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A REVIEW OF 'JUMPS' IN MACROECONOMIC MODELS: WITH SPECIAL REFERENCE TO THE CASE WHEN EIGENVALUES ARE COMPLEX-VALUED

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ABSTRACT

The dynamic properties of macroeconomic models are typically characterised by having a combination of stable and unstable eigenvalues. In a seminal paper, Blanchard and Kahn showed that, for linear models, in order to ensure a unique solution, the number of discontinuous or "jump" variables must equal the number of unstable eigenvalues in the economy. Assuming no zero eigenvalues, this also means that the number of predetermined variables, otherwise referred to as continuous or non-jump variables, must equal the number of stable eigenvalues. In this paper, we review the Blanchard and Kahn results when eigenvalues are real-valued and then establish that these results also carry through for linear dynamical systems where some of the eigenvalues are complex-valued. We show that the crucial reason why the results continue to hold for complex-valued eigenvalues is because, in order to ensure that the solutions for the endogenous variables are real-valued and thus have an economic interpretation, the coefficients associated with each complex conjugate pair of eigenvalues must also come in complex conjugate pairs. Examples with just one complex conjugate pair of stable eigenvalues and a general n-dimensional model have been presented for both the continuous-time and discrete-time cases.

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Keywords: Macroeconomics; Dynamical Systems; Complex-valued Eigenvalues; Saddlepath Instability; Jump Variables.

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1. INTRODUCTION

Macroeconomic models derived from optimising behaviour are typically characterised by the property of saddle-path instability, involving dynamics defined by a combination of stable and unstable eigenvalues. Turnovsky (2000) gives examples of a range of such models. When faced with an unanticipated shock, the standard approach in the literature is to assume that sufficient variables have discontinuities so that they "jump" at the time when new information hits the economic system, i.e. at the time of the unanticipated shock. These variables are assumed to jump so as to bring the economy back onto the stable path with this jumping behaviour being intuitively, but not rigorously, justified on the basis of underlying optimising behaviour of various agents in the economy.

In a seminal paper, Blanchard and Kahn (1980) showed that, for linear models, in order to ensure a unique solution, the number of jump variables must equal the number of unstable eigenvalues in the economy. Assuming no zero eigenvalues, this also means that the number of predetermined variables, otherwise referred to as continuous or non-jump variables, must equal the number of stable eigenvalues. Blanchard and Kahn's results are typically applied to models with real-valued eigenvalues. This paper focuses on the validity of their results when at least some of the stable eigenvalues are complex-valued.

While models of dynamical systems with complex-valued eigenvalues are well represented in various areas of the engineering literature, dynamic macro-models with complex-valued eigenvalues have received scant treatment in the economics literature.[1] This means that there has been virtually no treatment of "jump" variables, which are important in macroeconomics but have no real counterpart in the

[1] For an example of the application of complex-valued eigenvalues in a macroeconomic framework, see Stemp and Herbert (2003).

engineering literature. One way to handle the issue of jump variables in the case of models with complex-valued eigenvalues is just to calculate the number of unstable eigenvalues, whether real-valued or complex-valued, and then to appeal loosely to the Blanchard and Kahn results to justify the same number of jump variables as unstable eigenvalues. The problem with this approach is that Blanchard and Kahn made no explicit reference to complex-valued eigenvalues. In this paper, we review the Blanchard and Kahn results when eigenvalues are real-valued and then demonstrate that their results do carry through to the case of complex-valued eigenvalues, but only because of a particular property of the solution in these cases.

In the following sections of this paper, we derive the Blanchard and Kahn results for macroeconomic models when all eigenvalues are real-valued and then consider the case when some eigenvalues are complex-valued, considering the case of both continuous-time and discrete-time models. Section 2 considers the continuous-time example of Dornbusch (1976) Model, augmented by sluggish adjustment in net exports so that there is the possibility of complex-valued eigenvalues. Section 3 considers the case of the n-dimensional linear dynamical system in continuous-time. Sections 4 and 5 consider analogous models to those considered in Sections 2 and 3 but in discrete-time. Section 6 provides a conclusion.

2. AN EXAMPLE IN CONTINUOUS TIME

The basic model

Consider the following model:

$$\dot{p} = \alpha_1(x - x^*) - \alpha_2(r - r^*), \alpha_1 > 0, \alpha_2 > 0 \tag{1a}$$

$$\bar{m} - p = -\beta(r - r^*), \beta > 0 \tag{1b}$$

$$r = r^* + \dot{e} \tag{1c}$$

$$\dot{x} = \eta[\delta(e-p)-x], \eta > 0 \qquad (1d)$$

where parameters are given by Greek symbols, all variables are functions of time, and

p = price level (expressed in logarithms);
x = net exports;
r = nominal interest rate;
r^* = nominal interest rate in rest of world;
e = the nominal exchange rate (expressed in logarithms so that an increase in e denotes a depreciation); and
$\overline{m}$ = nominal money supply (assumed exogenous and fixed, except at point of initial shock).

This is essentially the Dornbusch (1976) model augmented to include sluggish adjustment of net exports in response to changes in the real exchange rate. The model can be reduced to the following set of equations:

$$\dot{p} = \alpha_1(x - x^*) + \frac{\alpha_2}{\beta}(\overline{m} - p) \qquad (2a)$$

$$\dot{e} = -\frac{1}{\beta}(\overline{m} - p) \qquad (2b)$$

$$\dot{x} = \eta[\delta(e-p)-x] \qquad (2c)$$

This can be expressed in matrix form as:

$$\begin{pmatrix} \dot{p} \\ \dot{e} \\ \dot{x} \end{pmatrix} = \mathbf{A} \begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} \qquad (3a)$$

where an asterisk denotes the corresponding steady-state value, and where

$$\mathbf{A} = \begin{pmatrix} -\frac{\alpha_2}{\beta} & 0 & \alpha_1 \\ \frac{1}{\beta} & 0 & 0 \\ -\eta\delta & \eta\delta & -\eta \end{pmatrix} \qquad (3b)$$

Real-valued and complex-valued eigenvalues

When $\beta < \infty$, the characteristic equation is given by:

$$c(\lambda) = \begin{vmatrix} -\frac{\alpha_2}{\beta} - \lambda & 0 & \alpha_1 \\ \frac{1}{\beta} & -\lambda & 0 \\ -\eta\delta & \eta\delta & -\eta - \lambda \end{vmatrix} \tag{4}$$

