ISSN 0819-2642
ISBN 0 7340 2531 9

THE UNIVERSITY OF MELBOURNE

DEPARTMENT OF ECONOMICS

RESEARCH PAPER NUMBER 876

MARCH 2003

HART EFFECT AND EQUILIBRIUM IN INCOMPLETE MARKETS I

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First Version, December 2002
Second Version, January 2003
This Version, March 2003

ABSTRACT. In this paper I reconsider the problem of the existence for GEI models. It is well known that existence for this kind of models can fail since there are bad spot prices. Using tools from Algebraic Geometry I show that the set of bad spot prices is typically empty if the information tree, the number of securities and the number of tradable goods at each node satisfy a well defined inequality. Even if such inequality is not satisfied I show that the set of bad spot price can be fully characterized and described using an algorithm and I give an universal bound for the cardinality of this set. Moreover, using these results, I prove that the equilibrium always exists for all endowment profiles.

Date: December 16, 2002.
2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Keyword one, keyword two, keyword three.
Thanks for Author One.
I thank Bob Anderson, Peter Barsdley, Suran Besov, Michele Boldrin, Larry Jones, Max Pak, Chris Shannon, Max Stinchcombe, Frank Sottile, Rabee Tourkey, A. van den Nouweland for useful comments. The usual disclaimer applies.

1. Introduction

We study the classical GEI models where securities are claims to commodity bundles that are contingent on the state of nature, and are insufficient to span all state contingent claim. As Hart (1975) has shown these economies do not necessarily have equilibria, even when the utilities of agents satisfy smooth assumption as in Debreu (1970) . Hart's counterexample is based on the collapse in the span of security markets that occurs on an exceptional set of 'bad" spot prices. Many attempts to resurrect the existence of equilibria have then concentrated on showing that the existence is there, but it fails on a set of measure zero. Duffie and Shafer proved that the intuition of 'bad' or "exceptional' spot prices was correct and they gave the existence proof for an two-period economy (see [7]) and, in a later paper (see [8]), for every fully stochastic economy i.e. an economy where the information is revealed slowly and there are more then two trading dates.

So it is natural to investigate the 'bad' spot prices side of the story little bit further. This is the goal of this paper. The main conclusion I reach is rather surprising. The "bad" spot prices set is typically empty if the information tree, the set of tradable securities satisfies a certain inequality. Moreover, even in the case that set is not empty it is possible to give an upper bound for the cardinality of this sets which is independent from the specific financial structure. Using this result I am able to reprove the existence theorems and I able to show that the existence is there for any endowment profile and to show that there exists an algorithm to find all the isolated bad spot prices.

The intuition behind this result is quite simple. The collapse of the rank of matrix is equivalent to the existence of a solution set for a well defined system of polynomial equations. Since every polynomial defines an algebraic variety the problem is naturally translated in an intersection problem of algebraic varieties. Clearly if the number of varieties is big enough -i.e. the a dimensional restriction is satisfied- the intersection is typically empty. I will show that the tree-event structure, the number of securities and the number of tradable goods are all we need to know to determine the number and structure of equations.

The paper is organized as it fallows. In the Section 2 I write down the model in two periods and I fix the notation and I explain the main results. In Section 3 I prove the basic results. In Section 4 I show how to extend the result to a general stochastic economy i.e. an economy where there are at least two trading dates. In the final Section I construct an algorithm to show how to localize the bad spot prices.

2. The Security Market Model

I start my investigation analyzing a GEI model in two periods and then I will extend the result to general GEI models so, in this Section, we outline the economic setting, stating the definition of an equilibrium with incomplete markets in a two-period model with uncertainty over the states of nature in the second period. I only write down the essential feature of the model the reader should also compare to [7], [10],[16],[17].

In period 0 there are spot markets for commodities, ad security markets for assets that pay bundles of commodities in period 1, the bundle paying depending on the state of nature. In period 1 agents cash in their portfolios of assets and their endowment, trading the proceeds on spot markets for commodities. There are ℓ

commodity types, S possible states of nature in period 1 and J assets. So we have the following matrix

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S) \end{bmatrix}$$

Note that for any j, $1 \leq j \leq J$,and for any s with $1 \leq s \leq S$ we have $\mathbf{a}^j(s) \in \mathbb{R}^\ell$. An agent holding one unit of asset j after trading at period 0 is entitled to the vector $\mathbf{a}^j(s)$ of commodities in period 1 if the state s occurs. A portfolio $\theta = (\theta_1, ..., \theta_J) \in \mathbb{R}^J$ of assets thus represents a claim to the vector $\mathbf{a}(s)\theta$ where, of course, $\mathbf{a}(s)$ is the following

$$\mathbf{a}(s) = \begin{bmatrix} \mathbf{a}^1(s) & \cdots & \mathbf{a}^J(s) \end{bmatrix}$$

$\ell \times J$ matrix.

Let $x_0 \in \mathbb{R}^\ell_{++}$ denote a period 0 consumption vector of a typical agent and let $x_1(s) \in \mathbb{R}^\ell_{++}$ denote the consumption vector of an agent in period 1, at the state s. So we can write $x = (x_0,(x_1(s)_{s=1}^S) \in \mathbb{R}^{\ell\times(S+1)}_{++}$ as a consumption vector. Similarly, $p = (p_0,(p_1(s)_{s=1}^S) \in \mathbb{R}^{\ell\times(S+1)}_{++}$ represents a collection of spot market price vectors. Each agent i, $1 \leqq i \leqq I$, is characterized by an initial endowment vector $w^i \in \mathbb{R}^{\ell\times(S+1)}_{++}$ and a utility function $u_i : \mathbb{R}^{\ell\times(S+1)}_{++} \to \mathbb{R}$ satisfying

(i) u_i is C^∞,

(ii) $Du_i(x) \in \mathbb{R}^{\ell\times(S+1)}_{++}$ for all $x \in \mathbb{R}^{\ell\times(S+1)}_{++}$,

(iii) $h^T Du_i(x)h < 0$ for all $h \neq 0$ such that $hDu_i(x) = 0$,

(iv) $\left\{x \in \mathbb{R}^{\ell\times(S+1)}_{++} : u_i(x) \geqq u_i(\overline{x})\right\}$ is closed in $\mathbb{R}^{\ell\times(S+1)}_{++}$ for all $\overline{x} \in \mathbb{R}^{\ell\times(S+1)}_{++}$.

Given a spot price system $p \in \mathbb{R}^{\ell\times(S+1)}_{++}$ and a portfolio $\theta \in \mathbb{R}^J$ the market value of the portfolio in state s is $p_1(s)^T a(s)\theta$. Denote by $V(p,a)$ the $S \times J$ returns matrix whose sth row is $p_1(s)^T a(s)$, so that $V(p,a)\theta \in \mathbb{R}^S$ is the vector of dividends across the S states in period 1 generated by a portfolio θ. For any $x = (x_0,(x_1(s)_{s=1}^S) \in \mathbb{R}^{\ell\times(S+1)}_{++}$, let $p_1 \Box x_1$denote the vector $(p_1(s) \cdot x_1(s))_{s=1}^S$of units of account required ***.

