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Implementation in Mixed Nash Equilibrium[*]

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Abstract

A mechanism implements a social choice correspondence f in mixed Nash equilibrium if, at any preference profile, the set of *all* (pure and mixed) Nash equilibrium outcomes coincides with the set of f-optimal alternatives for all cardinal representations of the preference profile. Unlike Maskin's definition, our definition does not require each optimal alternative to be the outcome of a *pure* equilibrium. We show that set-monotonicity, a weakening of Maskin's monotonicity, is necessary for mixed Nash implementation. With at least three players, set-monotonicity and no-veto power are sufficient. Important correspondences that are not Maskin monotonic can be implemented in mixed Nash equilibrium.

Keywords: Implementation, Maskin monotonicity, pure and mixed Nash equilibrium, set-monotonicity, social choice correspondence.

JEL Classification Numbers: C72; D71.

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1 Introduction

This paper studies the problem of implementation in mixed Nash equilibrium. According to our definition, a mechanism implements an ordinal social choice correspondence f in mixed Nash equilibrium if, at any preference profile, the set of all (pure and mixed) equilibrium outcomes corresponds to the set of f-optimal alternatives for all cardinal representations of the preference profile. Crucially, and unlike the classical definition of implementation, this definition of implementation does not give a predominant role to pure equilibria: an f-optimal alternative does not have to be the outcome of a pure Nash equilibrium. At the same time, we maintain an entirely ordinal approach. We assume that a social choice correspondence f maps profiles of preference orderings over alternatives into subsets of alternatives (not lotteries) and we require that a given mechanism implements f irrespective of which cardinal representation is chosen.

Most of the existing literature on Nash implementation does not consider equilibria in mixed strategies (see Jackson, 2001, and Maskin and Sjöström, 2002, for excellent surveys).[1] Perhaps, the emphasis on pure equilibria expresses a discomfort with the classical view of mixing as deliberate randomizations on the part of players. However, it is now accepted that even if players do not randomize but choose definite actions, a mixed strategy may be viewed as a representation of the other players' uncertainty about a player's choice (e.g., see Aumann and Brandenburger, 1995). Moreover, almost all mixed equilibria can be viewed as pure Bayesian equilibria of nearby games of incomplete information, in which players are uncertain about the exact profile of preferences, as first suggested in the seminal work of Harsanyi (1973). This view acknowledges that games with commonly known preferences are an idealization, a limit of near-complete information games. This interpretation is particularly important for the theory of implementation in Nash equilibrium, whereby the assumption of common knowledge of preferences, especially on large domains, is at best a simplifying assumption.

Furthermore, recent evidence in the experimental literature suggests that equilibria in mixed strategies are good predictors of behavior in some classes of games e.g., coordination games and chicken games (see chapters 3 and 7 of Camerer, 2003). Since, for some preference profiles, a mechanism can induce one of those games, paying attention to mixed equilibria is important if we want to describe or predict players' behavior. While we find no compelling reasons to

[1] Two notable exceptions are Maskin (1999) for Nash implementation and Serrano and Vohra (2010) for Bayesian implementation. These authors do consider mixed equilibria, but still require each f-optimal alternative to be the outcome of a *pure* equilibrium; pure equilibria are given a special status. While Maskin (1999) shows that eliminating unwanted mixed strategy equilibria imposes no additional restriction to Nash implementation, Serrano and Vohra (2010) show that significant additional restrictions are required to implement social choice correspondences in Bayesian equilibrium.

give pure Nash equilibria a special status, we follow an ordinal approach because we believe it imposes a welcome degree of robustness on society's preferences and the mechanism adopted. Thus, we require that the set of f-optimal outcomes only depends on players' ordinal preferences, and that the mechanism adopted implements f for all possible cardinal representations of those ordinal preferences.

Our definition of mixed Nash implementation yields novel insights. We demonstrate that the condition of Maskin monotonicity is not necessary for full implementation in mixed Nash equilibrium. Intuitively, consider a profile of preferences and an alternative, say a, that is f-optimal at that profile of preferences. According to Maskin's definition of implementation, there must exist a pure Nash equilibrium with equilibrium outcome a. Thus, any alternative a player can obtain by unilateral deviations must be less preferred than a. Now, if we move to another profile of preferences where a does not fall down in the players' ranking, then a remains an equilibrium outcome and must be f-optimal at that new profile of preferences. This is the intuition behind the necessity of Maskin monotonicity for Nash implementation. Unlike Maskin's definition of implementation, our definition does not require a to be a pure equilibrium outcome. So, suppose that there exists a mixed equilibrium with a as an equilibrium outcome.[2] The key observation to make is that the mixed equilibrium induces a *lottery* over optimal alternatives. Thus, when we move to another profile of preferences where a does not fall down in the players' ranking, the original profile of strategies does not have to be an equilibrium at the new state. In fact, we show that a much weaker condition, *set-monotonicity*, is necessary for implementation in mixed Nash equilibrium. Set-monotonicity states that the set $f(\theta)$ of optimal alternatives at state θ is included in the set $f(\theta')$ of optimal alternatives at state θ' whenever, for all players, either all alternatives in $f(\theta)$ are top-ranked at state θ', or the weak *and* strict lower contour sets at state θ of *all* alternatives in $f(\theta)$ are included in their respective weak and strict lower contour sets at state θ'. Moreover, we show that set-monotonicity and no-veto power are sufficient for implementation of social choice correspondences in environments with at least three players.

To substantiate our claim that set-monotonicity is a substantially weaker requirement than Maskin monotonicity, we show that the strong Pareto and the strong core correspondences are set-monotonic on the domain of single-top preferences, while they are not Maskin monotonic. Similarly, on the domain of strict preferences, the top-cycle correspondence is set-monotonic, but not Maskin monotonic.

The next section illustrates our approach with an example. Section 3 defines mixed Nash implementation. Section 4 defines set-monotonicity and shows that it is necessary for mixed

[2]More precisely, let σ^* be the mixed Nash equilibrium and $\mathbb{P}_{\sigma^*,g}$ the distribution over alternatives induced by the strategy profile σ^* and the allocation rule g. Then a belongs to the support of $\mathbb{P}_{\sigma^*,g}$.

Nash implementation. Section 5 proves that set-monotonicity and no-veto power are sufficient with at least three players. Section 6 builds a bridge between our approach and the standard approach with ordinal preferences. Section 7 concludes by applying our results to some well known social choice correspondences.

2 A Simple Example

Example 1 There are two players, 1 and 2, two states of the world, θ and θ', and four alternatives, a, b, c, and d. Players have state-dependent preferences represented in the table below. For instance, player 1 ranks b first and a second in state θ, while a is ranked first and b last in state θ'. Preferences are strict.

θ		θ'	
1	2	1	2
b	c	a	c
a	a	d	d
c	b	c	a
d	d	b	b

The designer aims to implement the social choice correspondence f, with $f(\theta) = \{a\}$ and $f(\theta') = \{a, b, c, d\}$. We say that alternative x is f-optimal at state θ if $x \in f(\theta)$.

We first argue that the social choice correspondence f is not implementable in the sense of Maskin (1999). Maskin's definition of Nash implementation requires that for each f-optimal alternative at a given state, there exists a *pure* Nash equilibrium (of the game induced by the mechanism) corresponding to that alternative. So, for instance, at state θ', there must exist a pure Nash equilibrium with b as equilibrium outcome. Maskin requires, furthermore, that no such equilibrium must exist at state θ. However, if there exists a pure equilibrium with b as equilibrium outcome at state θ', then b will also be an equilibrium outcome at state θ, since b moves up in every players' ranking when going from state θ' to state θ. Thus, the correspondence f is not implementable in the sense of Maskin. In other words, the social choice correspondence f violates Maskin monotonicity, a necessary condition for implementation in the sense of Maskin.

In contrast with Maskin, we do *not* require that for each f-optimal alternative at a given state, there exists a pure Nash equilibrium corresponding to that alternative. We require instead that the set of f-optimal alternatives coincides with the set of *mixed* Nash equilibrium outcomes. So, at state θ', there must exist a mixed Nash equilibrium with b corresponding to an action profile in the support of the equilibrium. With our definition of implementation, the correspondence f is implementable. To see this, consider the mechanism where each player has two messages m_1 and m_2, and the allocation rule is represented in the table below. (Player 1 is the row player.) For example, if both players announce m_1, the chosen alternative is a.

	m_1	m_2
m_1	a	b
m_2	d	c

At state θ, (m_1, m_1) is the unique Nash equilibrium, with outcome a. At state θ', both (m_1, m_1) and (m_2, m_2) are pure Nash equilibria, with outcomes a and c. Moreover, there exists a mixed Nash equilibrium that puts strictly positive probability on each action profile (since preferences are strict), hence on each outcome.[3] Therefore, f is implementable in mixed Nash equilibrium regardless of the cardinal representation chosen for the two players, although it is not implementable in the sense of Maskin.

We conclude this section with an important observation. Alternative d is f-optimal at state θ', and it moves down in player 1's ranking when moving from θ' to θ. This preference reversal guarantees the set-monotonicity of the correspondence f, which, as we shall see, is a necessary condition for implementation in mixed Nash equilibrium.

3 Preliminaries

An environment is a triplet $\langle N, X, \Theta \rangle$ where $N := \{1, \ldots, n\}$ is a set of n players, X a finite set of alternatives, and Θ a finite set of states of the world.[4] Associated with each state θ is a preference profile $\succcurlyeq^\theta := (\succcurlyeq_1^\theta, \ldots, \succcurlyeq_n^\theta)$, where $\succcurlyeq_i^\theta$ is player i's preference relation over X at state θ. The asymmetric and symmetric parts of $\succcurlyeq_i^\theta$ are denoted $\succ_i^\theta$ and $\sim_i^\theta$, respectively.