In order to derive the full analytic solution to this model, we must first demonstrate that it is possible for the dynamical system defined by equations (3a-3b) to have complex-valued eigenvalues. Letting $\beta \to \infty$, the characteristic equation of this system satisfies:

$$c(\lambda) = \begin{vmatrix} -\lambda & 0 & \alpha_1 \\ 0 & -\lambda & 0 \\ -\eta\delta & \eta\delta & -\eta - \lambda \end{vmatrix}$$

$$= -\lambda^3 - \eta\lambda^2 - \eta\alpha_1\delta\lambda$$

$$= -\lambda[\lambda^2 + \eta\lambda + \eta\alpha_1\delta] \tag{5a}$$

Hence, as $\beta \to \infty$, the eigenvalues of the system are given by:

$$\lambda_1 = 0\text{; and } \lambda_2, \lambda_3 = \frac{-\eta \pm \sqrt{\eta^2 - 4\eta\alpha_1\delta}}{2}. \tag{5b}$$

In continuous-time models, an eigenvalue is associated with stable dynamics if and only if the real part of the eigenvalue is negative. Note that λ_2 and λ_3 both have negative real parts and that both λ_2 and λ_3 are complex-valued whenever $4\alpha_1\delta > \eta$. Accordingly, if the system has at least one complex-valued eigenvalue then it will have two complex-valued eigenvalues each with negative real parts and these two eigenvalues will come as a complex conjugate pair. This means that, if $\lambda_2 = -\mu_a + i\mu_b$, where μ_a and μ_b are real numbers, then $\lambda_3 = -\mu_a - i\mu_b$. We denote the complex conjugate of a complex number by a bar over that number, so that $\overline{\lambda}_2 = \lambda_3$ and $\overline{\lambda}_3 = \lambda_2$.

If all other parameters are fixed, the size of the imaginary part of the complex-valued eigenvalues can be determined by changing the value of η. For high value of η, both λ_2 and λ_3 will be real-valued. For lower value of η, both λ_2 and λ_3 will be complex-valued. Using continuity in β, it follows that complex-valued eigenvalues will also arise for finite but large values of β.

Also, when $\beta < \infty$, for the system given by equations (3a-3b), the sum of the eigenvalues is given by trace **A** and the product of the eigenvalues is given by det **A**. Hence,

$$\lambda_1 + \lambda_2 + \lambda_3 = -\left(\eta + \frac{\alpha_2}{\beta}\right) < 0 \qquad (6a)$$

$$\lambda_1 \lambda_2 \lambda_3 = \frac{\eta \delta \alpha_1}{\beta} > 0 \qquad (6b)$$

From equation (6b), the real parts of the eigenvalues are either (i) all positive, or (ii) two negative and one positive. Then, from (6a), at least one eigenvalue has negative real part. Hence the signs of the real parts of the eigenvalues are: two negative and one positive.

It follows from the above that for β sufficiently large, the dynamical system given by equations (3a-3b) will have three non-zero eigenvalues, one of which is real-valued and positive, and two of which will be complex-valued, with negative real parts.

Table 1
Values of Eigenvalues for Different Values of η: Example in Continuous Time
(Unstable eigenvalues are preceded by an asterisk)

η	λ_1	λ_2	λ_3
0.25	*0.52	-0.63-0.61i	-0.63+0.61i
0.5	*0.63	-0.81-0.78i	-0.81+0.78i
1.0	*0.74	-1.12-0.95i	-1.12+0.95i
2.0	*0.84	-1.67-1.01i	-1.67+1.01i
4.0	*0.92	-2.10	-3.31

Next, the model is calibrated using the following parameter values: $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, $\beta = 1.0$, and $\delta = 1.0$. We allow the parameter, η, to vary over a range of values. This generates the values for eigenvalues summarised in Table 1, which, consistent with the earlier results, shows that it is possible to generate complex-valued eigenvalues over a range of parameter values and that these complex-valued eigenvalues come in complex conjugate pairs.

Without loss of generality, we will denote the positive real-valued eigenvalue by λ_1 and the other two eigenvalues by λ_2 and λ_3. Irrespective of whether λ_2 and λ_3 are real-valued or complex-valued, the eigenvectors of the system are given by:

$$\mathbf{v}(\lambda_i) = \begin{pmatrix} \alpha_1 \beta \lambda_i \\ \alpha_1 \\ \lambda_i [\alpha_2 + \beta \lambda_i] \end{pmatrix} \tag{7}$$

The eigenvectors are linearly independent so that no eigenvector can be written as a linear combination of the other eigenvectors.[2] The general solution to the system is then given by:

$$\begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \mathbf{v}(\lambda_3) \end{bmatrix} \begin{pmatrix} C_1 \exp(\lambda_1 t) \\ C_2 \exp(\lambda_2 t) \\ C_3 \exp(\lambda_3 t) \end{pmatrix} \tag{8a}$$

where C_1, C_2 and C_3 are yet-to-be-determined constants, which may be real-valued or may be complex-valued.

As shown above, it is always the case that precisely one eigenvalue, λ_1, has positive real part (the unstable eigenvalue) and two eigenvalues, λ_2 and λ_3, have negative real part (the stable eigenvalues). When λ_2 and λ_3 are real-valued, the

[2] For a proof, see Simon and Blume (1994), Theorem 23.5.

standard approach to solving this model is to assume that, when there is a sudden unanticipated shock to the economy, an appropriate number of variables "jump" to the stable solution, while other variables evolve continuously from their historically-determined positions. Blanchard and Kahn (1980) have shown that, in order to achieve a unique solution, there must be exactly the same number of jump variables as there are unstable eigenvalues. Put another way, in order to achieve a unique solution and assuming that there are no zero eigenvalues, there must be exactly the same number of predetermined variables as there are stable eigenvalues.

Since λ_1 has positive real part, the stable solution is given by setting $C_1 = 0$ so that:

$$\begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} = \left[\mathbf{v}(\lambda_2) \quad \mathbf{v}(\lambda_3) \right] \begin{pmatrix} C_2 \exp(\lambda_2 t) \\ C_3 \exp(\lambda_3 t) \end{pmatrix} \qquad \text{(8b)}$$

If we assume that *e* is the jump variable then the constants, C_2 and C_3, should be chosen consistent with the initial values of *p* and *x*. When the eigenvalues are all real-valued, C_2 and C_3 are real-valued constants and equation (8b) fully determines the analytic solution of the model. In the following paragraphs, we investigate whether the Blanchard and Kahn results carry over to the case when some of the eigenvalues are allowed to be complex-valued.