Given prices $(p,q) \in \mathbb{R}^{\ell\times(S+1)}_{++} \times \mathbb{R}^J$, the agent i is therefore faced with the problem

$$\max_{(x,\theta)} \quad u_i(x) \quad \text{s.t.} \quad \begin{array}{l} p_0 \cdot (x_0 - w_0^i) + q \cdot \theta \leqq 0 \\ \\ p_1 \Box (x_1 - w_1^i) \leqq V(p,a)\theta \end{array} \qquad (P)$$

An *equilibrium* is thus a collection $((\overline{x}^i, \overline{\theta}^i), (\overline{p}, \overline{q}))$ such that

1. $(\overline{x}^i, \overline{\theta}^i)$ solves (P) given $(\overline{p}, \overline{q})$ for all i,and
2. $\sum_{i\in I} \overline{x}^i = \sum_{i\in I} \overline{w}^i$ and
3. $\sum_{i\in I} \overline{\theta}^i = 0$

It is well known that an economy of this type has not necessarily an equilibrium but it is possible (see [7] and [8]) to prove that there exists an open set $\Omega \subset \mathbb{R}^{\ell\times(S+1)\times I}_{++} \times \mathbb{R}^{\ell\times S\times J}$, with null complement, such that an equilibrium exists for any economy $(\omega, \mathbf{a})$ in Ω. So if we use the symbol dV to denote the Lebesgue measure on $\mathbb{R}_{++}^{\ell\times(S+1)\times I}\times\mathbb{R}^{\ell\times S\times J}$ the Theorem tells us that

$$dV(\mathbb{R}_{++}^{\ell\times(S+1)\times I}\times\mathbb{R}^{\ell\times S\times J}\backslash\Omega)=0$$

so, from the point of view of Lebesgue measure, we know that the set where the equilibrium does not exists is very small, actually is so small that has measure zero. The reason why we can't have existence in any situation is the fact that the span of securities can collapse. I will call this the *H-effect*, after Hart. On the other hand it would be desirable to know more about the magnitude of the H-effect. In this paper I study, with the help of algebraic geometry techniques, the structure of the H-effect points.

The basic idea behind this results is quite simple. The collapse of the rank of matrix is equivalent of the existence of a solution set for a well defined system of polynomial equations. Since every polynomial defines an algebraic variety the problem is naturally translated in an intersection problem of algebraic varieties. Clearly if the number of varieties is big enough -i.e. the a dimensional restriction is satisfied- the intersection is typically empty. The reader should keep in mind that the set of zeros of a polynomial is a rather rigid set with a well precise geometric structure. Just to name an example we can think of the intersection of two curves in the plane. If the only condition we assume is that the defining equations are C^∞ function then we can not say much about the intersection locus since for every closed set Z in the plane there exists a C^∞ function f such $f(q)=0$ if and only if $q\in Z$ (see [3] Theorem 1 pp. 47). But if the curves are polynomial of a fixed degree, let say 2, then the intersection can be described in a very precise way.

Using this guiding idea I will show that the tree-event structure, the number of securities and the number of tradable goods are all we need to know to determine the number of equations.

The surprising result is the following: the points which create the H-effect have a very precise mathematical structure, in fact are an algebraic variety and a finite set of isolated points. Moreover, these sets are typically empty sets. More precisely I can prove the following:

Theorem 1. *Given the economy described in the first section with the financial structure given by the following*

$$\mathbf{a}=\begin{bmatrix}\mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1)\\ \vdots & & \vdots\\ \vdots & & \vdots\\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S)\end{bmatrix}$$

Note that for any j, for $1\leq j\leq J$,and for any s for $1\leq s\leq S$ we have $\mathbf{a}^j(s)\in\mathbb{R}^\ell$. If

$$\begin{pmatrix} S\\ J\end{pmatrix}>\ell S$$

Then the following are true:

1. The set of 'bad' spots prices, that we denote with the symbol M_H, is disjoint union of two sets M_H^A and M_H^i where M_H^A is an algebraic manifold and M_H^i is a finite set of isolated points so we can write

$$M_H(\mathbf{a})=M_H^A(\mathbf{a})\cup M_H^D(\mathbf{a})$$

2. There exists a closed set Ω_a of measure zero such that if $\mathbf{a} \notin \Omega_a$ *then*

$$M_H^A(\mathbf{a}) \cap \mathbb{R}_{++}^{\ell S} = M_H^D(\mathbf{a}) \cap \mathbb{R}_{++}^{\ell S} = \varnothing$$

3. There exists a set $\Omega_a' \subset \Omega_a$ closed and of measure zero such that

$$M_H^A(\mathbf{a}) \cap \mathbb{R}_{++}^{\ell S} = \varnothing$$

This implies that the event that the algebraic part is empty is more frequent that the fact that the discrete part is non empty.

4 There exists a positive integer number, that we denote with symbol n such that for any financial structure $\mathbf{a}$ *if we denote the cardinality of the set $M_H^i(\mathbf{a})$ with the symbol $\#(M_H^i(\mathbf{a}))$ then the following*

$$\#(M_H^D(\mathbf{a})) \leq J^{\ell S}$$

holds. Such integer is universal in the sense that it does not depend on the the individual financial structure.

5. Moreover, there exists algorithm to determine each element of M_H^D.

Theorem 2. *There exists an open set $\Omega \subset \mathbb{R}^2$, with null complement, such that an equilibrium always exists for any endowment profile $\omega \in \mathbb{R}_+^{\ell \times (S+1) \times I}$ and for any financial structure* $\mathbf{a}$ *with $P(\mathbf{a}) \subset \Omega$.*

Moreover I am able to prove the perfect analogue of this results for a fully Stochastic economy and I will do this in Section 4.

3. Proofs of the Main results

In this section we are going to prove our main results for a two-period economy. As I said in the Introduction our main tool is going to be the study of certain systems of polynomials in several variables which are naturally linked the economy. The proofs use, in an essential way, Algebraic Geometry. In the Appendix I collect the main results that I am going to use and I will give exact references for the reader interested in this type of techniques. For the moment I only give few definitions which are necessary to understand the proof of our results. We denote with the symbol $\mathbb{C}[x_1, ...x_n]$ the set of all polynomials with complex coefficients in n variables. If $f \in \mathbb{C}[x_1, ...x_n]$ then we can write

$$f(x_1, ...x_n) = \sum_{(\alpha_1 ... \alpha_n) \in \mathbb{N}} c_{\alpha_1 ... \alpha_n} x_1^{\alpha_1} ... x_n^{\alpha_n}$$

and we call support of f the set of all $(\alpha_1 ... \alpha_n)$ such that $c_{\alpha_1 ... \alpha_n} \neq 0$. Note that we can consider the vector $(\alpha_1 ... \alpha_n)$ as an element in $\mathbb{N}^n$ so the support of a polynomial in n variables is a finite subset of $\mathbb{N}^n$. I will denote the support of the polynomial f with the symbol $S(f)$.If we consider a system of m polynomials in n equations as the following

$$\begin{cases} f_1(x_1, ..., x_n) = 0 \\ f_2(x_1, ..., x_n) = 0 \\ \\ f_m(x_1, ..., x_n) = 0 \end{cases} \qquad (\mathcal{F})$$

we can associate at this system the collection of suppports $\{S(f_1), ..., S(f_n)\} \subset \mathbb{N}^n$.