We denote with $L_i(x, \theta) := \{y \in X : x \succcurlyeq_i^\theta y\}$ player i's lower contour set of x at state θ, and $SL_i(x, \theta) := \{y \in X : x \succ_i^\theta y\}$ the strict lower contour set. For any (i, θ) in $N \times \Theta$ and $Y \subseteq X$, define $\max_i^\theta Y$ as $\{x \in Y : x \succcurlyeq_i^\theta y \text{ for all } y \in Y\}$.

We assume that any preference relation $\succcurlyeq_i^\theta$ is representable by a utility function $u_i(\cdot, \theta) : X \to \mathbb{R}$, and that each player is an expected utility maximizer. We denote with $\mathcal{U}_i^\theta$ the set of all possible cardinal representations $u_i(\cdot, \theta)$ of $\succcurlyeq_i^\theta$ at state θ, and let $\mathcal{U}^\theta := \times_{i \in N} \mathcal{U}_i^\theta$.

A social choice correspondence $f : \Theta \to 2^X \setminus \{\emptyset\}$ associates with each state of the world θ, a non-empty subset of alternatives $f(\theta) \subseteq X$. Two classic conditions for Nash implementation are

[3] Note that at state θ, a is the unique rationalizable outcome, while all outcomes are rationalizable at state θ', so that f is implementable in rationalizable outcomes. See Bergemann, Morris and Tercieux (2011) for the study of social choice *functions* implementable in rationalizable outcomes.

[4] In some applications (e.g., exchange economies) it is natural to work with more general outcome spaces. As in Abreu and Sen (1991), our results extend to the case in which X is a separable metric space and the social choice correspondence maps Θ into a countable, dense subset of X.

Maskin monotonicity and no-veto power. A social choice correspondence f is *Maskin monotonic* if for all (x, θ, θ') in $X \times \Theta \times \Theta$ with $x \in f(\theta)$, we have $x \in f(\theta')$ whenever $L_i(x, \theta) \subseteq L_i(x, \theta')$ for all $i \in N$. Maskin monotonicity is a necessary condition for Nash implementation (à la Maskin). A social choice correspondence f satisfies *no-veto power* if for all $\theta \in \Theta$, we have $x \in f(\theta)$ whenever $x \in \max_i^\theta X$ for all $i \in N^* \subseteq N$, with N^* having at least $n-1$ elements. Maskin monotonicity and no-veto power are sufficient conditions for Nash implementation (à la Maskin) when there are at least three players.

For any subset $Y \subseteq X$, let $\Delta(Y)$ be the set of all probability measures over Y. We view $\Delta(Y)$ as a subset of $\Delta(X)$ with the property that $P(x) = 0$ for all $x \in X \setminus Y$ if $P \in \Delta(Y)$. A mechanism (or game form) is a pair $\langle (M_i)_{i \in N}, g \rangle$ with M_i the set of messages of player i, and $g : \times_{i \in N} M_i \to \Delta(X)$ the allocation rule. Let $M := \times_{j \in N} M_j$ and $M_{-i} := \times_{j \in N \setminus \{i\}} M_j$, with m and m_{-i} generic elements.

A mechanism $\langle (M_i)_{i \in N}, g \rangle$, a state θ and a profile of cardinal representations $(u_i(\cdot, \theta))_{i \in N}$ of $(\succcurlyeq_i^\theta)_{i \in N}$ induce a strategic-form game as follows. There is a set N of n players. The set of pure actions of player i is M_i, and player i's expected payoff when he plays m_i and his opponents play m_{-i} is

$$U_i(g(m_i, m_{-i}), \theta) := \sum_{x \in X} g(m_i, m_{-i})(x) u_i(x, \theta),$$

where $g(m_i, m_{-i})(x)$ is the probability that x is chosen by the mechanism when the profile of messages (m_i, m_{-i}) is announced. The induced strategic-form game is thus $G(\theta, u) := \langle N, (M_i, U_i(g(\cdot), \theta))_{i \in N} \rangle$. Let σ be a profile of mixed strategies. We denote with $\mathbb{P}_{\sigma, g}$ the probability distribution over alternatives in X induced by the allocation rule g and the profile of mixed strategies σ.[5]

Definition 1 *The mechanism $\langle (M_i)_{i \in N}, g \rangle$ implements the social choice correspondence f in mixed Nash equilibrium if for all $\theta \in \Theta$, for all cardinal representations $u(\cdot, \theta) \in \mathcal{U}^\theta$ of $\succcurlyeq^\theta$, the following two conditions hold:*

(i) For each $x \in f(\theta)$, there exists a Nash equilibrium σ^* of $G(\theta, u)$ such that x is in the support of $\mathbb{P}_{\sigma^*, g}$, and

(ii) if σ is a Nash equilibrium of $G(\theta, u)$, then the support of $\mathbb{P}_{\sigma, g}$ is included in $f(\theta)$.

It is important to contrast our definition of implementation in mixed Nash equilibrium with Maskin (1999) definition of Nash implementation. First, part (i) of Maskin's definition requires that for each $x \in f(\theta)$, there exists a pure Nash equilibrium m^* of $G(\theta, u)$ with equilibrium outcome x, while part (ii) of his definition is identical to ours. In contrast with Maskin, we allow

[5] Formally, the probability $\mathbb{P}_{\sigma, g}(x)$ of $x \in X$ is $\sum_{m \in M} \sigma(m) g(m)(x)$ if M is countable. If M is uncountable, a similar expression applies.

for mixed strategy Nash equilibria in part (i) and, thus, restore a natural symmetry between parts (i) and (ii). Yet, our definition respects the spirit of full implementation in that only optimal outcomes can be observed by the designer as equilibrium outcomes.

Second, as in Maskin, our concept of implementation is ordinal as all equilibrium outcomes have to be optimal, regardless of the cardinal representation chosen. Also, our approach parallels the approach of Gibbard (1977). Gibbard characterizes the set of strategy-proof probabilistic social choice functions (i.e., mappings from profiles of preferences to lotteries over outcomes). Importantly to us, Gibbard requires each player to have an incentive to truthfully reveal his preference, regardless of the cardinal representation chosen to evaluate lotteries (and announcements of others).[6]

Third, we allow the designer to use mechanisms that randomize among the optimal alternatives in equilibrium. This is a natural assumption given that players can use mixed strategies. Although a random mechanism introduces uncertainty about the alternative to be chosen, the concept of mixed Nash equilibrium already encapsulates the idea that players are uncertain about the messages sent to the designer and, consequently, about the alternative to be chosen.[7]

Finally, from our definition of mixed Nash implementation, it is immediate to see that if a social choice correspondence is Nash implementable (i.e., à la Maskin), then it is implementable in mixed Nash equilibrium. The converse is false, as shown by Example 1. The goal of this paper is to characterize the social choice correspondences implementable in mixed Nash equilibrium. The next section provides a necessary condition.

4 A Necessary Condition

In this section, we introduce a new condition, called *set-monotonicity*, which we show to be necessary for the implementation of social choice correspondences in mixed Nash equilibrium.

Definition 2 *A social choice correspondence f is set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $i \in N$, one of the following two conditions holds: either (1) $f(\theta) \subseteq \max_i^{\theta'} X$ or (2) for all $x \in f(\theta)$, (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq$*

[6] See also Barberà et al. (1998) and Abreu and Sen (1991) for further discussions of the ordinal approach.

[7] In the literature on (exact) Nash implementation, Benoît and Ok (2008) and Bochet (2007) have studied mechanisms in which randomization by the designer can only occur out of equilibrium; unlike us, they do not attempt to rule out mixed strategy equilibria with undesirable outcomes. (See also Vartiainen, 2007, for the use of random mechanisms in the implementation of correspondences in (pure) subgame perfect equilibrium.) Our approach also differs from the use of random mechanisms in the literature on virtual implementation (e.g., see Matsushima, 1998, and Abreu and Sen, 1991), which heavily exploits the possibility of selecting undesirable alternatives with positive probability in equilibrium.

$SL_i(x, \theta')$.[8]

Set-monotonicity is a weakening of Maskin monotonicity. As the state changes from θ to θ', it restricts the set $f(\theta')$ to contain the set $f(\theta)$ only when, for all players, *all* alternatives in $f(\theta)$ either are top ranked at θ' or they do not move down in the weak and strict rankings. Maskin monotonicity, on the contrary, restricts $f(\theta')$ to include each *single* alternative $x \in f(\theta)$ which does not move down in any player's weak ranking when moving from θ to θ'.

Theorem 1 *If the social choice correspondence f is implementable in mixed Nash equilibrium, then it satisfies set-monotonicity.*

Proof We begin by proving the following claim.

Claim C. Suppose $L_i(x, \theta) \subseteq L_i(x, \theta')$ and $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$. Then, given any cardinal representation $u_i(\cdot, \theta)$ of $\succcurlyeq_i^\theta$, there exists a cardinal representation $u_i(\cdot, \theta')$ of $\succcurlyeq_i^{\theta'}$ such that $u_i(x, \theta') \leq u_i(x, \theta)$ for all $x \in X$ and $u_i(x, \theta') = u_i(x, \theta)$ for all $x \in f(\theta)$.