Solving the model with complex-valued eigenvalues

It can be shown that, if λ_2 and λ_3 are complex-valued then they are complex conjugates and the corresponding eigenvectors are also complex conjugates; in other words, if λ_2 and λ_3 are complex-valued then $\bar{\lambda}_2 = \lambda_3$, and $\mathbf{v}(\bar{\lambda}_2) = \mathbf{v}(\lambda_3) = \bar{\mathbf{v}}(\lambda_2)$.[3]

[3] See Simon and Blume (1994), Theorem 23.13.

Assuming that $\lambda_2 = -\mu_a + i\mu_b$ and that $\lambda_3 = -\mu_a - i\mu_b$, then, from DeMoivre's Theorem,

$$\exp(i\mu_b t) = \cos(\mu_b t) + i\sin(\mu_b t) \qquad (9a)$$

and
$$\exp(-i\mu_b t) = \cos(\mu_b t) - i\sin(\mu_b t) \qquad (9b)$$

Hence,

$$\exp(\lambda_2 t) = \exp(-\mu_a t + i\mu_b t) = \exp(-\mu_a t)\exp(i\mu_b t)$$

$$= \exp(-\mu_a t)[\cos(\mu_b t) + i\sin(\mu_b t)] \qquad (10a)$$

and
$$\exp(\lambda_3 t) = \exp(\bar{\lambda}_2 t) = \exp(-\mu_a t)[\cos(\mu_b t) - i\sin(\mu_b t)] \qquad (10b)$$

so that $\exp(\lambda_2 t)$ and $\exp(\lambda_3 t)$ are also complex conjugates.

Let
$$\mathbf{w_a} + i\mathbf{w_b} = \exp(\lambda_2 t)\mathbf{v}(\lambda_2) \qquad (11a)$$

Then
$$\mathbf{w_a} - i\mathbf{w_b} = \exp(\lambda_3 t)\mathbf{v}(\lambda_3) = \exp(\bar{\lambda}_2 t)\mathbf{v}(\bar{\lambda}_2) \qquad (11b)$$

Furthermore, let $C_2 = C_{2a} + iC_{2b}$ and $C_3 = C_{3a} + iC_{3b}$. Then, from equation (8b), in order that the variables, *p*, *e* and *x* are real-valued, it must be the case that the imaginary part of the following $(\mathbf{w_a} + i\mathbf{w_b})(C_{2a} + iC_{2b}) + (\mathbf{w_a} - i\mathbf{w_b})(C_{3a} + iC_{3b})$ is zero. Hence,

$$\mathbf{w_a}(C_{2b} + C_{3b}) + \mathbf{w_b}(C_{2a} - C_{3a}) = 0$$

(11c)

Since $\mathbf{w_a}$ and $\mathbf{w_b}$ are linear combinations of $\mathbf{v}(\lambda_2)$ and $\mathbf{v}(\lambda_3)$, it follows from the linear independence of $\mathbf{v}(\lambda_2)$ and $\mathbf{v}(\lambda_3)$ that:

$$C_{2b} + C_{3b} = 0 \qquad (12a)$$

and
$$C_{2a} - C_{3a} = 0 \qquad (12b)$$

In other words, in order for the solutions to *p*, *e* and *x* to be real-valued, it is necessary and sufficient that C_2 and C_3 also form a complex conjugate pair and so are of the form:

$$C_2 = D_a + iD_b \tag{13a}$$

$$C_3 = D_a - iD_b \tag{13b}$$

Hence, the general solution in the case of complex-valued eigenvalues can be written in the form:

$$\begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_2) & \mathbf{v}(\bar{\lambda}_2) \end{bmatrix} \begin{pmatrix} C_2 \exp(\lambda_2 t) \\ \bar{C}_2 \exp(\bar{\lambda}_2 t) \end{pmatrix} \tag{14a}$$

Letting $\mathbf{v}(\lambda_2) = \mathbf{u_a} + i\mathbf{u_b}$, $\lambda_2 = -\mu_a + i\mu_b$ and $C_2 = D_a + iD_b$, equation (14a) can be rewritten in the form:

$$\begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{u_a} + i\mathbf{u_b} & \mathbf{u_a} - i\mathbf{u_b} \end{bmatrix} \begin{pmatrix} (D_a + iD_b)\exp(-\mu_a t)[\cos(\mu_b t) + i\sin(\mu_b t)] \\ (D_a - iD_b)\exp(-\mu_a t)[\cos(\mu_b t) - i\sin(\mu_b t)] \end{pmatrix} \tag{14b}$$

For the model examined in this Section, equation (14b) can be rewritten as:

$$\begin{pmatrix} p - p^* \\ e - e^* \\ x - x^* \end{pmatrix} = \begin{pmatrix} -\alpha_1(1-\gamma)\mu_a + i\alpha_1(1-\gamma)\mu_b & -\alpha_1(1-\gamma)\mu_a - i\alpha_1(1-\gamma)\mu_b \\ \alpha_2(\mu_a^2 - \mu_b^2) + \mu_a - i(2\alpha_2\mu_a\mu_b + \mu_b) & \alpha_2(\mu_a^2 - \mu_b^2) + \mu_a + i(2\alpha_2\mu_a\mu_b + \mu_b) \\ -\eta\alpha_2\mu_a - 1 + i(\eta\alpha_2\mu_b) & -\eta\alpha_2\mu_a - 1 - i(\eta\alpha_2\mu_b) \end{pmatrix}$$

$$\cdot \begin{pmatrix} (D_a + iD_b)\exp(-\mu_a t)[\cos(\mu_b t) + i\sin(\mu_b t)] \\ (D_a - iD_b)\exp(-\mu_a t)[\cos(\mu_b t) - i\sin(\mu_b t)] \end{pmatrix} \tag{14c}$$

Next, let $\mathbf{u_a} = \begin{pmatrix} u_{a1} \\ u_{a2} \\ u_{a3} \end{pmatrix}$ and $\mathbf{u_b} = \begin{pmatrix} u_{b1} \\ u_{b2} \\ u_{b3} \end{pmatrix}$ (15a)

The constants, D_a and D_b, can then be chosen consistent with the initial values of *p* and *x*. For example, if at time $t=0$, $p(0)=p_0$ and $x(0)=x_0$, then, from equation (14b),

$$\begin{pmatrix} p_0-p^* \\ x_0-x^* \end{pmatrix} = \begin{pmatrix} u_{a1}+iu_{b1} & u_{a1}-iu_{b1} \\ u_{a3}+iu_{b3} & u_{a3}-iu_{b3} \end{pmatrix} \begin{pmatrix} D_a+iD_b \\ D_a-iD_b \end{pmatrix} \tag{15b}$$