Moreover it is possible to prove the following that the volume (i.e. the Lebesgue measure) of the linear combination of non-empty convex compact set $K_1, ..., K_s$

with non-negative coefficient $\lambda_1, ..., \lambda_s$ is a homogenous polynomial of degree n with respect to $\lambda_1, ..., \lambda_s$:

$$Vol(\lambda_1 K_1 + ... + \lambda_s K_s) = \sum_{i_1}^{s} ... \sum_{i_n}^{s} Vol(K_{i_1}, ..., K_{i_n})\lambda_{i_1}...\lambda_{i_n}$$

where it is assumed that for products of the λ_i which differ in the order of the factors the coefficients have the same numerical value. The following theorem relates the Minkowski sum of sets with ordinary euclidean volume. It also important to observe that the operation is continuous in an appropriate topology.

It is well know that the reason why the equilibrium could fail to exist is the fact that the rank of the matrix $V(p, a)$ is not constant but it is a function of the spot prices for the good bundles. This has been discuss in depth in several excellent papers and books and interested the reader should read those if she wants to learn more about this problem.

Let us concentrate on the matrix $V(p, a)$. I remind to the reader that, by definition, given the spot prices $p = (p_0,(p_1(s))_{s=1}^{S}) \in \mathbb{R}_{++}^{\ell \times (S+1)}$ and the assets structures $\mathbf{a}$ we have

$$V(p, a) = \left[p_1(s) \cdot \mathbf{a}^j(s)\right]_{\substack{s=1,...,S \\ j=1,...,J}}$$

and we need to study the rank of this $S \times J$ matrix. The reader should keep in mind that we are interested in Incomplete Markets and this implies that $S \geqq J$.

Let us suppose that we specify any $n \leqq J$ different rows an the same number of different columns. The elements appearing at the intersections of these rows and columns form a square matrix of order n. The determinant of this matrix is called a *minor of order* n of the original matrix $V(p, \mathbf{a})$ and it will be denoted by the symbol

$$M(p, \mathbf{a}) = M_{j_1,...,j_k}^{i_1,...,i_k}(p, \mathbf{a})$$

where $i_1, ..., i_k$ are the numbers of the selected rows, and $j_1, ..., j_k$ are the numbers of the selected columns. Of course, we are interested to the minors of order J. In fact, the rank of A is r if A has a nonvanishing minor of order r and all its minor of order $r+1$ and higher vanish. So, in our case the matrix $V(p, \mathbf{a})$ has rank J at p if and only if there exists a minor of order J which does not vanishes at p. Since this simple fact has special role in our analysis we record it in the following

Lemma 1. *Given the economy described in the first section with the financial structure given by the following*

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S) \end{bmatrix}$$

Note that for any j, for $1 \leq j \leq J$,and for any s for $1 \leq s \leq S$ we have $\mathbf{a}^j(s) \in \mathbb{R}^\ell$. Then it is possible to construct a system of polynomial that we denote with the symbol $\mathbb{P}(\mathbf{a})$ with the following properties

1. The economy has an H-effect point if and only if the system $\mathbb{P}(\mathbf{a})$ has a strictly positive solution.

2. All the polynomials appearing in the system $\mathbb{P}(\mathbf{a})$ have degree J, the number of tradable assets, and they are homogenous. Moreover, each coefficient is a $J \times J$ minor of the matrix $\mathbf{a}$.

Proof. To start let us observe that the matrix $V(p, \mathbf{a})$ has rank J at p if and only if there exists a minor of order J which does not vanish at p. This means that for any p there exists a choice of $i_1, ..., i_J$ rows, with $1 \leqq i_1 < ... < i_J \leqq S$, such that

$$M^{i_1,...,i_J}_{1,...,J}(p, \mathbf{a}) \neq 0.$$

Then we can conclude that the matrix $V(p, \mathbf{a})$ will fail to have rank J for some prices if and only if the following system

$$\left\{ \; M^{i_1,...,i_J}_{1,...,J}(p, \mathbf{a}) = 0 \qquad \text{for all} \quad 1 \leqq i_1 < ... < i_J \leqq S \right. \qquad (\mathcal{F})$$

has a strictly positive solution.

In order to proceed we need to give a closer look at $M^{i_1,...,i_J}_{1,...,J}(p, \mathbf{a})$. By definition, we have

$$M^{i_1,...,i_J}_{1,...,J}(p, \mathbf{a}) = \text{Det} \left[\sum_{k=1}^{\ell} p_{1,\ell}(s) \cdot \mathbf{a}^j_\ell(s) \right]_{\substack{s = i_1, ..., i_J \\ j = 1, ..., J}}$$

this a homogeneous polynomial of degree J and there are involved $J \times \ell$ variables namely

$$\begin{matrix} p_{1,1}(i_1) & ... & p_{1,\ell}(i_1) \\ p_{1,1}(i_2) & ... & p_{1,\ell}(i_2) \\ \vdots & & \vdots \\ p_{1,1}(i_J) & ... & p_{1,\ell}(i_J) \end{matrix}$$

Finally, let us note that

$$M^{i_1,...,i_J}_{1,...,J}(p, \mathbf{a}) = \text{Det } P^{i_1,...,i_J} A^{i_1,...,i_J}$$

where we denote with the symbol $P^{i_1,...,i_J}$ the matrix

$$\begin{bmatrix} p_1(i_1) & \cdots & p_\ell(i_1) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_1(i_2) & \cdots & p_\ell(i_2) & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & & & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & p_1(i_J) & \cdots & p_\ell(i_J) \end{bmatrix}$$

and with the symbol $A^{i_1,...,i_J}$ the matrix

$$\begin{bmatrix} \mathbf{a}^1(i_1) & \cdots & \mathbf{a}^J(i_1) \\ \mathbf{a}^1(i_2) & & \mathbf{a}^J(i_2) \\ \vdots & & \vdots \\ \mathbf{a}^1(i_J) & \cdots & \mathbf{a}^J(i_J) \end{bmatrix}.$$

Moreover it is well-known (see Theorem at pp 93 of [24]) that if X is an $m \times n$ matrix and Y is $n \times p$ matrix then every minor of order $k \leq n$ can be expressed this way

$$M^{\alpha_1,...,\alpha_k}_{\beta_1,...,\beta_k}(XY) = \sum M^{\alpha_1,...,\alpha_k}_{\ell_1,...,\ell_k}(X) M^{\ell_1,...,\ell k}_{\beta_1,...,\beta_k}(Y)$$

where the summation is over all distinct sets of indices $\ell_1, ..., \ell_k$.