Proof of Claim C. To prove our claim, consider any pair $(x, x') \in f(\theta) \times f(\theta)$ with $x \succcurlyeq_i^\theta x'$. Since $L_i(\hat{x}, \theta) \subseteq L_i(\hat{x}, \theta')$ for all $\hat{x} \in f(\theta)$, we have that $x \succ_i^\theta x'$ implies $x \succ_i^{\theta'} x'$ and $x \sim_i^\theta x'$ implies $x \sim_i^{\theta'} x'$. Hence, we can associate with each alternative in $f(\theta)$ the same utility at θ' as at θ. Now, fix an alternative $x \in f(\theta)$ and consider $y \in L_i(x, \theta)$. Since $L_i(x, \theta) \subseteq L_i(x, \theta')$, we must have $u_i(y, \theta') \leq u_i(x, \theta') = u_i(x, \theta)$. If $x \sim_i^\theta y$, then we can choose $u_i(y, \theta') \leq u_i(y, \theta) = u_i(x, \theta)$. If $x \succ_i^\theta y$, then we must have $x \succ_i^{\theta'} y$ since $SL_i(x, \theta) \subseteq SL_i(x, \theta')$; we can therefore choose $u_i(y, \theta')$ in the open set $(-\infty, u_i(y, \theta))$ and still represent $\succcurlyeq_i^{\theta'}$ by $u_i(\cdot, \theta')$. Finally, if $y \notin \cup_{x \in f(\theta)} L_i(x, \theta)$, we have that $u_i(y, \theta) > u_i(x, \theta)$ for all $x \in f(\theta)$. If $y \in L_i(x, \theta')$ for some $x \in f(\theta)$, then we can set $u_i(y, \theta') \leq u_i(x, \theta') = u_i(x, \theta) \leq \max_{x' \in f(\theta)} u_i(x', \theta) < u_i(y, \theta)$. If $y \notin \cup_{x \in f(\theta)} L_i(x, \theta')$, then we can choose $u_i(y, \theta')$ in the open set $(\max_{x' \in f(\theta)} u_i(x', \theta), u_i(y, \theta))$. This concludes the proof of our claim.

The proof proceeds by contradiction on the contrapositive. Assume that the social choice correspondence f does not satisfy set-monotonicity and yet is implementable in mixed Nash equilibrium by the mechanism $\langle M, g \rangle$.

Since f does not satisfy set-monotonicity, there exist x^*, θ, and θ' such that $x^* \in f(\theta) \setminus f(\theta')$, while $L_i(x, \theta) \subseteq L_i(x, \theta')$ and either $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$ or $f(\theta) \subseteq \max_i^{\theta'} X$, for all $i \in N$. Let $N^* := \{i \in N : f(\theta) \nsubseteq \max_i^{\theta'} X\}$.

Since f is implementable and $x^* \in f(\theta)$, for any cardinal representation $u(\cdot, \theta)$ of $\succcurlyeq^\theta$, there exists an equilibrium σ^* of the game $G(\theta, u)$ with x^* in the support of $\mathbb{P}_{\sigma^*, g}$. Furthermore, since

[8] Alternatively, a social choice correspondence f is set-monotonic if $x^* \in f(\theta) \setminus f(\theta')$ implies that there exists a triple (x, x', y) in $f(\theta) \times f(\theta) \times X$ and a player $i \in N$ such that $x' \notin \max_i^{\theta'} X$ and either (1) $x \succcurlyeq_i^\theta y$ and $y \succ_i^{\theta'} x$, or (2) $x \succ_i^\theta y$ and $y \succcurlyeq_i^{\theta'} x$.

$x^* \notin f(\theta')$, for all cardinal representations $u(\cdot,\theta')$ of $\succcurlyeq^{\theta'}$, for all equilibria σ of $G(\theta',u)$, x^* does not belong to the support of $\mathbb{P}_{\sigma,g}$. In particular, this implies that σ^* is not an equilibrium at θ' for all cardinal representations $u(\cdot,\theta')$. Thus, assuming that M is countable, there exist a player i, a message m_i^* in the support of σ_i^*, and a message m_i' such that:[9]

$$\begin{aligned}\sum_{m_{-i}} \left[U_i(g(m_i^*,m_{-i}),\theta) - U_i(g(m_i',m_{-i}),\theta)\right]\sigma_{-i}^*(m_{-i}) &\geq 0, \quad \text{and}\\ \sum_{m_{-i}} \left[U_i(g(m_i^*,m_{-i}),\theta') - U_i(g(m_i',m_{-i}),\theta')\right]\sigma_{-i}^*(m_{-i}) &< 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{m_{-i}} \left[U_i(g(m_i^*,m_{-i}),\theta) - U_i(g(m_i^*,m_{-i}),\theta')\right]\sigma_{-i}^*(m_{-i}) > \\ &\sum_{m_{-i}} \left[U_i(g(m_i',m_{-i}),\theta) - U_i(g(m_i',m_{-i}),\theta')\right]\sigma_{-i}^*(m_{-i})\end{aligned} \tag{1}$$

First, assume that $i \in N^*$. By Claim C, we can construct cardinal representations such that $u_i(x,\theta') \leq u_i(x,\theta)$ for all $x \in X$ and $u_i(x,\theta) = u_i(x,\theta')$ for all $x \in f(\theta)$. Since f is implementable, we have that the support of $\mathbb{P}_{\sigma^*,g}$ is included in $f(\theta)$. Therefore, $U_i(g(m_i^*,m_{-i}),\theta) = U_i(g(m_i^*,m_{-i}),\theta')$ for all m_{-i} in the support of σ_{-i}^*. Hence, the left-hand side of inequality (1) is zero. Furthermore, we have that $U_i(g(m_i',m_{-i}),\theta) \geq U_i(g(m_i',m_{-i}),\theta')$ for all m_{-i}. Hence, the right-hand side of (1) is non-negative, a contradiction.

Second, assume $i \in N \setminus N^*$. Since $f(\theta) \subseteq \max_i^{\theta'} X$, it follows that for all $x \in f(\theta)$, $u_i(x,\theta') \geq u_i(x',\theta')$ for all $x' \in X$, regardless of the cardinalization $u_i(\cdot,\theta')$ of $\succcurlyeq_i^{\theta'}$. Consequently, player i has no profitable deviation at state θ', a contradiction. This completes the proof. □

Note first that Theorem 1 remains valid if we restrict ourselves to deterministic mechanisms, so that set-monotonicity is a necessary condition for implementation in mixed Nash equilibrium, regardless of whether we consider deterministic or random mechanisms. Second, it is easy to verify that set-monotonicity is also a necessary condition for implementation if we require the Nash equilibria to be in pure strategies, but allow random mechanisms. Third, while we have restricted attention to von Neumann-Morgenstern preferences, the condition of set-monotonicity remains necessary if we consider larger classes of preferences that include the von Neumann-Morgenstern preferences. This is because we follow an ordinal approach and require that f be implemented by all admissible preference representations. Fourth, set-monotonicity is related to the concepts of almost monotonicity (Sanver, 2006) and quasimonotonicity (Cabrales and Serrano, 2011). Quasimonotonicity and almost monotonicity restrict f when a *single*

[9] If the mechanism is uncountable, a similar argument holds with appropriate measurability conditions.

alternative in $f(\theta)$ moves up in the rankings of all players. The social choice correspondence f is quasimonotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$ and $x \in f(\theta)$, we have $x \in f(\theta')$ whenever for all $i \in N$, $SL_i(x, \theta) \subseteq SL_i(x, \theta')$. The social choice correspondence f is almost monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$ and $x \in f(\theta)$, we have $x \in f(\theta')$ whenever for all $i \in N$, the following two conditions hold: (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$. On the unrestricted domain of preferences, set-monotonicity is neither weaker nor stronger than quasimonotonicity or almost monotonicity. Note, however, that for social choice *functions*, set monotonicity is weaker than quasimonotonicity and almost monotonicity. For an example of a set-monotonic social choice correspondence that is neither quasimonotonic nor almost monotonic, see Example 1. Conversely, for an example of a quasimonotonic (and almost monotonic) social choice correspondence that is not set-monotonic, see Example 2.

Example 2 There are three players, 1, 2 and 3, two states of the world, θ and θ', and three alternatives a, b and c. Preferences are represented in the table below.

θ			θ'		
1	2	3	1	2	3
b	a	c	b	$a \sim b$	c
c	b	a	c		a
a	c	b	a	c	b

The social choice correspondence is $f(\theta) = \{a\}$ and $f(\theta') = \{b\}$. It is not set-monotonic since $L_i(a, \theta) \subseteq L_i(a, \theta')$ for all $i \in \{1, 2, 3\}$, $SL_i(a, \theta) \subseteq SL_i(a, \theta')$ for all $i \in \{1, 3\}$ and $a \in \max_2^{\theta'}\{a, b, c\}$ and yet $a \notin f(\theta')$. However, it is quasimonotonic and almost monotonic.

Yet, if $\max_i^{\theta} X$ is a singleton for each $i \in N$, for each $\theta \in \Theta$, then set-monotonicity is weaker than quasimonotonicity and almost monotonicity. Indeed, whenever $\max_i^{\theta} X$ is a singleton for each $i \in N$, for each $\theta \in \Theta$, the requirement that $f(\theta)$ be nested in $f(\theta')$ in the definition of set-monotonicity is equivalent to: For all $i \in N$, for all $x \in f(\theta)$, (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$. We refer to this domain of preferences as the *single-top preferences.*[10] As we shall see in Section 7, important correspondences, like the strong Pareto correspondence, the strong core correspondence and the top-cycle correspondence are set-monotonic on the domain of single-top preferences, while they fail to be Maskin monotonic.