Hence,

$$\begin{pmatrix} p_0-p^* \\ x_0-x^* \end{pmatrix} = \begin{pmatrix} 2u_{a1} & -2u_{b1} \\ 2u_{a3} & -2u_{b3} \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix} \tag{15c}$$

We can refer to D_a and D_b as the two parameters that need to be calculated to solve the model. It will be observed that the number of parameters (two) is precisely the same as the number of stable eigenvalues. As long as the 2x2 matrix in equation (15c) is invertible, this establishes the property that there needs to be a 1-1 relationship between the number of parameters, the number of stable eigenvalues and the number of predetermined variables. This is the result of Blanchard and Kahn extended to a model with two complex-valued eigenvalues.

Hence,

$$\begin{pmatrix} D_a \\ D_b \end{pmatrix} = \begin{pmatrix} -2\alpha_1(1-\gamma)\mu_a & -2\alpha_1(1-\gamma)\mu_b \\ -2\eta\alpha_2\mu_a-2 & -2\eta\alpha_2\mu_b \end{pmatrix}^{-1} \begin{pmatrix} p_0-p^* \\ x_0-x^* \end{pmatrix}$$

$$= \frac{1}{\alpha_1(1-\gamma)\mu_b} \begin{pmatrix} \eta\alpha_2\mu_b & -\alpha_1(1-\gamma)\mu_b \\ -\eta\alpha_2\mu_a-1 & \alpha_1(1-\gamma)\mu_a \end{pmatrix} \begin{pmatrix} p_0-p^* \\ x_0-x^* \end{pmatrix} \tag{15d}$$

Equations (14c) and (15d) fully determine the analytic solution of the model with complex-valued eigenvalues, providing real-valued solutions for *p*, *e* and *x*. It will be observed that the solution given by equation (14c) includes the terms $\cos(\mu_b t)$ and $\sin(\mu_b t)$, thus demonstrating that the analytic solution has cyclic properties and, in particular, that the frequency of the cycles increases as the absolute value of μ_b

increases. Also, since μ_a is positive, $\exp(-\mu_a t) \to 0$, so that *p, e* and *x* all converge to their steady-state values.

The requirement that, in order for the endogenous variables, *p, e* and *x,* to be real-valued, it is necessary and sufficient for C_2 and C_3 to form a complex conjugate pair is crucial for the Blanchard and Kahn results to carry over in this case. When C_2 and C_3 form a complex conjugate pair, then their values are determined by only two parameters, D_a and D_b, in the above. Hence, by taking the initial values of only two predetermined variables, *p* and x, it is generally possible to specify jumps to a unique path, where this unique path is given by equation (14c). If C_2 and C_3 did not have to be complex conjugates, but instead could be chosen from anywhere in the complex plane, then initial values for only two variables, such as *p* and *x,* would not be sufficient to ensure that the economy jumped to a unique path after a shock.

3. GENERAL CASE IN CONTINUOUS TIME

We next consider the general continuous-time linear dynamical system of the form:

$$\underset{nx1}{\dot{\mathbf{y}}} = \underset{nxn}{\mathbf{M}}(\mathbf{y} - \underset{nx1}{\mathbf{y}^*}) \qquad (16)$$

It is assumed throughout that the *n* eigenvalues of **M** are all distinct. Hence, the corresponding eigenvectors are also distinct and linearly independent. We again focus on the complex-valued eigenvalues and the associated eigenvectors. The following propositions are extensions of related ideas and theorems presented in Section 2.

Proposition 1: *For any matrix, **M**, if $\lambda(= \mu_a + i\mu_b)$ is a complex-valued eigenvalue of **M**, then the complex conjugate $\bar{\lambda}(= \mu_a - i\mu_b)$ is also an eigenvalue of **M**. Hence,* *any matrix, **M,** must have an even number of complex-valued eigenvalues and these complex-valued eigenvalues must always come in complex conjugate pairs.*

Proposition 2: *If* λ *and* $\bar{\lambda}$ *are a complex conjugate pair of complex-valued eigenvalues of **M**, then the corresponding eigenvectors,* $\mathbf{v}(\lambda)(=\mathbf{u_a}(\lambda)+i\mathbf{u_b}(\lambda))$ *and* $\mathbf{v}(\bar{\lambda})(=\mathbf{u_a}(\lambda)-i\mathbf{u_b}(\lambda))$ *also form a complex conjugate pair.*

Proposition 3: *If* λ *and* $\bar{\lambda}$ *are a complex conjugate pair of complex-valued numbers, then* $\exp(\lambda t)$ *and* $\exp(\bar{\lambda} t)$ *also form a complex conjugate pair of complex-valued numbers.*

Assume that $\mathbf{M}$ has *k* real-valued eigenvalues and *2l* complex-valued eigenvalues, making a total of $n = k + 2l$ distinct eigenvalues and eigenvectors. Then the general solution to the system of equations given by (16) is of the form:

$$\begin{pmatrix} y_1 - y_1^* \\ \vdots \\ y_k - y_k^* \\ y_{k+1,1} - y_{k+1,1}{}^* \\ y_{k+1,2} - y_{k+1,2}{}^* \\ \vdots \\ y_{k+l,1} - y_{k+l,1}{}^* \\ y_{k+l,2} - y_{k+l,2}{}^* \end{pmatrix} =$$

$$\begin{bmatrix} \mathbf{v}(\lambda_1) & \cdots & \mathbf{v}(\lambda_k) & \mathbf{v}(\lambda_{k+1}) & \mathbf{v}(\bar{\lambda}_{k+1}) & \cdots & \mathbf{v}(\lambda_{k+l}) & \mathbf{v}(\bar{\lambda}_{k+l}) \end{bmatrix} \begin{pmatrix} C_1 \exp(\lambda_1 t) \\ \vdots \\ C_k \exp(\lambda_k t) \\ C_{k+1,1} \exp(\lambda_{k+1} t) \\ C_{k+1,2} \exp(\bar{\lambda}_{k+1} t) \\ \vdots \\ C_{k+l,1} \exp(\lambda_{k+l} t) \\ C_{k+l,2} \exp(\bar{\lambda}_{k+l} t) \end{pmatrix}$$

(17a)