If we apply this result to our case we obtain

$$M^{i_1,\ldots,i_J}_{1,\ldots,J}(p,\mathbf{a}) = \sum M^{\ell_1,\ldots,\ell_J}_{1,\ldots,J}(A^{i_1,\ldots,i_J})p_{\ell_1}(i_1)\ldots p_{\ell_J}(i_J)$$

and we are done. □

At this point we have can prove Theorem 1

Proof. (Theorem 1)

1. Last Lemma implies that the matrix $V(p,\mathbf{a})$ will fail to have rank J if and only if the following

$$\left\{ M^{i_1,\ldots,i_J}_{1,\ldots,J}(p,\mathbf{a}) = 0 \qquad \text{for all} \quad 1 \leqq i_1 < \ldots < i_J \leqq S \right. \qquad (\mathcal{F})$$

system has a strictly positive solution. Moreover we know that the polynomial $M^{i_1,\ldots,i_J}_{1,\ldots,J}(p,\mathbf{a})$ is defined as above then we have the following

$$M^{i_1,\ldots,i_J}_{1,\ldots,J}(p,\mathbf{a}) = \sum M^{\ell_1,\ldots,\ell_J}_{1,\ldots,J}(A^{i_1,\ldots,i_J})p_{\ell_1}(i_1)\ldots p_{\ell_J}(i_J)$$

holds. Now that we know how each equation in the system look like then we ask ourself another simple question. Namely we should count how many equations are involved in the system. The answer is clearly

$$\binom{S}{J} = \frac{S!}{J!(S-J)!}$$

So we have the following system

$$\left\{ M^{i_1,\ldots,i_J}_{1,\ldots,J}(p,\mathbf{a}) = 0 \qquad \text{for all} \quad 1 \leqq i_1 < \ldots < i_J \leqq S \right. \qquad (\mathcal{F})$$

with $\binom{S}{J}$ equations. The reader should notice that since we assume that the rank of $\mathbf{a}$ is J then the system is not identically zero. Moreover, since we assume that

$$\binom{S}{J} > \ell S$$

if we select ℓS different equations

$$\left\{ M^{i_1^{(k)},\ldots,i_J^{(k)}}_{1,\ldots,J}(p,\mathbf{a}) = 0 \right\}_{k=1,\ldots,\ell S}$$

then, for each choice, we can construct the following

$$\left\{ M^{i_1^{(k)},\ldots,i_J^{(k)}}_{1,\ldots,J} = 0 \qquad \text{for all} \quad 1 \leqq k \leqq \ell S \right. \qquad (\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}})$$

and if we denote solution set with the symbol the $Z(\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}})$ in $(\mathbb{R}^{2*})^n$ we have.

$$Z(\mathcal{F}) = \bigcap Z(\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}})$$

It is a well-known fact in Algebraic Geometry (see [4] and [19]) that the a system of the type above has a solution set of the following form

$$\begin{aligned} Z(\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}}) &= M(Z(\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}}))^A \cup \\ & M(Z(\mathcal{F}_{\{i_1^{(k)},\ldots,i_J^{(k)}\}_{k=1,\ldots,\ell S}}))^D \end{aligned}$$

where

$$M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^A$$

is a algebraic variety and

$$M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D$$

is a finite set. Finally, let us observe that since we assume

$$\binom{S}{J} > \ell S$$

then we can construct at least n system of this type where

$$n \geq \binom{S}{J}$$

Since intersection of varieties is a variety and intersections of finite sets is, at most, a finite set we have completed the proof of the first claim.

2. We have just proved that

$$\begin{aligned} Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}) &= M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^A \cup \\ & M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D \end{aligned}$$

since, by Bernstein's Theorem, there exists a set of measure zero in $\mathbb{C} = \mathbb{R}^2$ such that if the coefficients of one the system are not there then the solution set of that system is just a set of finite points we can conclude that typically

$$M_H^A(\mathbf{a}) = \bigcap M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^A = \varnothing$$

so, a fortiori,

$$M_H^A(\mathbf{a}) = \bigcap M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^A \cap \mathbb{R}_{++}^{\ell S} = \varnothing$$

For the discrete component we have that it is true in general that intersection of finite sets in generic position is empty if each set is just a finite set so we have

$$\bigcap M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D \cap \mathbb{R}_{++}^{\ell S} = \varnothing$$

and this complete the proof of the second claim.

3. By Bernstein Theorem ([4]) there exists a set of measure zero such that if the coefficients of the polynomials are not there then solution set has not manifold component and clearly this implies that

$$Z(\mathcal{F}) = \bigcap M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D$$

Each of the set that appears in the right part of the equation is a finite set so we have proved that typically the solution set of the system is just a problem of intersection of sets of finite points. In this sense the algebraic part of the solution appears less frequently then the discrete part.

4 . To show the claim we can just use the classical Bezout's Theorem (see [4] and [9]). This Theorem implies that

$$\#(M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D) \leq J^{\ell S}$$

in fact it is to check that each $M_{1,\ldots,J}^{i_1,\ldots,i_J}(p,\mathbf{a})$ is just an homogenous polynomial of degree J and since

$$M_H^A(\mathbf{a}) = \bigcap M(Z(\mathcal{F}_{\left\{i_1^{(k)},\ldots,i_J^{(k)}\right\}_{k=1,\ldots,\ell S}}))^D$$

we can conclude that $\#\left(M_H^A(\mathbf{a})\right) \leq J^{\ell S}$. The reader should observe that we have been using properties that are true in general for any system of polynomials of the given form so the bound is independent from any particular financial structure.

We also notice that the bound is not very sharp. In fact it is possible to give a sharper bound, this will be done for the proof of the existence of the algorithm.