Lastly, the condition of set-monotonicity is related to the condition of extended monotonicity of Bochet and Maniquet (2010). They consider the problem of virtual implementation in pure-strategy Nash equilibrium, with the additional restriction that the approximate lottery correspondence to be exactly implemented has a restricted support h. Roughly, it means that

[10]Note that the domain of strict preferences is a subset of the domain of single-top preferences.

for all $\theta \in \Theta$, for all $x \in f(\theta)$, the lottery that is ε-close to $\mathbf{1}_x$ must have support $h(\theta, x)$. In particular, if we impose that $h(\theta, x) = f(\theta)$ for all $x \in f(\theta)$, then the approximate lottery correspondence can assign positive probability only to f-optimal alternatives, and virtual implementation comes to resemble, to some extent, mixed strategy implementation as we define it. Under the restriction of strict preferences, they show that extended monotonicity with respect to the admissible support h is necessary and sufficient for virtual implementation with support h. On the domain of strict preferences, the condition of set-monotonicity implies the condition of extended monotonicity with respect to the support f, and thus virtual implementation with support f. In general, however, the two conditions are quite different, because unlike set-monotonicity, extended monotonicity does not require nestedness of the strict lower contour sets. For instance, when f is a function (i.e., single valued) extended monotonicity with respect to f is equivalent to Maskin monotonicity, while set-monotonicity is less restrictive.

5 A Sufficient Condition

We now show that in any environment with at least three players, set-monotonicity and no veto-power are sufficient for implementation in mixed Nash equilibrium.

Theorem 2 *Let $\langle N, X, \Theta \rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is set-monotonic and satisfies no-veto power, then it is implementable in mixed Nash equilibrium.*

Proof Let $\mathcal{U} = \cup_{\theta \in \Theta} \mathcal{U}^\theta$ and define the set $\Theta\mathcal{U}$ as $\{(\theta, u) \in \Theta \times \mathcal{U} : u \in \mathcal{U}^\theta\}$. Consider the following mechanism $\langle M, g \rangle$. For each player $i \in N$, the message space M_i is $\Theta\mathcal{U} \times \{\alpha^i : \alpha^i : X \times \Theta^2 \to X\} \times X \times \mathbb{Z}_{++}$. In words, each player announces a state of the world and a profile of cardinal representations consistent with that state of the world, a function from alternatives and pairs of states into alternatives, an alternative, and a strictly positive integer. A typical message m_i for player i is $((\theta^i, u^i), \alpha^i, x^i, \mathbf{z}^i)$. (Note that we denote any integer $\mathbf{z}$ in bold.) Let $M := \times_{i \in N} M_i$ with typical element m.

Let $\{f_1(\theta), \ldots, f_{K^\theta}(\theta)\} = f(\theta)$ be the set of f-optimal alternatives at state θ; note that $K^\theta = |f(\theta)|$. For any $\theta \in \Theta$, for any $u \in \mathcal{U}^\theta$, let $1 > \varepsilon_u > 0$ be such that for all $i \in N$, for all pairs $(x, y) \in X \times X$ with $x \succ_i^\theta y$, we have $u_i(x, \theta) \geq (1 - \varepsilon_u) u_i(y, \theta) + \varepsilon_u \max_{w \in X} u_i(w, \theta) + \varepsilon_u (|X| - 1)(\max_{w \in X} u_i(w, \theta) - \min_{w \in X} u_i(w, \theta))$. Since X and N are finite, such an ε_u exists. Let $\mathbf{1}[x] \in \Delta(X)$ be the lottery that puts probability one on outcome $x \in X$. The allocation rule g is defined as follows:

Rule 1: If $m_i = ((\theta, u), \alpha, x, 1)$ for all $i \in N$ (i.e., all agents make the same announcement m_i) and $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, then $g(m)$ is the "uniform" lottery over alternatives in $f(\theta)$; that is,

$$g(m)=\frac{1}{K^{\theta}}\sum_{k=1}^{K^{\theta}}\mathbf{1}\left[f_k(\theta)\right].$$

Rule 2: If there exists $j\in N$ such that $m_i=((\theta,u),\alpha,x,1)$ for all $i\in N\setminus\{j\}$, with $\alpha(f_k(\theta),\theta,\theta)=f_k(\theta)$ for all $f_k(\theta)\in f(\theta)$, and $m_j=((\theta^j,u^j),\alpha^j,x^j,\mathbf{z}^j)\neq m_i$, then $g(m)$ is the lottery:

$$\frac{1}{K^{\theta}}\sum_{k=1}^{K^{\theta}}\left\{\delta_k(m)(1-\varepsilon_k(m))\mathbf{1}\left[\alpha^j(f_k(\theta),\theta,\theta^j)\right]+\delta_k(m)\varepsilon_k(m)\mathbf{1}\left[x^j\right]+(1-\delta_k(m))\,\mathbf{1}\left[f_k(\theta)\right]\right\},$$

with

$$\delta_k(m)=\begin{cases}\delta & \text{if } \alpha^j(f_k(\theta),\theta,\theta^j)\in L_j(f_k(\theta),\theta)\\ 0 & \text{if } \alpha^j(f_k(\theta),\theta,\theta^j)\notin L_j(f_k(\theta),\theta)\end{cases}$$

for $1>\delta>0$, and

$$\varepsilon_k(m)=\begin{cases}\varepsilon_u & \text{if } \alpha^j(f_{k'}(\theta),\theta,\theta^j)\in SL_j(f_{k'}(\theta),\theta) \text{ for some } k'\in\left\{1,...,K^{\theta}\right\}\\ 0 & \text{if } \alpha^j(f_{k'}(\theta),\theta,\theta^j)\notin SL_j(f_{k'}(\theta),\theta) \text{ for all } k'\in\left\{1,...,K^{\theta}\right\}\end{cases}.$$

That is, suppose all players but player j send the same message $((\theta,u),\alpha,x,1)$ with $\alpha(f_k(\theta),\theta,\theta)$ $=f_k(\theta)$ for all $k\in\{1,\ldots,K^{\theta}\}$. Let $m_j=((\theta^j,u^j),\alpha^j,x^j,\mathbf{z}^j)$ be the message sent by player j. Consider the set $F_{L_j}(\theta)$ of all outcomes $f_k(\theta)$ such that $\alpha^j(f_k(\theta),\theta,\theta^j)\in L_j(f_k(\theta),\theta)$. First, suppose that for some $k'\in\left\{1,...,K^{\theta}\right\}$, $\alpha^j(f_{k'}(\theta),\theta,\theta^j)$ selects an alternative x in player j' strict lower-contour set $SL_j(f_{k'}(\theta),\theta)$ of $f_{k'}(\theta)$ at state θ. Then, the designer modifies the uniform lottery by replacing each outcome in the set $F_{L_j}(\theta)$ with the lottery that attaches probability $\delta(1-\varepsilon_u)$ to $\alpha^j(f_k(\theta),\theta,\theta^j)$, probability $\delta\varepsilon_u$ to x^j, and probability $(1-\delta)$ to $f_k(\theta)$. Second, suppose that for all $k'\in\left\{1,...,K^{\theta}\right\}$, $\alpha^j(f_k(\theta),\theta,\theta^j)$ selects an alternative x that is not in $SL_j(f_k(\theta),\theta)$. Then, the designer replaces the outcomes in the set $f_{L_j}(\theta)$ with with the lottery that attaches probability δ to x and probability $(1-\delta)$ to $f_k(\theta)$.

Rule 3: If neither rule 1 nor rule 2 applies, then $g\left(((\theta^i,u^i),\alpha^i,x^i,\mathbf{z}^i)_{i\in N}\right)=x^{i^*}$, with i^* a player announcing the highest integer $\mathbf{z}^{i^*}$. (If more than one player i selects the highest integer, then g randomizes uniformly among their selected x^i.)

Fix a state θ^* and a cardinal representation $u_i^*\in\mathcal{U}_i^{\theta^*}$ of $\succcurlyeq_i^{\theta^*}$ for each player i. Let u^* be the vector of cardinal representations. We divide the rest of the proof in several steps.

Step 1. We first show that for any $x\in f(\theta^*)$, there exists a Nash equilibrium σ^* of $G(\theta^*,u^*)$ such that x belongs to the support of $\mathbb{P}_{\sigma^*,g}$. Consider a profile of strategies σ^* such that $\sigma_i^*=((\theta^*,u^*),\alpha,x,1)$ for all $i\in N$, so that rule 1 applies. The (pure strategy) profile σ^* is a

