the cost function depends only on z' , the integrand in the definition of π increases by ε . If $0 \notin \Gamma$, one can find such values of δ and ε such that new participation region is a superset of the initial one. Otherwise, some points may drop out of the participation region, but their Lebesgue measure will be $O(\varepsilon^m)$. In any case, $\pi(s') > \pi(s^*)$ for sufficiently small δ and ε . This implies that $z_i^* = \partial v / \partial \alpha_i(\alpha, z^*)$, which completes the proof of the existence.

Proof of the uniqueness is exactly the same as in Rochet and Chone [1998].

REFERENCES

- J. A. Adams, and J. L. Yellen, 1976, Commodity Bundling and the Burden of Monopoly, *Quarterly Journal of Economics*, 90, 475-498.
- M. Armstrong, 1996, Multiproduct Nonlinear Pricing, *Econometrica* 64, 51-75.
- M. Armstrong, 1999, Price Discrimination by a Many-Product Firm, *Review of Economic Studies* 66, 151-168.
- S. Basov, 2001, Hamiltonian Approach to Multi-dimensional Screening, *Journal of Mathematical Economics*, 36, 77-94.
- G. Carlier, 2001, A General Existence Result for the Principal-Agent Problem with Adverse Selection, *Journal of Mathematical Economics*, 129-150.
- G. Carlier, 2002, Duality and Existence for a Class of Mass Transportation Problems and Economic Applications,
<http://www.sci.unich.it/~scarsini/erice/pdffiles/CarlierErice.pdf>.
- P. Funk, 1962, *Variationsrechnung und ihre Anwendung in Physik und Technik*, (Springer-Verlag, Berlin).
- D. Kinderlehrer, and G. Stampacchia, 1980, *An Introduction to Variational Inequalities*, Academic Press, Boston, MA.
- Laffont, J. J., E. Maskin, and J. C. Rochet, 1987, *Optimal Nonlinear Pricing*

with Two-Dimensional Characteristics, in: T. Groves, R. Radner, and S. Reiter, eds., *Information, Incentives, and Economic Mechanisms*, (Minneapolis: University of Minnesota Press), 256-266.

Mas-Colell, A., M. D. Whinston, and J. R. Green. *Microeconomic Theory*, Oxford University Press, 1995.

P. Milgrom, and I. Segal, 2002, Envelope Theorems for Arbitrary Choice Sets, *Econometrica*; 70, 583-601..

M. Mussa, and S. Rosen, 1978, Monopoly and Product Quality, *Journal of Economic Theory* 18, 301-317.

J. C. Rochet, 1985, The Taxation Principle and Multitime Hamilton-Jacobi Equations, *Journal of Mathematical Economics* 14, 113-128.

J. C. Rochet, 1987, A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context, *Journal of Mathematical Economics* 16, 191-200.

J. C. Rochet, and P.Chone, 1998, Ironing, Sweeping and Multidimensional Screening, *Econometrica* 66, 783-826.

I. Singer, 1997, *Abstract Convex Analysis*, 1997, (John Willey & Sons).

R. Wilson, 1993, *Non Linear Pricing*, (Oxford University Press).

E. Zeidler, 1984, *Nonlinear Functional Analysis and its Applications, III. Variational Methods and Optimization*. (Springer-Verlag, Berlin).