5. I am working on this. □

From the last Theorem it is clear that we can associate to any financial structure the set of coefficients of the polynomials which are construct in the system , we denote such a set of points with the symbol $P(\mathbf{a}) \subset \mathbb{C}$. To summarize what I did I will consider the following

$$\begin{array}{ccccccc}
 & & I_{C_J^S}(\mathbb{C}^{\ell S}) & \stackrel{\Lambda_{\mathbb{C}}}{\to} & \prod^{n(C_J^S,\ell S)} I_{\ell S}(\mathbb{C}^{\ell S}) & \stackrel{\Psi_{\mathbb{C}}}{\to} & \prod^{S} \mathbb{C}^* \\
 & \stackrel{\Phi_{\mathbb{C}}}{\nearrow} & \cup & & \cup & & \cup \\
\mathbb{R}^{\ell\times S\times J} & \stackrel{\Phi}{\to} & I_{C_J^S}(\mathbb{R}^{\ell S}) & \stackrel{\Lambda}{\to} & \prod^{n(C_J^S,\ell S)} I_{\ell S}(\mathbb{R}^{\ell S}) & \stackrel{\Psi}{\to} & \prod^{S} \Delta_\ell
\end{array}$$

Where:

(1) $\mathbb{R}^{\ell\times S\times J}$ is the space of assets
(2) The symbol C_J^S is

$$C_J^S = \binom{S}{J}$$

(3) $I_{C_J^S}(\mathbb{R}^{\ell S})$ (resp. $I_{C_J^S}(\mathbb{C}^{\ell S})$) is the set on systems of polynomials in ℓS real (resp. complex) variables and C_J^S equations
(4) $I_{\ell S}(\mathbb{R}^{\ell S})$ (resp. $I_{\ell S}(\mathbb{R}^{\ell S})$) is the set on systems of polynomials in ℓS real (resp. complex) variables and ℓS equations
(5) Δ_ℓ is just the standard simplex
(6) Φ is the map that associate to the financial structure the polynomial system
(7) Λ is the map that singles out the subsystem used to apply Bernstein's Theorem
(8) Ψ is the map that take the intersection sets and gives us the set of points of the H-effect.

Note that the if we call H the correspondence that associate to a financial structure $\mathbf{a}$ the set of points which generates the hart effect then, given the above definitions, we can write $H = \Psi \circ \Lambda \circ \Phi$. The Bernstein Theorem gives us a sharp estimate for the cardinality of $\Psi_{\mathbb{C}} \circ \Lambda_{\mathbb{C}} \circ \Phi_{\mathbb{C}}(\mathbf{a})$ and the fact that real numbers are inside the complex field gives also an upper bound on the set $H(\mathbf{a}) \subset \Psi_{\mathbb{C}} \circ \Lambda_{\mathbb{C}} \circ \Phi_{\mathbb{C}}(\mathbf{a})$.

The main idea of the theorem is to use the fact that the solution sets are perfectly controlled since the equations are polynomials which are extremely well-behaved maps and to use the fact that if the information set is reach enough then the number of condition that need to be satisfied to allow a collapse or the presence of the Hart-effect is way too high.

Last Theorem can be refined in order to remove genericity part of the statement. If we are dealing with a concrete example of Incomplete Markets structure it could be important to be able to establish if the set of bad spot prices is finite or has an manifold component. In fact, if it has only a discrete part - and usually this should be the case even if the set of bad spot prices is not empty -it could be that the system has an equilibrium. To address this kind of necessity we need to introduce few new definitions.

To start we will use the symbol $\mathbb{R}^{n*}$ to denote the dual of $\mathbb{R}^n$ i.e. when we think of any element of $\mathbb{R}^n$ as a linear function defined on $\mathbb{R}^n$.

Definition 1. *If we have a system of polynomials*

$$\begin{cases} f_1(x_1,...,x_n) = \sum_{(\alpha_1...\alpha_n)} c_{1,(\alpha_1...\alpha_n)} x_1^{\alpha_1}...x_n^{\alpha_n} = 0 \\ f_2(x_1,...,x_n) = \sum_{(\alpha_1...\alpha_n)} c_{2,(\alpha_1...\alpha_n)} x_1^{\alpha_1}...x_n^{\alpha_n} = 0 \\ \\ f_n(x_1,...,x_n) = \sum_{(\alpha_1...\alpha_n)} c_{n,(\alpha_1...\alpha_n)} x_1^{\alpha_1}...x_n^{\alpha_n} = 0 \end{cases} \qquad (\mathcal{F})$$

and $\omega \in \mathbb{R}^{n*}$ then the initial form of f_i with respect to ω, denoted by $init_\omega(f_i)$, is, by definition, the sum over all the terms $c_{i,(\alpha_1...\alpha_n)} x_1^{\alpha_1}...x_n^{\alpha_n}$ with $c_{i,(\alpha_1...\alpha_n)} \neq 0$ for which the inner product $\langle \omega, (\alpha_1...\alpha_n) \rangle$ is minimized.

From the last Theorem is clear that we can associate to any financial structure the set of coefficients of the polynomials which are construct in the system , we denote such a set of points with the symbol $P(\mathbf{a}) \subset \mathbb{C}$.With this definition in mind we can give prove the following

Theorem 3. *Given the economy described in the first section with the financial structure given by the following matrix following*

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S) \end{bmatrix}$$

If there exists choice of equations $\left\{ M_{1,...,J}^{i_1^{(k)},...,i_J^{(k)}}(p,\mathbf{a}) = 0 \right\}_{k=1,...,\ell S}$ such that the system

$$\left\{ \; init_\omega \left(M_{1,...,J}^{i_1^{(k)},...,i_J^{(k)}} \right)(x) = 0 \qquad for\ all \quad 1 \leqq k \leqq \ell S \right.$$

has not zero in $(\mathbb{C}^*)^{\ell S}$ then the following are true:

1. The set of 'bad' spots prices has form

$$M_H(\mathbf{a}) = M_H^D(\mathbf{a})$$

this means that there is no manifold component.

2. The cardinality of the set $M_H^D(\mathbf{a})$ is given by the following formula

$$\#(M_H^D(\mathbf{a})) \leqq_{\mathcal{F}} \underset{\left\{ i_1^{(k)},...,i_J^{(k)} \right\}_{k=1,...,\ell S}}{Min} \mathcal{M}(co(M_{1,...,J}^{i_1^{(1)},...,i_J^{(1)}}(p,\mathbf{a})),...,co(M_{1,...,J}^{i_1^{(\ell S)},...,i_J^{(\ell S)}}(p,\mathbf{a})))$$

where, by definition (see Appendix), we have

$$\mathcal{M}(co(M_{1,...,J}^{i_1^{(1)},...,i_J^{(1)}}(p,\mathbf{a})),...,co(M_{1,...,J}^{i_1^{(\ell S)},...,i_J^{(\ell S)}}(p,\mathbf{a})))$$

is the coefficient of the term $\lambda_1 ... \lambda_n$ *in the polynomial*

$$V(\lambda_1 co(M_{1,...,J}^{i_1^{(1)},...,i_J^{(1)}}(p,\mathbf{a})) + ... + \lambda_n co(M_{1,...,J}^{i_1^{(\ell S)},...,i_J^{(\ell S)}}(p,\mathbf{a}))$$

and the this bound is always less then the Bezout's bound.