We next introduce the following definition:

Definition 1: *Consider the dynamical system given by* $\dot{\mathbf{y}} = \mathbf{M}(\mathbf{y} - \mathbf{y}^*)$, *where* $\mathbf{y}$ *is an nx1 matrix and* $\mathbf{M}$ *is an nxn matrix. If* $\mathbf{M}$ *has n distinct eigenvalues given by*

$\lambda_1, \lambda_2, ..., \lambda_n$ and n corresponding eigenvectors, $\mathbf{v}(\lambda_1), \mathbf{v}(\lambda_2), ..., \mathbf{v}(\lambda_n)$, then the solution of the dynamical system is given by:

$$\mathbf{y}(t) - \mathbf{y}^* = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \cdots & \mathbf{v}(\lambda_n) \end{bmatrix} \begin{pmatrix} C_1 \exp(\lambda_1 t) \\ C_2 \exp(\lambda_2 t) \\ \vdots \\ C_n \exp(\lambda_n t) \end{pmatrix}$$

where $C_1, C_2, ..., C_n$ are arbitrary constants determined by the boundary values of the dynamical system. For each i, we say that the arbitrary constant, C_i, is associated with the eigenvalue, λ_i.

These arbitrary constants have particular properties as summarized in the following propositions:

Proposition 4: *If λ is a real-valued eigenvalue of **M**, then the arbitrary constant, C, associated with λ must also be real-valued.*

Proposition 5: *If λ_1 and λ_2 are a complex conjugate pair of complex-valued eigenvalues of **M**, then the arbitrary constants, C_1 and C_2, associated with λ_1 and λ_2, also form a complex conjugate pair. Hence, there exist two parameters, D_a and D_b, such that $C_1 = D_a + iD_b$ and $C_2 = D_a - iD_b$.*

Hence, the last matrix in equation (17a) can be rewritten as:

$$\begin{pmatrix} C_1 \exp(\lambda_1 t) \\ \vdots \\ C_k \exp(\lambda_k t) \\ C_{k+1} \exp(\lambda_{k+1} t) \\ \bar{C}_{k+1} \exp(\bar{\lambda}_{k+1} t) \\ \vdots \\ C_{k+l} \exp(\lambda_{k+l} t) \\ \bar{C}_{k+l} \exp(\bar{\lambda}_{k+l} t) \end{pmatrix} \qquad (17b)$$

It is usual practice to assume a No-Ponzi Game solution, so that unstable paths are eliminated from the set of acceptable solutions. In order to eliminate the unstable paths it is first necessary to discuss the stability properties associated with the different eigenvalues.

Definition 2: *Consider the dynamical system in continuous time given by* $\dot{\mathbf{y}} = \mathbf{M}(\mathbf{y} - \mathbf{y}^*)$. *If* λ *is an eigenvalue of* ***M****, then we say that* λ *is a stable eigenvalue if the real part of* λ *is negative. We say that* λ *is an unstable eigenvalue if the real part of* λ *is positive.*

Assume that there are $k_s (\leq k)$ stable real-valued eigenvalues and $l_s (\leq l)$ complex conjugate pairs of stable complex-valued eigenvalues. In order to ensure stability we can then set to zero the arbitrary constants associated with the unstable eigenvalues in equation (17a), thus eliminating all unstable eigenvalues and associated eigenvectors.

Let us first consider the case when all stable eigenvalues are real-valued. Then, after elimination of the unstable eigenvalues, the solution to this dynamical system can be written in the form:

$$\begin{pmatrix} y_1 - y_1^* \\ y_2 - y_2^* \\ \vdots \\ y_{k_s} - y_{k_s}^* \\ y_{k_s+1} - y_{k_s+1}^* \\ \vdots \\ y_k - y_k^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \cdots & \mathbf{v}(\lambda_{k_s}) \end{bmatrix} \begin{pmatrix} C_1 \exp(\lambda_1 t) \\ C_2 \exp(\lambda_2 t) \\ \vdots \\ C_{k_s} \exp(\lambda_{k_s} t) \end{pmatrix} \qquad (17c)$$

where the C_i's are all real-valued. In general, we can use the predetermined values of k_s variables, say $y_1, y_2, ..., y_{k_s}$, to determine values for the arbitrary constants, $C_1, C_2, ..., C_{k_s}$. Following an unanticipated shock, the steady state value of $\mathbf{y}^*$ will change and the remaining variables, $y_{k_s+1}, y_{k_s+2}, ..., y_k$, will jump instantaneously to the new stable solution. This is the result of Blanchard and Kahn, which can be summarized in the following proposition:

Proposition 6: *If there are m real-valued stable eigenvalues then there are m corresponding arbitrary constants, associated with these stable eigenvalues. Since each of these arbitrary constants is real-valued and hence determined by one* *parameter, these m arbitrary constants can be uniquely determined by precisely m predetermined variables.*

We next consider the case when all stable eigenvalues are complex-valued and hence come in complex conjugate pairs. Then, after elimination of the unstable eigenvalues, the solution to this dynamical system can be written in the form:

$$\begin{pmatrix} y_{1,1}-y_{1,1}^* \\ y_{1,2}-y_{1,2}^* \\ \vdots \\ y_{l_s,1}-y_{l_s,1}^* \\ y_{l_s,2}-y_{l_s,2}^* \\ y_{2l_s+1}-y_{2l_s+1}^* \\ \vdots \\ y_{2l}-y_{2l}^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\bar{\lambda}_1) & \cdots & \mathbf{v}(\lambda_{l_s}) & \mathbf{v}(\bar{\lambda}_{l_s}) \end{bmatrix} \begin{pmatrix} C_1\exp(\lambda_1 t) \\ \bar{C}_1\exp(\bar{\lambda}_1 t) \\ \vdots \\ C_{l_s}\exp(\lambda_{l_s} t) \\ \bar{C}_{l_s}\exp(\bar{\lambda}_{l_s} t) \end{pmatrix} \qquad (17d)$$

The arbitrary constants, the C_i 's, come in complex conjugate pairs with the value of each pair being determined by two arbitrary constants. Hence, in general, we can use the predetermined values of $2l_s$ variables, say $y_{1,1}, y_{1,2}, ..., y_{l_s,1}, y_{l_s,2}$, to determine values for the arbitrary constants, $C_1, \bar{C}_1, ..., C_{l_s}, \bar{C}_{l_s}$. This is the extension of the Blanchard and Kahn results to complex-valued eigenvalues and can be summarized in the following proposition:

Proposition 7: *If there are m complex conjugate pairs of stable eigenvalues (making a total of 2m complex-valued stable eigenvalues) then there are m corresponding complex conjugate pairs of arbitrary constants (making a total of 2m arbitrary constants) associated with these 2m stable eigenvalues. Since the arbitrary constants come in complex conjugate pairs and hence each pair is determined by two parameters, these 2m arbitrary constants can be uniquely determined by precisely 2m predetermined variables.*

The Blanchard and Kahn results also carry over to the situation where there are both real-valued and complex-valued eigenvalues. Then, in order to ensure a unique solution, the same number of predetermined variables will be required as the sum of both stable real-valued and stable complex-valued eigenvalues.