Nash equilibrium at state θ^*. By deviating, each player i can trigger rule 2, but none of these possible deviations are profitable. Clearly, if player i's deviation is such that $\alpha^i(f_{k'}(\theta^*), \theta^*, \theta^i) \notin SL_i(f_{k'}(\theta^*), \theta^*)$ for all $k' \in \{1, \ldots, K^{\theta^*}\}$, player i's deviation cannot be profitable. So, let us assume that player i's deviation is such that $\alpha^i(f_{k'}(\theta^*), \theta^*, \theta^i) \in SL_i(f_{k'}(\theta^*), \theta^*)$ for some k'. It follows that $\varepsilon_k = \varepsilon_{u^*} > 0$ for all $k \in \{1, \ldots, K^{\theta^*}\}$. There are three cases to consider. First, for all $k \in \{1, \ldots, K^{\theta^*}\}$ such that $\alpha^i(f_k(\theta^*), \theta^*, \theta^i) \notin L_i(f_k(\theta^*), \theta^*)$, we have $\delta_k = 0$ and, thus, there is no shift in probability from $f_k(\theta^*)$. Second, for all $k \in \{1, \ldots, K^{\theta^*}\}$ such that $\alpha^i(f_k(\theta^*), \theta^*, \theta^i) \sim_i^{\theta^*} f_k(\theta^*)$, we have $\delta_k = \delta$ and, thus, there is a probability shift from $f_k(\theta^*)$ to a lottery with mass $\delta(1 - \varepsilon_{u^*})$ on the alternative $\alpha^i(f_k(\theta^*), \theta^*, \theta^i)$ indifferent to $f_k(\theta^*)$ and mass $\delta\varepsilon_{u^*}$ on x^i. In particular, if $x^i \succ_i^{\theta^*} f_k(\theta^*)$, the shift in probabilities leads to a lottery strictly preferred over $f_k(\theta^*)$. Third, for all $k \in \{1, \ldots, K^{\theta^*}\}$ such that $f_k(\theta^*) \succ_i^{\theta^*} \alpha^i(f_k(\theta^*), \theta^*, \theta^i)$, we again have $\delta_k = \delta$ and, thus, there is a probability shift from $f_k(\theta^*)$ to a lottery with mass $\delta(1 - \varepsilon_{u^*})$ on the alternative $\alpha^i(f_k(\theta^*), \theta^*, \theta^i)$ strictly less preferred than $f_k(\theta^*)$ and mass $\delta\varepsilon_{u^*}$ on x^i. By definition of ε_{u^*}, this shift of probabilities leads to a lottery worse than $f_k(\theta^*)$. It is important to note that this last case exists since we have assumed that $\alpha^i(f_{k'}(\theta^*), \theta^*, \theta^i) \in SL_i(f_{k'}(\theta^*), \theta^*)$ for some k'. Thus, the best deviation for player i consists in choosing α^i such that $\alpha^i(f_{k^*}(\theta^*), \theta^*, \theta^i) \in SL_i(f_{k^*}(\theta^*), \theta^*)$ for a unique k^*, $\alpha^i(f_k(\theta^*), \theta^*, \theta^i) \sim_i^{\theta^*} f_k(\theta^*)$ for all other k, and $x^i \in \max_i^{\theta^*} X$. The maximal difference in payoffs between playing σ_i^* and deviating is therefore (up to the multiplicative term δ/K^{θ^*}):

$$u_i^*(f_{k^*}(\theta^*), \theta^*) - (1 - \varepsilon_{u^*})u_i^*(\alpha^i(f_{k^*}(\theta^*), \theta^*, \theta^i), \theta^*) - \varepsilon_{u^*}u_i^*(x^i, \theta^*) + \sum_{k \neq k^*} \varepsilon_{u^*}\left[u_i^*(f_k(\theta^*), \theta^*) - u_i^*(x^i, \theta^*)\right], \tag{2}$$

which is positive by construction of ε_{u^*}. Therefore, player i has no profitable deviation. Lastly, under σ^*, the support of $\mathbb{P}_{\sigma^*,g}$ is $f(\theta^*)$. Hence, for any $x \in f(\theta^*)$, there exists an equilibrium that implements x.

Step 2. Conversely, we need to show that if σ^* is a mixed Nash equilibrium of $G(\theta^*, u^*)$, then the support of $\mathbb{P}_{\sigma^*,g}$ is included in $f(\theta^*)$. Let m be a message profile and denote with $g^O(m)$ the set of alternatives that occur with strictly positive probability when m is played: $g^O(m) = \{x \in X : g(m)(x) > 0\}$. Let us partition the set of messages M into three subsets corresponding to the three allocation rules. First, let R_1 be the set of message profiles such that rule 1 applies, i.e., $R_1 = \{m : m_j = ((\theta, u), \alpha, x, 1)$ for all $j \in N$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)\}$. Second, if all agents $j \neq i$ send some message $m_j = ((\theta, u), \alpha, x, 1)$ with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, while agent i sends a different message $m_i = \left((\theta^i, u^i), \alpha^i, x^i, \mathbf{z}^i\right)$, then rule 2 applies and agent i is the only agent differentiating his message. Let R_2^i be the set of these message profiles and define $R_2 = \cup_{i \in N} R_2^i$. Third, let R_3 be the set of message profiles such that rule 3 applies (i.e., R_3 is the complement of $R_1 \cup R_2$ in M).

Consider an equilibrium σ^* of $G(\theta^*, u^*)$ and let M_i^* be the set of message profiles that occur with positive probability under σ_i^*. (M_i^* is the support of σ_i^*.) We need to show that $g^O(m^*) \subseteq f(\theta^*)$ for all $m^* \in M^* := \times_{i \in N} M_i^*$.

<u>Step 3.</u> For any player $i \in N$, for all $m_i^* = ((\theta^i, u^i), \alpha^i, x^i, \mathbf{z}^i) \in M_i^*$, define the (deviation) message $m_i^D(m_i^*) = ((\theta^i, u^i), \alpha^D, x^D, \mathbf{z}^D)$, where: 1) α^D differs from α^i in at most the alternatives associated with elements $(f_k(\theta), \theta, \theta)$ for all $\theta \in \Theta$, for all $k \in \{1, \ldots, K^\theta\}$; that is, we can only have $\alpha^D(f_k(\theta), \theta, \theta) \neq \alpha^i(f_k(\theta), \theta, \theta)$ for some $k \in \{1, \ldots, K^\theta\}$ and some $\theta \in \Theta$, while we have $\alpha^D(x, \theta', \theta'') = \alpha^i(x, \theta', \theta'')$, otherwise, 2) $x^D \in \max_i^{\theta^*} X$, and 3) $\mathbf{z}^D > \mathbf{z}^i$ and for $1 > \mu \geq 0$, the integer $\mathbf{z}^D$ is chosen strictly larger than the integers $\mathbf{z}^j$ selected by all the other players $j \neq i$ in all messages $m_{-i}^* \in M_{-i}^*$, except possibly a set of message profiles $M_{-i}^\mu \subseteq M_{-i}^*$ having probability of being sent less than μ. (Note that μ can be chosen arbitrarily small, but not necessarily zero because other players may randomize over an infinite number of messages.) Consider the following deviation σ_i^D for player i from the equilibrium strategy σ_i^*:

$$\sigma_i^D(m_i) = \begin{cases} \sigma_i^*(m_i^*) & \text{if } m_i = m_i^D(m_i^*) \text{ for some } \ m_i^* \in M_i^* \\ 0 & \text{otherwise} \end{cases}.$$

<u>Step 4.</u> First, note that under $(\sigma_i^D, \sigma_{-i}^*)$, the set of messages sent is a subset of $R_2^i \cup R_3$: either rule 2 applies and all players but player i send the same message or rule 3 applies. Second, whenever rule 3 applies, player i gets his preferred alternative at state θ^* with arbitrarily high probability $(1-\mu)$. Third, suppose that under σ^*, there exists $m^* \in R_2^j$ with $j \neq i$. Under $(\sigma_i^D, \sigma_{-i}^*)$, with the same probability that m^* is played, $(m_i^D(m_i^*), m_{-i}^*) \in R_3$ is played (rule 3 applies) and with probability at most μ, the lottery $g((m_i^D(m_i^*), m_{-i}^*))$ under $(m_i^D(m_i^*), m_{-i}^*)$ might be less preferred by player i than the lottery $g(m^*)$. (With probability $1-\mu$, $g((m_i^D(m_i^*), m_{-i}^*)) = \max_i^{\theta^*} X$.) Yet, since μ can be made arbitrarily small and utilities are bounded, the loss can be made arbitrarily small. Consequently, by setting $\alpha^D(f_k(\theta), \theta, \theta^i) \succsim_i^{\theta^*} \alpha^i(f_k(\theta), \theta, \theta^i)$ for all θ and all $k \in \{1, \ldots, K^\theta\}$, player i can guarantee himself an arbitrarily small, worst-case loss of $\bar{u}$, in the event that $m^* \in \cup_{j \neq i} R_2^j$ under σ^*.

<u>Step 5.</u> Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_1$; that is, for all $j \neq i$, $m_j^* = m_i^* = ((\theta, u), \alpha, x, 1)$. Assume that there exists $k' \in \{1, \ldots, K^\theta\}$ such that $f_{k'}(\theta) \notin \max_i^{\theta^*} X$. In the event the message sent by all others is m_j^*, player i strictly gains from the deviation if $\alpha^D(f_{k'}(\theta), \theta, \theta) \in L_i(f_{k'}(\theta), \theta)$ and either (1) $\alpha^D(f_{k'}(\theta), \theta, \theta) \succ_i^{\theta^*} f_{k'}(\theta)$ or (2) $\alpha^D(f_{k'}(\theta), \theta, \theta) \in SL_i(f_{k'}(\theta), \theta)$, $\alpha^D(f_{k'}(\theta), \theta, \theta) \succsim_i^{\theta^*} f_{k'}(\theta)$ (since $\varepsilon_u > 0$ and $f_{k'}(\theta) \notin \max_i^{\theta^*} X$). Now assume that, in addition, there exists $k'' \in \{1, \ldots, K^\theta\}$ such that $f_{k''}(\theta) \in \max_i^{\theta^*} X$. Player i strictly gains from the deviation if (3) $\alpha^D(f_{k''}(\theta), \theta, \theta) \in SL_i(f_{k''}(\theta), \theta)$ and $\alpha^D(f_{k''}(\theta), \theta, \theta) \sim_i^{\theta^*} f_{k''}(\theta)$, since the deviation shifts probability mass from $f_{k'}(\theta) \notin \max_i^{\theta^*} X$ to $x^D \in \max_i^{\theta^*} X$. (Note that player i would not gain from the deviation if $f(\theta) \subseteq \max_i^{\theta^*} X$.)