Proof. In progress. □

Proof. We remind to the reader that if agent 1 solves the problem $\left(P'\right)$

$$\max_x u_1(x) \qquad \text{s.t.} \qquad p \cdot (x - w^1) = 0$$

and each agent $i \geq 2$ in the economy solves the problem $\left(P''\right)$

$$\max_{(x,\theta)} \quad u_i(x) \quad \text{s.t.} \qquad p \cdot (x - w^i) = 0$$

$$p_1 \square (x_1 - w_1^i) \in L \qquad \dim L = k \text{ and } V(p,a) \subset L$$

then a *pseudoequilibrium* for the economy described in Section 2 is a collection $((\overline{x}^i), \overline{p}, L)$ such that

1. $(\overline{x}^i)$ solves (P') if $i = 1$ and solves $(P")$ for all $i \geq 2$,
2. $\sum_{i \in I} \overline{x}^i = \sum_{i \in I} \overline{w}^i$.

We know that the pseudoequilibrium always exists (see DS) and using the last Corollary we can conclude that the pseudoequilibrium is always an equilibrium if, given the profile of endowments, we choose the asset space to belong to a the appropriate set. □

4. Stochastic Economies

In this Section I extend the results of the previous Section to a fully stochastic economy i.e. an economy where there is more then one trading date. The extension is quite straightforward so I will just describe the economy, I will state the Theorems and I will add few simple comments (the reader should see [8], [18]).

As usual the underlying economy can be described by modelling the uncertainty via an event-tree. There is a finite set of state of nature $\mathbf{S} = \{1, ..., S\}$ and a collection of partitions of $\mathbf{S}$ $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T)$ where $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$ for all $1 \leq t \leq T-1$ and $\mathcal{F}_0 = \{\mathbf{S}, \emptyset\}$ and $\mathcal{F}_T = \{\{s\}\}_{s \in \mathbf{S}}$. $\mathcal{F}$ defines an information structure in that at each date $t = 0, ..., T$ exactly one of the events has occurred. If $\sigma \in \mathcal{F}_t$ has occurred the possible events $\sigma' \in \mathcal{F}_{t+1}$ that occurs at $t+1$ are those satisfying $\sigma' \subset \sigma$. Of course the filtration defines an event-tree as follows. Let $\mathbf{D} = \cup_{t=0}^{T} \mathcal{F}_t$ be the set of nodes. For each node $\xi \in \mathbf{D}$ there is exactly one $t \in \mathbf{T}$ and one $\sigma \in \mathcal{F}_t$ such that $\xi = (t, \sigma)$. The unique node for each $t \in \mathbf{T}$ and for each $\sigma \in \mathcal{F}_t$ the pair (t, σ) is called a node. For each node $\xi \in \mathbf{D}$ we denote with the symbol ξ^+ the set of immediate successor of ξ.The cardinality of ξ^+ is called the branching number and it is denoted with the symbol $b(\xi)$. With this notation we denote with the symbol $C_{\mathcal{F}}(\mathrm{D}, \mathbb{R}^{\ell})$ we denote of all functions $f : \mathbf{D} \to \mathbb{R}^{\ell}$ which are stochastic processes.

If $\sigma \in \mathbf{D}$ we denote with the symbol $\mathbf{D}(\sigma)$ the branch of the tree-event starting at σ.

There are I and each agent i, $1 \leqq i \leqq I$, is characterized by an initial stochastic endowment vector $w^i \in C_{\mathcal{F}}(\mathbf{D}, \mathbb{R}^{\ell})_{++}$ and a utility function $u_i : \mathbb{R}_{++}^{\ell \times \#(\mathbf{D})} \to \mathbb{R}$ satisfying

(i) u_i is C^∞,
(ii) $Du_i(x) \in \mathbb{R}_{++}^{\ell \times \#(\mathbf{D})}$ for all $x \in \mathbb{R}_{++}^{\ell \times \#(\mathbf{D})}$,
(iii) $h^T Du_i(x) h < 0$ for all $h \neq 0$ such that $hDu_i(x) = 0$,
(iv) $\left\{ x \in \mathbb{R}_{++}^{\ell \times \#(\mathbf{D})} : u_i(x) \geqq u_i(\overline{x}) \right\}$ is closed in $\mathbb{R}_{++}^{\ell \times \#(\mathbf{D})}$ for all $\overline{x} \in \mathbb{R}_{++}^{\ell \times \#(\mathbf{D})}$.

There are J real assets so, for any $j = 1, ..., J$, $\mathbf{a}^j : \mathbf{D} \to \mathbb{R}^\ell$ is an adapted process with $\mathbf{a}^j(\xi_0) = 0$. One unit of asset $\mathbf{a}^j$ held at the node ξ promises to deliver the commodity vector $\mathbf{a}^j(\xi') \in \mathbb{R}^\ell$ at each node $\xi' \in \mathbf{D}(\xi)$. If $\mathbf{a}(\xi) = \left[\mathbf{a}^1(\xi), ..., \mathbf{a}^J(\xi)\right]$ and $p \in C_{\mathcal{F}}(\mathbf{D}, \mathbb{R}^\ell)_+$ is a stochastic price process that

$$\begin{cases} \max\limits_{(x,\theta)} \quad u_i(x) & \text{s.t.} \\ p(\xi_0)\cdot(x(\xi_0) - w(\xi_0)) = -q(\xi_0)\cdot\theta(\xi_0) & \\ p(\xi)\cdot(x(\xi) - w(\xi)) = \left[p(\xi)\mathbf{a}(\xi) + q(\xi)\right]\theta(\xi^-) - q(\xi)\cdot\theta(\xi) & \xi \in \mathbf{D}\backslash\xi_0 \text{ and } \xi \notin \mathbf{D}_T \\ p(\xi)\cdot(x(\xi) - w(\xi)) = \left[p(\xi)\mathbf{a}(\xi) + q(\xi)\right]\theta(\xi^-) & \xi \in \mathbf{D}_T \end{cases}$$

Definition 2. *An equilibrium is thus a collection* $((\overline{x}^i, \overline{\theta}^i), (\overline{p}, \overline{q}))$ *such that*

1. $(\overline{x}^i, \overline{\theta}^i)$ solves (P) given $(\overline{p}, \overline{q})$ for all i,and
2. $\sum_{i \in I} \overline{x}^i = \sum_{i \in I} \overline{w}^i$ and
3. $\sum_{i \in I} \overline{\theta}^i = 0$

This type of equilibria also called GEI equilibria.