4. EXAMPLE IN DISCRETE TIME

The basic model of Section 2 can be rewritten in discrete time as follows:

$$p_{t+1} - p_t = \alpha_1(x_t - x^*) + \frac{\alpha_2}{\beta}(\overline{m} - p_t) \tag{18a}$$

$$e_{t+1} - e_t = -\frac{1}{\beta}(\overline{m} - p_t) \tag{18b}$$

$$x_{t+1} - x_t = \eta[\delta(e_t - p_t) - x_t] \tag{18c}$$

This can be expressed in matrix form as:

$$\begin{pmatrix} p_{t+1} - p^* \\ e_{t+1} - e^* \\ x_{t+1} - x^* \end{pmatrix} = \mathbf{B} \begin{pmatrix} p_t - p^* \\ e_t - e^* \\ x_t - x^* \end{pmatrix} \tag{19a}$$

where an asterisk denotes the corresponding steady-state value, and where

$$\mathbf{B} = \begin{pmatrix} 1 - \frac{\alpha_2}{\beta} & 0 & \alpha_1 \\ \frac{1}{\beta} & 1 & 0 \\ -\eta\delta & \eta\delta & 1-\eta \end{pmatrix} \tag{19b}$$

We can now proceed as we did for the continuous-time case. First we consider the general case when $\beta < \infty$ where the characteristic equation is given by:

$$c(\lambda) = \begin{vmatrix} 1 - \frac{\alpha_2}{\beta} - \lambda & 0 & \alpha_1 \\ \frac{1}{\beta} & 1-\lambda & 0 \\ -\eta\delta & \eta\delta & 1-\eta-\lambda \end{vmatrix} \tag{20}$$

We first focus on the limiting case as $\beta \to \infty$ where the characteristic equation satisfies:

$$c(\lambda) = \begin{vmatrix} 1-\lambda & 0 & \alpha_1 \\ 0 & 1-\lambda & 0 \\ -\eta\delta & \eta\delta & 1-\eta-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(1-\eta-\lambda) + \eta\alpha_1\delta(1-\lambda)$$

$$= (1-\lambda)[\lambda^2 + (\eta-2)\lambda + (1-\eta+\eta\alpha_1\delta)] \qquad \text{(21a)}$$

Hence, as $\beta \to \infty$, the eigenvalues of the system are given by:

$$\lambda_1 = 1; \text{ and } \lambda_2, \lambda_3 = \frac{2-\eta \pm \sqrt{\eta^2 - 4\eta\alpha_1\delta}}{2}. \qquad \text{(21b)}$$

It will again be observed that if the eigenvalues are complex-valued then they will come as a complex conjugate pair. The last two eigenvalues, λ_2 and λ_3, are complex-valued whenever $\eta^2 - 4\eta\alpha_1\delta < 0$, that is whenever $\eta < 4\alpha_1\delta$. In discrete-time models, an eigenvalue is associated with stable dynamics if and only if the absolute value of the eigenvalue is strictly less than one. In the case when λ_2 and λ_3 are complex valued, their absolute values are also strictly less than one whenever $\frac{(2-\eta)^2 + 4\eta\alpha_1\delta - \eta^2}{4} = 1 - \eta + \eta\alpha_1\delta < 1$, that is, whenever $\alpha_1\delta < 1$. Hence, as $\beta \to \infty$, λ_2 and λ_3 are stable and complex-valued whenever $\eta < 4\alpha_1\delta < 4$. This means that stable, complex-valued eigenvalues are also likely to arise under appropriate parameter configurations when $\beta < \infty$.

Table 2
Values of Eigenvalues for Different Values of η: Example in Discrete Time
(Unstable eigenvalues are preceded by an asterisk)

η	λ_1	λ_2	λ_3
0.25	*1.52	0.37-0.61i	0.37+0.61i
0.5	*1.63	0.18-0.78i	0.18+0.78i
1.0	*1.74	-0.12-0.95i	-0.12+0.95i
2.0	*1.84	*-0.67-1.01i	*-0.67+1.01i
4.0	*1.92	*-1.10	*-2.31

We can investigate this further by applying the same parameter values as were used to generate Table 1 and considering the eigenvalues generated as η is allowed to vary. The results in Table 2 demonstrate that, in the discrete time case, it is possible to generate unstable eigenvalues when η is chosen too large. However it is still possible to generate stable complex-valued eigenvalues over a range of parameter values and, once again, these complex-valued eigenvalues come in complex conjugate pairs.

The rest of the analysis is restricted to the case where there is one unstable real-valued eigenvalue (λ_1) and two stable complex-valued eigenvalues $(\lambda_2$ and $\lambda_3)$. Irrespective of whether the λ_i 's are real-valued or complex-valued, the eigenvectors of the system are given by:

$$\mathbf{v}(\lambda_i) = \begin{pmatrix} \alpha_1\beta(\lambda_i - 1) \\ \alpha_1 \\ (\lambda_i - 1)[\alpha_2 + \beta\lambda_i - \beta] \end{pmatrix} \tag{22}$$

and the general solution to the system is given by:

$$\begin{pmatrix} p_t - p^* \\ e_t - e^* \\ x_t - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \mathbf{v}(\lambda_3) \end{bmatrix} \begin{pmatrix} C_1(\lambda_1)^t \\ C_2(\lambda_2)^t \\ C_3(\lambda_3)^t \end{pmatrix} \tag{23a}$$

where C_1, C_2 and C_3 are yet-to-be-determined constants, which may be real-valued or may be complex-valued.