Since the expected gain in this event can be made greater than $\bar{u}$ by appropriately choosing μ, (1) and (2) cannot hold for any player i and any k^* such that $f_{k'}(\theta) \notin \max_i^{\theta^*} X$. Similarly, (3) cannot hold for any player i and any k'' such that $f_{k''}(\theta) \in \max_i^{\theta^*} X$ unless $f(\theta) \subseteq \max_i^{\theta^*} X$.

It follows that for σ^* to be an equilibrium, we must have (1) $L_i(f_k(\theta), \theta) \subseteq L_i(f_k(\theta), \theta^*)$ for all k and (2) either $SL_i(f_k(\theta), \theta) \subseteq SL_i(f_k(\theta), \theta^*)$ for all k or $f(\theta) \subseteq \max_i^{\theta^*} X$. Therefore, by set-monotonicity of f, we must have $f(\theta) \subseteq f(\theta^*)$. This shows that $g(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_1$.

<u>Step 6.</u> Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_2^i$; that is, for all $j \neq i$, $m_j^* = ((\theta, u), \alpha, x, 1) \neq m_i^*$. In this case, any player $j \neq i$ strictly gains from the deviation σ_j^D whenever $\mathbf{z}^D$ is the largest integer, which occurs with a probability of at least $1 - \mu$, unless $g(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$. Since μ can be made arbitrarily small, it must be $g(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$ for all $j \neq i$. Therefore, by no-veto power, it must be $g(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_2^i$.

<u>Step 7.</u> It only remains to consider messages $(m_i^*, m_{-i}^*) \in R_3$. For such messages the argument is analogous to messages in R_2^i. For no player i to be able to profit from the deviation σ_i^D, it must be $g(m_i^*, m_{-i}^*) \subseteq \max_i^{\theta^*} X$ for all $i \in N$. Therefore, the condition of no-veto power implies $g(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_3$. □

Some remarks are in order. First, the mechanism constructed in the proof is inspired by the mechanism in the appendix of Maskin (1999), but ours is a random mechanism. As we have already explained, we believe this is natural given that we consider the problem of implementation in mixed Nash equilibrium.

Second, the assumption of von Neumann-Morgenstern utility functions is not easily dispensed with. For instance, we might just assume that preferences over lotteries are monotone in probabilities, so that shifts in probability mass to strictly preferred alternatives yield preferred lotteries. However, as in Abreu and Sen (1991, Section 5), this is not sufficient, since our mechanism involves adding and subtracting *non-degenerate* lotteries, thus creating compound lotteries that are not comparable if we just assume that preferences are monotone in probability. For a concrete example, suppose that there are three alternatives x, y, and z, that player i at state θ prefers x to y to z and $f(\theta) = \{y\}$. According to rule 1, if all players report $(\theta, \alpha, y, 1)$ with $\alpha(y, \theta, \theta) = y$, player i faces the lottery $(0, 1, 0)$. Now, according to rule 2, if player i deviates to $(\theta, \alpha^i, x, 1)$ with $\alpha^i(y, \theta, \theta) = z$, he faces the lottery $(\varepsilon\delta, 1 - \delta, (1 - \varepsilon)\delta)$. The two lotteries are not comparable if we only impose the axiom of monotonicity in probabilities. Whether we can design an implementing mechanism that only requires the axiom of monotonicity in probabilities is an open issue.

Lastly, Theorem 2 strongly relies on the condition of set-monotonicity, a weakening of Maskin monotonicity, which is relatively easy to check in applications. We have not tried to look for necessary and sufficient conditions for mixed Nash implementation. We suspect that such a characterization will involve conditions that are hard to check in practice, as it is the case for Nash implementation à la Maskin (e.g., condition μ of Moore and Repullo, 1990, condition M of Sjöström, 1991, condition β of Dutta and Sen, 1991, or essential monotonicity of Danilov, 1992).

6 Cardinal vs Ordinal: a Bridge

Following the suggestion of a referee, this section builds a bridge between the ordinal approach adopted in this paper and a cardinal approach. To do so, we adopt an alternative definition of mixed Nash implementation, proposed by the referee. This definition, which we call mixed* Nash implementation, replaces part (i) in Definition 1 with: (i^*) For each $x \in f(\theta)$, there exists a Nash equilibrium σ^* of $G(\theta, u)$ such that the support of $\mathbb{P}_{\sigma^*,g}$ is x. This alternative definition is intermediate between our definition of mixed Nash implementation and Maskin's definition of Nash implementation. With mixed* Nash implementation, a necessary and almost sufficient condition for implementation is cardinal monotonicity.

Definition 3 *A social choice correspondence f is cardinally monotonic if for all triples $(x, \theta, \theta') \in X \times \Theta \times \Theta$ with $x \in f(\theta)$, we have that $x \in f(\theta')$ whenever for each player $i \in N$, there exists a cardinal representation $u_i(\cdot, \theta)$ of $\succcurlyeq_i^\theta$ and a cardinal representation $u_i(\cdot, \theta')$ of $\succcurlyeq_i^{\theta'}$ such that for all $P \in \Delta(X)$,*

$$u_i(x, \theta) \geq \sum_{x \in X} P(y) u_i(y, \theta) \Longrightarrow u_i(x, \theta') \geq \sum_{y \in X} P(y) u_i(y, \theta'). \quad (3)$$

The necessity of cardinal monotonicity is clear. We now provide a sketch of the proof that cardinal monotonicity together with no-veto power is sufficient (with at least three players). For simplicity, we consider social choice functions. Fix a social choice function $f : \Theta \to X$. Define the extended state space $\mathcal{U} = \cup_{\theta \in \Theta} \mathcal{U}^\theta$, and let $\tilde{f} : \mathcal{U} \to X$ be the social choice function such that for any $\theta \in \Theta$, for any $u \in \mathcal{U}^\theta$, $\tilde{f}(u) = f(\theta)$. Define the lower contour set $\tilde{L}_i(x, u)$ of x at u as $\{P \in \Delta(X) : u_i(x) \geq \sum_{y \in X} u_i(y) P(y)\}$.[11] First, we show that if f is cardinally monotonic and satisfies no-veto power, then $\tilde{f}$ is Maskin monotonic and satisfies no-veto power. Assume that f is cardinally monotonic and consider any two states of the world u and u' such that $L_i(\tilde{f}(u), u) \subseteq L_i(\tilde{f}(u), u')$ for each $i \in N$. We want to show that $\tilde{f}(u) = \tilde{f}(u')$. This is clearly true if there exists $\theta \in \Theta$ such that $u \in \mathcal{U}^\theta$ and $u' \in \mathcal{U}^\theta$. So, assume that $u \in \mathcal{U}^\theta$ and $u' \in \mathcal{U}^{\theta'}$

[11] It is important that we define lower-contour sets in the space of lotteries, as opposed to the space of alternatives.

with $\theta \neq \theta'$. We thus have two cardinal representations $u(\cdot, \theta) = u \in \mathcal{U}^{\theta}$ and $u(\cdot, \theta') = u' \in \mathcal{U}^{\theta'}$ such that

$$u_i(f(u), \theta) \geq \sum_{y \in X} P(y) u_i(y, \theta) \Longrightarrow u_i(f(u), \theta') \geq \sum_{y \in X} P(y) u_i(y, \theta').$$

Cardinal monotonicity then implies that $\tilde{f}(u(\cdot, \theta)) = f(\theta) = f(\theta') = \tilde{f}(u(\cdot, \theta'))$ and, consequently, $\tilde{f}(u) = \tilde{f}(u')$ (since $\tilde{f}$ is ordinal). Thus, $\tilde{f}$ is Maskin monotonic. Second, since $\tilde{f}$ clearly satisfies no-veto power, we can apply Maskin's theorem and conclude that it is Nash implementable. The final step is to observe that f can be implemented in mixed* Nash equilibrium by the mechanism used to implement $\tilde{f}$ in Nash equilibrium. This is because, for each $\theta \in \Theta$ and for each $u \in \mathcal{U}^{\theta}$, the set of equilibrium outcomes of the game induced at u by the mechanism implementing $\tilde{f}$ in Nash equilibrium (e.g., the mechanism in Sjöström and Maskin, 2002) coincides with the singleton $f(\theta)$.

Several additional remarks are worth making. First, mixed* Nash implementation is stronger than mixed Nash implementation, as the simple example in Section 2 shows. In Example 1, the social choice correspondence is not cardinally monotonic, and yet it is mixed Nash implementable. Second, for social choice functions, set-monotonicity and mixed Nash implementation are equivalent to cardinal monotonicity and mixed* Nash implementation. Third, and most important, an appealing feature of our concept of mixed Nash implementation is that it simply stipulates that the set of equilibrium outcomes is $f(\theta)$ at each state of the world θ, and does not impose any additional restrictions. This contrasts sharply with Maskin's concept that requires the existence of a pure equilibrium for each f-optimal alternative at θ or with mixed* Nash implementation that requires the existence of equilibria that put probability one on each f-optimal alternative.

To clarify further the connection between the two concepts, we present an alternative characterization of set-monotonicity, which explicitly considers cardinal representations and lotteries over alternatives, and is clearly a weaker condition than cardinal monotonicity.