Theorem 4. *Given the economy described in the first section with the financial structure given, at each node* $\sigma \in \mathbf{D}$, *by the following matrix following*

$$\mathbf{a}_\sigma = \begin{bmatrix} \mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S) \end{bmatrix}$$

Note that for any j, *for* $1 \leq j \leq J$,*and for any* s *for* $1 \leq s \leq S$ *we have* $\mathbf{a}^j(s) \in \mathbb{R}^\ell$. *If* $\sigma \in \mathcal{F}_{t-1}$ *we set* $S_t(\sigma) \overset{def}{=} \#(\mathbf{D}(\sigma) \cap \mathcal{F}_t)$

$$\begin{pmatrix} S_t(\sigma) \\ J \end{pmatrix} > \ell S_t(\sigma)$$

Then the following are true:

1. The set of 'bad' spots prices at each node $\sigma \in \mathbf{D}$, *that we denote with the symbol* $M_H(\mathbf{a}, \boldsymbol{\sigma})$, *is disjoint union of two sets* $M_H^A(\mathbf{a}, \boldsymbol{\sigma})$ *and* $M_H^i(\mathbf{a}, \boldsymbol{\sigma})$ *where* $M_H^A(\mathbf{a}, \boldsymbol{\sigma})$ *is an algebraic manifold and* $M_H^i(\mathbf{a}, \boldsymbol{\sigma})$ *is a finite set of isolated points so we can write*

$$M_H(\mathbf{a}, \boldsymbol{\sigma}) = M_H^A(\mathbf{a}, \boldsymbol{\sigma}) \cup M_H^D(\mathbf{a}, \boldsymbol{\sigma})$$

2. There exists a closed set $\Omega_a \subset \mathbb{R}^2$ *of measure zero such that if* $\mathbf{a} \notin \Omega_a$ *then*

$$M_H^A(\sigma) \cap \mathbb{R}_{++}^{\ell S} = M_H^D(\sigma) \cap \mathbb{R}_{++}^{\ell \mathbf{S}} = \varnothing$$

3. There exists a set $\Omega'_a \subset \Omega_a$ closed and of measure zero such that

$$M_H^A(\mathbf{a},\boldsymbol{\sigma}) \cap \mathbb{R}_{++}^{\ell S} = \varnothing$$

This implies that the event that the algebraic part is not empty is more frequent that the fact that the discrete part is non empty.

4 There exists a positive integer number, that we denote with symbol n such that for any financial structure $\mathbf{a}$ *if we denote the cardinality of the set* $M_H^i(\mathbf{a},\boldsymbol{\sigma})$ *with the symbol* $\#(M_H^i(\mathbf{a}))$ *then the following*

$$\#(M_H^D(\mathbf{a},\boldsymbol{\sigma})) \leq J_t^{\ell S_t(\sigma)}$$

holds. Such integer is universal in the sense that it does not depend on the the individual financial structure.

5. Moreover, there exists algorithm to determine each element of $M_H^D(\mathbf{a},\boldsymbol{\sigma})$.

The proof of the Theorem is just as the proof of the Theorem the previous Section, In fact to prove the theorem I just used the property of the algebraic variety without even mentioning to trading dates.

In the same way it is possible to reprove the following

Theorem 5. *Given the economy described in the first section with the financial structure given, at each node* $\sigma \in \mathbf{D}$, *by the following matrix*

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^1(1) & \cdots & \mathbf{a}^J(1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \mathbf{a}^1(S) & \cdots & \mathbf{a}^J(S) \end{bmatrix}$$

Note that for any j, *for* $1 \leq j \leq J$,*and for any* s *for* $1 \leq s \leq S$ *we have* $\mathbf{a}^j(s) \in \mathbb{R}^\ell$.
If $S_t(\sigma) \stackrel{def}{=} \#(\mathbf{D}(\sigma))$ *and*

$$\begin{pmatrix} S_t(\sigma) \\ J \end{pmatrix} > \ell S_t(\sigma)$$

If there exists choice of equations $\left\{ M_{1,\dots,J}^{i_1^{(k)},\dots,i_J^{(k)}}(p,\mathbf{a}) = 0 \right\}_{k=1,\dots,\ell S}$ *such that the system*

$$\left\{ \; init_\omega \left(M_{1,\dots,J}^{i_1^{(k)},\dots,i_J^{(k)}} \right)(x) = 0 \qquad \text{for all} \quad 1 \leqq k \leqq \ell S \right.$$

has not zero in $(\mathbb{C}^*)^{\ell S}$ *then the following are true:*

1. The set of 'bad' spots prices has form

$$M_H(\mathbf{a},\boldsymbol{\sigma}) = M_H^D(\mathbf{a},\boldsymbol{\sigma})$$

this means that there is no manifold component.

2. The cardinality of the set $M_H^D(\mathbf{a},\boldsymbol{\sigma})$ *is given by the following formula*

$$\#(M_H^D(\mathbf{a},\boldsymbol{\sigma})) \leqq_{\mathcal{F}_{\left\{i_1^{(k)},\dots,i_J^{(k)}\right\}_{k=1,\dots,\ell S}}} Min \quad \mathcal{M}(co(M_{1,\dots,J}^{i_1^{(1)},\dots,i_J^{(1)}}(p,\mathbf{a})),\dots,co(M_{1,\dots,J}^{i_1^{(\ell S)},\dots,i_J^{(\ell S)}}(p,\mathbf{a})))$$

where, by definition (see Appendix), we have

$$\mathcal{M}(co(M_{1,\dots,J}^{i_1^{(1)},\dots,i_J^{(1)}}(p,\mathbf{a})),\dots,co(M_{1,\dots,J}^{i_1^{(\ell S)},\dots,i_J^{(\ell S)}}(p,\mathbf{a})))$$

is the coefficient of the term $\lambda_1 ... \lambda_n$ *in the polynomial*

$$Vol(\lambda_1 co(M_{1,...,J}^{i_1^{(1)},...,i_J^{(1)}}(p,\mathbf{a})) + ... + \lambda_n co(M_{1,...,J}^{i_1^{(\ell S)},...,i_J^{(\ell S)}}(p,\mathbf{a}))$$

and the this bound is always less then the Bezout's bound.

Theorem 6. *There exists an open set* $\Omega \subset \mathbb{R}^2$*, with null complement, such that an equilibrium always exists for any economy* $\omega\mathbb{R}$

Finally we are able to prove the following existence result which, as his analogue in the last Section, removes the genericity in the endowment part

The reader should note that we prove in the last theorem more then we stated. Non only the existence is there for any profile but the

5. Appendix

In this section we explain how to find the solution sets for system of polynomials. We only report the fundamental definitions and facts that we need. The interested reader should see Chapter 7 of [4] where a very interesting exposition of these and similar results can be found with proofs and relevant bibliography.

We denote with the symbol $\mathbb{C}[x_1, ...x_n]$ the set of all polynomials with complex coefficients in n variables. If $f \in \mathbb{C}[x_1, ...x_n]$ then we can write

$$f(x_1, ...x_n) = \sum_{\alpha_1 ... \alpha_n} c_{\alpha_1 ... \alpha_n} x_1^{\alpha_1} ... x_n^{\alpha_n}$$

and we call support of f the set of all $\alpha_1 ... \alpha_n$ such that $c_{\alpha_1 ... \alpha_n} \neq 0$. Note that we can consider vector $(\alpha_1 ... \alpha_n)$ as an element in $\mathbb{Z}^n$ the support of a polynomial in n variables as a finite subset of $\mathbb{Z}^n$. I will denote the support of the polynomial f with the symbol $S(f)$.Therefore if we consider a system of n polynomial in n equations as the following (see Ewald)

$$\begin{cases} f_1(x_1, ..., x_n) = 0 \\ f_2(x_1, ..., x_n) = 0 \\ \\ f_m(x_1, ..., x_n) = 0 \end{cases} \qquad (\mathcal{F})$$

then we can associate at this system the collection $\{S(f_1), ..., S(f_n)\} \subset \mathbb{Z}^n$.