Since λ_1 has absolute value greater than one and so is unstable, the stable solution is given by setting $C_1 = 0$ so that:

$$\begin{pmatrix} p_t - p^* \\ e_t - e^* \\ x_t - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_2) & \mathbf{v}(\lambda_3) \end{bmatrix} \begin{pmatrix} C_2(\lambda_2)^t \\ C_3(\lambda_3)^t \end{pmatrix} \tag{23b}$$

In Section 2, we discussed how, whenever λ_2 and λ_3 are complex-valued, then they come as a complex conjugate pair and the corresponding eigenvectors also come as a complex conjugate pair. Assuming that $\lambda_2 = -\mu_a + i\mu_b$ and that $\lambda_3 = -\mu_a - i\mu_b$, we can write λ_2 and λ_3 in the form:

$$\lambda_2 = \rho\exp(i\theta) = \rho(\cos\theta + i\sin\theta) \tag{24a}$$

and

$$\lambda_3 = \rho\exp(-i\theta) = \rho(\cos\theta - i\sin\theta) \tag{24b}$$

where $\rho = \sqrt{\mu_a^2 + \mu_b^2}$ and $\theta = \arctan\left(\frac{\mu_b}{\mu_a}\right)$

Then,

$$(\lambda_2)^t = \rho^t\exp(i\theta t) = \rho^t(\cos\theta t + i\sin\theta t) \tag{25a}$$

$$(\lambda_3)^t = \rho^t\exp(-i\theta t) = \rho^t(\cos\theta t - i\sin\theta t) \tag{25b}$$

So that $(\lambda_2)^t$ and $(\lambda_3)^t$ are also complex conjugates. By a similar argument to the continuous-time example, it is necessary and sufficient that C_2 and C_3 also form a complex conjugate pair in order for the solutions to *p*, *e* and *x* to be real-valued.

Hence, the solution can be written in the form:

$$\begin{pmatrix} p_t - p^* \\ e_t - e^* \\ x_t - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_2) & \mathbf{v}(\bar{\lambda}_2) \end{bmatrix} \begin{pmatrix} (D_a + iD_b)(\lambda_2)^t \\ (D_a - iD_b)(\bar{\lambda}_2)^t \end{pmatrix} \tag{26}$$

As in the continuous time case, the constants, D_a and D_b, can then be chosen consistent with the initial values of *p* and *x*, ensuring a unique solution. Once again, as a consequence of the complex eigenvalues, this solution will have cyclic properties. It is important to note that, as for the continuous-time case, the Blanchard and Kahn results only carry through in the discrete time case with complex-valued eigenvalues because of the requirement that, in order for the endogenous variables, *p*, *e* and *x*, to be real-valued, it is necessary and sufficient for the coefficients, C_2 and C_3, to form a complex conjugate pair.

5. GENERAL CASE IN DISCRETE TIME

Now consider the general discrete-time linear dynamical system of the form:

$$\underset{nx1}{\mathbf{y}_{t+1}} - \mathbf{y}^* = \underset{nxn}{\mathbf{M}}(\underset{nx1}{\mathbf{y}_t} - \mathbf{y}^*) \qquad (27)$$

Once again, we assume that the n eigenvalues of **M** are all distinct so that the corresponding eigenvectors are also distinct and linearly independent.

The properties of complex-valued eigenvalues described in Propositions 1 and 2 remain relevant and valid. In order to accommodate the way that eigenvalues come explicitly into the solution in the discrete case, Proposition 3 is replaced by Proposition 3':

Proposition 3': *If λ and $\bar{\lambda}$ are a complex conjugate pair of complex-valued numbers, then $(\lambda)^t$ and $(\bar{\lambda})^t$ also form a complex conjugate pair of complex-valued numbers.*

Assume that **M** has *k* real-valued eigenvalues and *2l* complex-valued eigenvalues, making a total of $n = k + 2l$ distinct eigenvalues and eigenvectors. Then the general solution to the system of equations given by (27) is of the form:

$$\begin{pmatrix} y_{1,t}-y_1^* \\ \vdots \\ y_{k,t}-y_k^* \\ y_{k+1,1,t}-y_{k+1,1}^* \\ y_{k+1,2,t}-y_{k+1,2}^* \\ \vdots \\ y_{k+l,1,t}-y_{k+l,1}^* \\ y_{k+l,2,t}-y_{k+l,2}^* \end{pmatrix} =$$

$$\begin{bmatrix} \mathbf{v}(\lambda_1) & \cdots & \mathbf{v}(\lambda_k) & \mathbf{v}(\lambda_{k+1}) & \mathbf{v}(\bar{\lambda}_{k+1}) & \cdots & \mathbf{v}(\lambda_{k+l}) & \mathbf{v}(\bar{\lambda}_{k+l}) \end{bmatrix} \begin{pmatrix} C_1(\lambda_1)^t \\ \vdots \\ C_k(\lambda_k)^t \\ C_{k+1,1}(\lambda_{k+1})^t \\ C_{k+1,2}(\bar{\lambda}_{k+1})^t \\ \vdots \\ C_{k+l,1}(\lambda_{k+l})^t \\ C_{k+l,2}(\bar{\lambda}_{k+l})^t \end{pmatrix}$$

(28a)

We can associate a coefficient to a particular eigenvalue in an analogous way to that defined in Definition 1. Propositions 4 and 5 continue to remain valid so that the coefficients associated with real-valued eigenvalues are real-valued and the coefficients associated with complex-valued eigenvalues come in complex conjugate pairs. The last matrix in equation (28a) can then be rewritten as:

$$\begin{pmatrix} C_1(\lambda_1)^t \\ \vdots \\ C_k(\lambda_k)^t \\ C_{k+1}(\lambda_{k+1})^t \\ \bar{C}_{k+1}(\bar{\lambda}_{k+1})^t \\ \vdots \\ C_{k+l}(\lambda_{k+l})^t \\ \bar{C}_{k+l}(\bar{\lambda}_{k+l})^t \end{pmatrix} \qquad (28b)$$

Stability is defined differently for discrete-time systems than it is for continuous-time systems as follows:

Definition 2': *Consider the dynamical system in discrete time given by* $\mathbf{y}_{t+1} - \mathbf{y}^* = \mathbf{M}(\mathbf{y}_t - \mathbf{y}^*)$. *If* λ *is an eigenvalue of* ***M****, then we say that* λ *is a stable eigenvalue if the absolute value of* λ *is strictly less than 1. We say that* λ *is an unstable eigenvalue if the absolute value of* λ *is strictly greater than 1.*

Once again, in order to consider the case of an unanticipated shock, all unstable solutions can be eliminated from equations (28a-28b) by setting the appropriate coefficients to zero. Then, as for the continuous-time case, we can consider separately the cases when, firstly, all stable eigenvalues are real-valued and, secondly, when all stable eigenvalues are complex-valued. This yields Propositions 6 and 7 as in the continuous-time case. In each case, a unique solution is achieved by choosing the same number of predetermined variables as there are stable eigenvalues. The Blanchard and Kahn results also carry over to the discrete-time case where there is a combination of both real-valued and complex-valued eigenvalues.