Definition 4 *A social choice correspondence f is cardinally set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for each player $i \in N$, there exists a cardinal representation $u_i(\cdot, \theta)$ of $\succsim_i^{\theta}$ and a cardinal representation $u_i(\cdot, \theta')$ of $\succsim_i^{\theta'}$ such that for all $P^{f(\theta)} \in \Delta(f(\theta))$, for all $P \in \Delta(X)$,*

$$\sum_{x \in X} P^{f(\theta)}(x) u_i(x, \theta) \geq \sum_{x \in X} P(x) u_i(x, \theta) \Longrightarrow \sum_{x \in X} P^{f(\theta)}(x) u_i(x, \theta') \geq \sum_{x \in X} P(x) u_i(x, \theta'). \tag{4}$$

According to Definition 4, the set of optimal alternatives $f(\theta)$ at state θ must be a subset of the set of optimal alternatives $f(\theta')$ at state θ', whenever there exist a cardinal representation of preferences at θ and a cardinal representation at θ' such that, for any lottery $P^{f(\theta)}$ with support in $f(\theta)$, the lower contour set of $P^{f(\theta)}$ for the cardinalization at state θ is nested in the lower contour set of $P^{f(\theta)}$ for the cardinalization at state θ', with the lower contour sets defined in the lottery space. In spite of the apparent differences between cardinal set-monotonicity and set-monotonicity, the next proposition shows that the two conditions are equivalent.

Proposition 1 *A social choice correspondence is set-monotonic if and only if it is cardinally set-monotonic.*

Proof (*Only if*) Suppose that f is set-monotonic. Consider two states θ and θ' and for each player $i \in N$, two cardinal representations $u_i(\cdot, \theta)$ and $u_i(\cdot, \theta')$ such that Equation (4) holds. We want to show that $f(\theta) \subseteq f(\theta')$. To prove this, we show that Equation (4) implies that for each player $i \in N$, either $f(\theta) \subseteq \max_i^{\theta'} X$ or (*i*) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (*ii*) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$. The conclusion then follows from the set-monotonicity of f.
Step 1: Equation (4) clearly implies that (*i*) holds for all $x \in f(\theta)$. To see this, consider $x_k \in f(\theta)$ and any $y \in L_i(x_k, \theta)$, so that $u_i(x_k, \theta) \geq u_i(y, \theta)$. Letting $P^{f(\theta)}(x_k) = 1$ and $P(y) = 1$, Equation (4) implies that $u_i(x_k, \theta') \geq u_i(y, \theta')$, i.e., $y \in L_i(x_k, \theta')$, as required.
Step 2: Assume that $f(\theta) \nsubseteq \max_i^{\theta'} X$ and consider any $f_k(\theta) \notin \max_i^{\theta'} X$. There exists $x_k \in X$ with $x_k \succ_i^{\theta'} f_k(\theta)$. From step 1, it must be $x_k \succ_i^{\theta} f_k(\theta)$. Contrary to (*ii*), suppose that there exists $y \in X$ such that $f_k(\theta) \succ_i^{\theta} y$, but $y \succcurlyeq_i^{\theta'} f_k(\theta)$. Then there exists $p \in (0, 1)$ such that $u_i(f_k(\theta), \theta) \geq pu_i(x_k, \theta) + (1-p)u_i(y, \theta)$. Equation (4) implies that $u_i(f_k(\theta), \theta') \geq pu_i(x_k, \theta') + (1-p)u_i(y, \theta')$, a contradiction. Therefore, it must be $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$ such that $x \notin \max_i^{\theta'} X$.
Step 3: Continue to assume that $f(\theta) \nsubseteq \max_i^{\theta'} X$ and consider any $f_k(\theta) \in \max_i^{\theta'} X$. Then there exists $f_{k'}(\theta) \notin \max_i^{\theta'} X$. Since $f_k(\theta) \succ_i^{\theta'} f_{k'}(\theta)$, by step 1 it is also $f_k(\theta) \succ_i^{\theta} f_{k'}(\theta)$. Contrary to (*ii*), suppose that there exists $y \in X$ such that $f_k(\theta) \succ_i^{\theta} y$, but $y \sim_i^{\theta'} f_k(\theta)$. Then there exists $p \in (0, 1)$ such that $pu_i(f_{k'}(\theta), \theta) + (1-p)u_i(f_k(\theta), \theta) \geq pu_i(f_k(\theta), \theta) + (1-p)u_i(y, \theta)$. Equation (4) then implies that $u_i(f_{k'}(\theta), \theta') \geq u_i(f_k(\theta), \theta')$, a contradiction. Therefore, it must be $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$ such that $x \in \max_i^{\theta'} X$.
Step 4: Equation (4) trivially holds whenever $f(\theta) \subseteq \max_i^{\theta'} X$. This completes the first part of the proof.

(*If*) Suppose that f is cardinally set-monotonic. Consider two states θ and θ' such that for each player $i \in N$, either $f(\theta) \subseteq \max_i^{\theta'} X$ or (*i*) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (*ii*) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$. We want to show that this implies the existence of two cardinal representations $u_i(\cdot, \theta)$ and $u_i(\cdot, \theta')$ such that Equation (4) holds. The conclusion then follows from the cardinal set-monotonicity of f.

First, consider any player $i \in N$ such that $f(\theta) \subseteq \max_i^{\theta'} X$. It is immediate that any cardinal representations $u_i(\cdot, \theta)$ and $u_i(\cdot, \theta')$ satisfy Equation (4).

Second, consider any player $i \in N$ such that $f(\theta) \nsubseteq \max_i^{\theta'} X$, but $L_i(x,\theta) \subseteq L_i(x,\theta')$ and $SL_i(x,\theta) \subseteq SL_i(x,\theta')$ for all $x \in f(\theta)$. The claim in the proof of Theorem 1 shows that, for any cardinal representation $u_i(\cdot,\theta)$ of $\succcurlyeq_i^{\theta}$, there exists a cardinal representation $u_i(\cdot,\theta')$ of $\succcurlyeq_i^{\theta'}$ such that $u_i(x,\theta) \geq u_i(x,\theta')$ for all $x \in X$ and $u_i(x,\theta) = u_i(x,\theta')$ for all $x \in f(\theta)$. It directly follows that Equation (4) holds. □

7 Conclusions

We conclude with a series of remarks in which we provide applications of our results to some important social choice rules.

Remark 1 On the domain of single-top preferences, the strong Pareto correspondence f^{PO} is set-monotonic and hence, by Theorem 2, is implementable in mixed Nash equilibrium; on the contrary, it fails to be Maskin monotonic.

The strong Pareto correspondence is defined as follows:

$$f^{PO}(\theta) := \{x \in X : \text{ there is no } y \in X \text{ such that } x \in L_i(y,\theta) \text{ for all } i \in N, \\ \text{and } x \in SL_i(y,\theta) \text{ for at least one } i \in N\}.$$

To see that f^{PO} is set-monotonic on the domain of single-top preferences, consider two states θ and θ' such that for all $i \in N$, either $f^{PO}(\theta) \subseteq \max_i^{\theta'} X$ or for all $x \in f^{PO}(\theta)$, (i) $L_i(x,\theta) \subseteq L_i(x,\theta')$ and (ii) $SL_i(x,\theta) \subseteq SL_i(x,\theta')$. Suppose that $x^* \in f^{PO}(\theta)$, but $x^* \notin f^{PO}(\theta')$. First, if $f^{PO}(\theta) \subseteq \max_i^{\theta'} X$, it follows from the single-top condition that $\{x^*\} = f^{PO}(\theta) = \max_i^{\theta'} X$ and, thus, $x^* \in f^{PO}(\theta')$. Second, assume that $f^{PO}(\theta) \nsubseteq \max_i^{\theta'} X$, so that (i) and (ii) above hold. At state θ', there must then exists $y \in X$ such that $y \notin SL_i(x^*,\theta')$ for all $i \in N$ and $y \notin L_i(x^*,\theta')$ for at least one $i \in N$. It follows that $y \notin SL_i(x^*,\theta)$ for all $i \in N$ and $y \notin L_i(x^*,\theta)$ for at least one $i \in N$, a contradiction with $x^* \in f^{PO}(\theta)$. Consequently, $f^{PO}(\theta) \subseteq f^{PO}(\theta')$ and f^{PO} is set-monotonic.

To see that the strong Pareto correspondence is not Maskin monotonic on the domain of single-top preferences (and, therefore, on the unrestricted domain), consider the following example. There are three players, 1, 2 and 3, and two states of the world θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
d	d	b	d	d	b
b	a	a	b	$a \sim b$	a
c	b	c	c		c
a	c	d	a	c	d

The strong Pareto correspondence is: $f^{PO}(\theta) = \{a, b, d\}$ and $f^{PO}(\theta') = \{b, d\}$. Maskin monotonicity does not hold since $L_2(a, \theta) \subseteq L_2(a, \theta')$ and yet $a \notin f^{PO}(\theta')$.[12]

Remark 2 Using arguments that parallel the ones used for the strong Pareto correspondence, it can be verified that on the domain of single-top preferences, the strong core correspondence f^{SC} is set-monotonic, while it is not Maskin monotonic.

A coalitional game is a quadruple $\langle N, X, \theta, v\rangle$, where N is the set of players, X is the finite set of alternatives, θ is a profile of preference relations, and $v : 2^N \setminus \{\emptyset\} \to 2^X$. An alternative x is weakly blocked by the coalition $S \subseteq N \setminus \{\emptyset\}$ if there is a $y \in v(S)$ such that $x \in L_i(y, \theta)$ for all $i \in S$ and $x \in SL_i(y, \theta)$ for at least one $i \in S$. If there is an alternative that is not weakly blocked by any coalition in $2^N \setminus \{\emptyset\}$, then $\langle N, X, \theta, v\rangle$ is a game with a non-empty strong core. A coalitional environment with non-empty strong core is a quadruple $\langle N, X, \Theta, v\rangle$, where Θ is a set of preference relations such that $\langle N, X, \theta, v\rangle$ has a non-empty strong core for all $\theta \in \Theta$.