Definition 3. *Given* s *non-empty convex compact sets* $A_1, ..., A_s$ *we denote with the symbol*

$$A_1 + ... + A_s = \{a_1 + ... + a_s \text{ such that } a_i \in A_i \text{ for } i = 1, ..., s\}$$

this set is called the Minkowski sum of $A_1, ..., A_s$.

Definition 4. *Given the set* A *we denote with the symbol* $co(A)$ *the following set*

$$co(A) = \left\{ \sum_{i=1}^{s} \lambda_i a_i \text{ such that } a_i \in A \text{ for } i = 1, ..., s \text{ and } \sum \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\}$$

this set is called convex hull of the set A.

Definition 5. *A polytope is the convex hull of a finite set of point in* $\mathbb{R}^n$. *If the points have integers coordinates the polytope is called a lattice polytope.*

The following Theorem is of paramount importance

Theorem 7. *(H. Minkowski) The volume of the linear combination of non-empty convex compact set $K_1, ..., K_s$ with non-negative coefficients $\lambda_1, ..., \lambda_s$*

$$Vol(\lambda_1 K_1 + ... + \lambda_s K_s)$$

is a homogenous polynomial of degree n with respect to $\lambda_1, ..., \lambda_s$.

Then we have the following definition

Definition 6. *The n-dimensional mixed volume of collection of polytopes $K_1, ..., K_n$, denoted by*

$$MVol(K_1, ..., K_n)$$

is the coefficients of the monomial $\lambda_1\lambda_2...\lambda_n$ in $Vol(\lambda_1 K_1 + ... + \lambda_n K_n)$.

It is important to observe that, surprisingly, $Vol(K_1, ..., K_n)$ is always an integer if the polytopes are lattice polytopes! (for a proof see

Proposition 1. *The mixed volume $Vol(K_1, ..., K_n)$ depends continuously on*

$$K_1, ..., K_n$$

in the Hausdorff distance.

Proof. In order to prove this property we observe that it is possible to prove that

$$\begin{aligned} n!Vol(K_1, ..., K_n) = {} & Vol(K_1 + ... + K_n) \\ & - \textstyle\sum_{i=1}^n Vol(K_1 + ... + K_{i-1} + K_{i+1} + ... + K_n) \\ & + \textstyle\sum_{i<j}^n Vol(K_1 + ... + K_{i-1} + K_{i+1} + ... + K_{j-1} \\ & + K_{j+1} + ... + K_n) \\ & \vdots \\ & + (-1)^{n-2} \textstyle\sum_{i<j}^n Vol(K_i + K_j) + (-1)^{n-1} \sum_{i=1}^n Vol(K_i) \end{aligned}$$

and since the operations involved are only continuous operations we are done. Using this Theorem it is possible to prove the following □

Theorem 8. *(Bernstein) For almost all the choices in $\mathbb{C}$ of coefficients the number of common zeros in the torus $(\mathbb{C}^*)^n$ of*

$$\begin{cases} f_1(x_1, ..., x_n) = 0 \\ f_2(x_1, ..., x_n) = 0 \\ \\ f_n(x_1, ..., x_n) = 0 \end{cases} \qquad (\mathcal{F})$$

equals the mixed volume $MVol(co(S(f_1)), ..., co(S(f_n)))$.

Few words to explain the idea behind Bernstein's Theorem. Let's assume we start with a a system

$$\begin{cases} f_1(x_1, ..., x_n) = 0 \\ f_2(x_1, ..., x_n) = 0 \\ \\ f_n(x_1, ..., x_n) = 0 \end{cases} \qquad (\mathcal{F})$$

then , since we restrict ourself to polynomial functions, we can write

$$f_j(x_1, ...x_n) = \sum_{(\alpha_1 ... \alpha_n) \in \mathbb{N}} c_{j,\alpha_1...\alpha_n} x_1^{\alpha_1} ... x_n^{\alpha_n}$$

for each polynomial we have the support $S(f_j) \subset \mathbb{N}^n$ so we can associate to the system the n-tuple of supports $\{S(f_1), ..., S(f_n)\}$. The central idea is to hold the n-tuple of supports $\{S(f_1), ..., S(f_n)\}$ fixed and treat the coefficients $c_{j,\alpha_1 \ldots \alpha_n}$ as parameters. The reader should keep in mind that we proved. see Lemma 1, that each polynomial which appears in our system has the form

$$M_{1,\ldots,J}^{i_1,\ldots,i_J}(p,\mathbf{a}) = \sum M_{1,\ldots,J}^{\ell_1,\ldots,\ell_J}(A^{i_1,\ldots,i_J}) p_{\ell_1}(i_1) \ldots p_{\ell_J}(i_J)$$

where $\left\{ M_{1,\ldots,J}^{\ell_1,\ldots,\ell_J}(A^{i_1,\ldots,i_J}) \right\}$ is, by definition, the set of $J \times J$ minors of the matrix

$$\begin{bmatrix} \mathbf{a}^1(i_1) & \cdots & \mathbf{a}^J(i_1) \\ \mathbf{a}^1(i_2) & & \mathbf{a}^J(i_2) \\ \vdots & & \vdots \\ \mathbf{a}^1(i_J) & \cdots & \mathbf{a}^J(i_J) \end{bmatrix}$$

so in our setting the coefficients are just wel defined multilinear functions of the asset payoffs.

6. Conclusions and Extensions

There are several observations which are in

1. We have been assuming that the number of goods at each node at each node. This is not necessary and we can prove the same results when the number of goods changes at each nodes. Of course, the statements of the theorems need to me modified accordingly.

2. The algorithm we have constructed is a god ones since exploits the projective structure and assures the fact that we can find all the solutions of the systems. However it seems possible that different kind of algorithms can be constructed. It seems plausible that different kind of elementary systems could be constructed. This is object of further research.

4. I did not investigate very deeply the structure of the manifold component but this an interesting part. In a new paper I want to investigate further to see if sharper anal;is possible.

5. In this paper I used principally the tools of complex algebraic geometry. I suspect that it is possible to get sharper results with the help of Real Algebraic Geometry. This aspect will be object of further research.

6. My results suggest that the excess demand can be studied at a more detailed level. This is the object of further research. Please see my paper "How bad can the excess demand be in GEI models?"

7. This techniques allow us extend the analysis for discrete version of continuous-time models in finance. This will be the object of a paper work with Bob Anderson (UC Berkeley).

8. Finally, the result have implication to study regular GEI economy and this is going to be the object of further research. An early draft is contained in my work "Regular GEI economies"

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