6. CONCLUSION

Blanchard and Kahn have shown that a linear dynamical system can have a unique solution if it has the same number of predetermined variables as it has stable eigenvalues and, hence, if it has the same number of "jump" variables as it has unstable eigenvalues. In this paper, we have established that these results also carry through for linear dynamical systems where some of the eigenvalues are complex-valued. We have shown that the crucial reason why the results continue to hold for complex-valued eigenvalues is because, in order to ensure that the solutions for the endogenous variables are real-valued and thus have an economic interpretation, the coefficients associated with each complex conjugate pair of eigenvalues must also come in complex conjugate pairs.

Since the conclusion of this paper supports the Blanchard and Kahn results it is reasonable to ask whether any reasonable interpretation of their paper could be consistent with them having demonstrated their result for complex-valued eigenvalues. Examination of Blanchard and Kahn's results shows that at no point did they explicitly address the possibility that the eigenvalues could be complex-valued. Furthermore, if we make the most generous interpretation of their analysis and assume that the eigenvalues and corresponding coefficients could be interpreted as complex-valued, then at no point do they demonstrate that their results only carry through in the complex case when the coefficients come in complex conjugate pairs. Hence, it is fair to conclude that Blanchard and Kahn's paper only satisfactorily proved their results for the case when all eigenvalues are real-valued.

This paper makes a contribution by using an extension of the Dornbusch (1976) model to provide a specific example of the calculation of "jumps" when eigenvalues are real-valued and then by explicitly demonstrating how the Blanchard and Kahn conclusions can be justified for dynamical systems with complex-valued eigenvalues. Examples with just one complex conjugate pair of stable eigenvalues and a general n-dimensional model have been presented for both the continuous-time and discrete-time cases. Readers should note that, while the results presented here are for the case of a sudden unanticipated shock, it is also possible to show, using a similar analysis to that presented here, that corresponding results carry through for an anticipated shock, which is announced now, but does not occur until some time in the future.

APPENDIX

There is one crucial idea required to establish that the Blanchard and Kahn results carry through to dynamic systems with complex-valued eigenvalues: the coefficients associated with a complex conjugate pair of eigenvalues must themselves be complex conjugates. This idea is summarized in Propositions 4 and 5. In this Appendix we establish these propositions for the general (n-dimensional) case.

Consider first the case where the solution has *k* real-valued eigenvalues and *2l* complex-valued eigenvalues and so is of the form given by equation (17a). Then the following vector is real valued:

$$\mathbf{u_h} = \exp(\lambda_h)\mathbf{v}(\lambda_h)\,, \quad \text{for } h = 1, 2, ..., k \tag{A.1a}$$

Furthermore, the following vectors are complex-valued

$$\mathbf{w_{aj}} + i\mathbf{w_{bj}} = \exp(\lambda_{k+j}t)\mathbf{v}(\lambda_{k+j}) \tag{A.1b}$$

and

$$\mathbf{w_{aj}} - i\mathbf{w_{bj}} = \exp(\overline{\lambda}_{k+j}t)\mathbf{v}(\overline{\lambda}_{k+j})\,, \quad \text{for } j = 1, 2, ..., l \tag{A.1c}$$

Furthermore, let

$$C_h = C_{ha} + iC_{hb}\,, \qquad \text{for } h = 1, 2, ..., k \tag{A.2a}$$

$$C_{k+j,1} = C_{j1a} + iC_{j1b}\,, \quad \text{for } j = 1, 2, ..., l \tag{A.2b}$$

$$C_{k+j,2} = C_{j2a} + iC_{j2b}\,, \quad \text{for } j = 1, 2, ..., l \tag{A.2c}$$

Then, in order that the solutions to the dynamical system are real-valued, it must be the case that the imaginary part of the following equation is zero:

$$\sum_{h=1}^{k}\mathbf{u_h}(C_{ha} + iC_{hb}) + \sum_{j=1}^{l}\left[(\mathbf{w_{aj}} + i\mathbf{w_{bj}})(C_{j1a} + iC_{j1b}) + (\mathbf{w_{aj}} - i\mathbf{w_{bj}})(C_{j2a} + iC_{j2b})\right] \tag{A.3a}$$

Hence,

$$\sum_{h=1}^{k}\mathbf{u_h}C_{hb} + \sum_{j=1}^{l}\left[\mathbf{w_{aj}}(C_{j1b} + C_{j2b}) + \mathbf{w_{bj}}(C_{j1a} - C_{j2a})\right] = \mathbf{0} \tag{A.3b}$$

Since $\mathbf{w_{aj}}$ and $\mathbf{w_{bj}}$ are linear combinations (with complex-valued coefficients) of $\mathbf{v}(\lambda_{k+j})$ and $\mathbf{v}(\overline{\lambda}_{k+j})$ it follows from the linear independence of $\mathbf{v}(\lambda_h)$, $\mathbf{v}(\lambda_{k+j})$ and $\mathbf{v}(\overline{\lambda}_{k+j})$ that:

$$C_{hb} = 0, \qquad \text{for } h = 1, 2, ..., k \qquad \text{(A.4a)}$$

$$C_{j1b} + C_{j2b} = 0, \qquad \text{for } j = 1, 2, ..., l \qquad \text{(A.4b)}$$

and

$$C_{j1a} - C_{j2a} = 0, \qquad \text{for } j = 1, 2, ..., l \qquad \text{(A.4c)}$$

In other words, to ensure real-valued solutions for all the endogenous variables, it is necessary and sufficient that C_h is real-valued for $h = 1, 2, ..., k$ and also that $C_{k+j,1}$ and $C_{k+j,2}$ form a complex conjugate pair and so are of the form:

$$C_{k+j,1} = D_{aj} + iD_{bj} \qquad \text{for } j = 1, 2, ..., l \qquad \text{(A.5a)}$$

$$C_{k+j,2} = D_{aj} - iD_{bj} \qquad \text{for } j = 1, 2, ..., l \qquad \text{(A.5b)}$$

This establishes Propositions 4 and 5 for the General Case in continuous time. Similar Propositions can be established for the General Case in discrete time.

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