The strong core correspondence f^{SC} is defined for all coalitional environments with non-empty strong core as follows:

$$f^{SC}(\theta) := \{x \in v(N) : \ x \text{ is not weakly blocked by any } \emptyset \neq S \subseteq N\}.$$

Remark 3 On the unrestricted domain of preferences, a Maskin monotonic social choice *function* must be constant (Saijo, 1988). It is simple to see that this is also true for a set-monotonic social choice function. Suppose, to the contrary, that $f(\theta) = x \neq y = f(\theta')$. Let θ'' be such that $\{x, y\} \subseteq \max_i^{\theta''} X$ for all $i \in N$. Then set-monotonicity implies $\{x, y\} \subseteq f(\theta'')$, contrary to the assumption that $f(\theta'')$ is a singleton.

Remark 4 On the domain of strict preferences, the top-cycle correspondence, an important voting rule, is set-monotonic, while it is not Maskin monotonic. Since it also satisfies no-veto power, Theorem 2 applies: on the domain of strict preferences, the top-cycle correspondence is implementable in mixed Nash equilibrium.

We say that alternative x defeats alternative y at state θ, written $x \gg^\theta y$, if the number of players who prefer x to y is strictly greater than the number of players who prefer y to x. At each state θ, the top-cycle correspondence selects the smallest subset of X such that any alternative in it defeats all alternatives outside it.

$$f^{TC}(\theta) := \cap\{X' \subseteq X : x' \in X', x \in X \setminus X' \text{ implies } x' \gg^\theta x\}.$$

To prove that the top-cycle correspondence is set-monotonic, assume to the contrary that there is at least an alternative x^* such that $x^* \in f^{TC}(\theta)$, $x^* \notin f^{TC}(\theta')$, and $L_i(x, \theta) \subseteq L_i(x, \theta')$

[12] On the unrestricted domain of preferences, f^{PO} is not set-monotonic. To see this, suppose alternative d is not available in the example. The strong Pareto correspondence is then $f(\theta) = \{a, b\}$ and $f(\theta') = \{b\}$. Set-monotonicity fails, since it is $L_2(a, \theta) \subseteq L_2(a, \theta')$, $\{a, b\} \subseteq \max_2^{\theta'} \{a, b, c\}$ and yet $a \notin f^{PO}(\theta')$.

for all $x \in f^{TC}(\theta)$, for all $i \in N$. (When preferences are strict $L_i(x,\theta) \setminus \{x\} = SL_i(x,\theta)$ and strong set-monotonicity coincides with set-monotonicity.) Clearly, if x^* is a Condorcet winner at θ so that $f^{TC}(\theta) = \{x^*\}$, then x^* is also a Condorcet winner at θ'; hence $x^* \in f^{TC}(\theta')$, a contradiction. Assume that x^* is not a Condorcet winner; that is, the set $f^{TC}(\theta)$ is not a singleton. Take any alternative $x \in f^{TC}(\theta)$ and any $y \notin f^{TC}(\theta)$. By definition of f^{TC}, it must be that $x \gg^{\theta} y$. Since $L_i(x,\theta) \subseteq L_i(x,\theta')$ for all $x \in f^{TC}(\theta)$, it must also be that $x \gg^{\theta'} y$. Hence, it must be $f^{TC}(\theta') \subseteq f^{TC}(\theta)$. For all $x^* \in f^{TC}(\theta) \setminus f^{TC}(\theta')$, it must be the case that $x \gg^{\theta'} x^*$ for all $x \in f^{TC}(\theta')$. Furthermore, since the lower contour sets of x^* satisfy $L_i(x^*,\theta) \subseteq L_i(x^*,\theta')$, the strict upper contour set of x^* at θ' is a subset of the strict upper contour set at θ for all $i \in N$, and hence it must be $x \gg^{\theta} x^*$ for all $x \in f^{TC}(\theta')$. This contradicts the assumption that $x^* \in f^{TC}(\theta)$ and $f^{TC}(\theta)$ is the smallest subset of X such that any alternative in it defeats all alternatives outside it at θ.

To see that f^{TC} is not Maskin monotonic, consider the following example with two states, three alternatives and three players.

θ			θ'		
1	2	3	1	2	3
a	c	b	a	c	c
b	a	c	b	a	b
c	b	a	c	b	a

We have that $f^{TC}(\theta) = \{a,b,c\}$ and $f^{TC}(\theta') = \{c\}$. Since $L_i(a,\theta) \subseteq L_i(a,\theta')$ for all $i \in N$, Maskin monotonicity and $f^{TC}(\theta) = \{a,b,c\}$ would require $a \in f^{TC}(\theta')$.

Remark 5 The Borda, Kramer and plurality voting rules fail to satisfy set-monotonicity.[13]

It is simple to see that the example in Maskin (1999, page 30) shows that the Borda rule fails to satisfy not only Maskin monotonicity, but also set-monotonicity. For the Kramer rule, consider the example in the table below with five players, three alternatives and two states θ and θ'.

θ					θ'				
1	2	3	4	5	1	2	3	4	5
a	a	a	a	c	a	a	a	a	b
b	b	c	c	b	b	b	b	b	c
c	c	b	b	a	c	c	c	c	a

The Kramer rule selects a at state θ and b at state θ', a violation of set-monotonicity. (Remember that strong set-monotonicity coincides with set-monotonicity when preferences are strict,

[13] The Kramer score of alternative x at state θ is $\max_{y \neq x} |\{i \in N : x \succ_i^{\theta} y\}|$; the Kramer rule selects the alternatives with the highest Kramer score at each state. The plurality rule selects the alternatives ranked top by the highest number of players.

as in the above example.) For the plurality rule, consider the table below with seven players, three alternatives and two states θ and θ'.

θ							θ'						
1	2	3	4	5	6	7	1	2	3	4	5	6	7
a	a	a	b	b	c	c	a	a	a	b	b	b	b
b	b	b	c	c	b	b	b	b	b	c	c	c	c
c	c	c	a	a	a	a	c	c	c	a	a	a	a

In this example, the plurality rule selects a at θ and b at θ', which also violates set-monotonicity. So, our results are not so permissive so as to imply that all "reasonable" social choice correspondences are implementable in mixed Nash equilibrium.

References

[1] Abreu Dilip and Hitoshi Matsushima, 1992, "Virtual Implementation in Iteratively Undominated Strategies: Complete Information," Econometrica, 60, pp. 993-1008.

[2] Abreu Dilip and Arunava Sen, 1991, "Virtual Implementation in Nash Equilibrium," Econometrica, 59, pp. 997-1022.

[3] Aumann Robert J. and Adam Brandenburger, 1995, "Epistemic Condition for Nash Equilibrium," Econometrica, 64, pp. 1161-1180.

[4] Barbera Salvador, Anna Bogomolnaia, and Hans van der Stel, 1998, "Strategy-proof Probabilistic Rules for Expected Utility Maximizers," Mathematical Social Sciences, 35, pp. 89-103.

[5] Benoît Jean-Pierre and Efe Ok, 2008, "Nash Implementation Without No-Veto Power," Games and Economic Behavior, 64, pp. 51-67.

[6] Bergemann Dirk, Stephen Morris and Olivier Tercieux, 2011, "Rationalizable Implementation," Journal of Economic Theory, 146, pp. 1253-1274.

[7] Bochet Olivier, 2007, "Nash Implementation with Lottery Mechanisms," Social Choice and Welfare, 28, pp. 111-125.

[8] Bochet Olivier and François Maniquet, 2010, "Virtual Nash Implementation with Admissible Support," Journal of Mathematical Economics, 46, pp. 99-108.

[9] Cabrales Antonio and Roberto Serrano, 2011, "Implementation in Adaptive Better-Response Dynamics: Towards a General Theory of Bounded Rationality in Mechanisms," Games and Economic Behavior, 73, pp. 360-374.

[10] Camerer Colin, 2003, "Behavioral Game Theory: Experiments on Strategic Interaction," Princeton: Princeton University Press.

[11] Danilov Vladimir, 1992, "Implementation via Nash Equilibria," Econometrica, 60, pp. 43-56.

[12] Dutta Bhaskar and Arunava Sen, 1991, "A Necessary and Sufficient Condition for Two-Person Nash Implementation," Review of Economic Studies, 58, pp. 121-28.

[13] Gibbard Allan, 1977, "Manipulation of Schemes that Mix Voting with Chance," Econometrica, 45, pp. 665-681.

[14] Harsanyi John C., 1973, "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points," International Journal of Game Theory, 2, pp. 1-23.

[15] Jackson Matthew O., 1991, "Bayesian Implementation," Econometrica, 59, pp. 461-477.

[16] Jackson Matthew O., 2001, "A Crash Course in Implementation Theory," Social Choice and Welfare, 18, pp. 655-708.

[17] Maskin Eric, 1999, "Nash Equilibrium and Welfare Optimality," Review of Economic Studies, 66, pp. 23-38.

[18] Maskin Eric and Tomas Sjöström, 2002, "Implementation Theory," in Amartya Sen and Kotaro Suzumura (eds), Handbook of Social Choice and Welfare, vol. 1, Amsterdam: North Holland, pp. 237-288.

[19] Moore John and Rafael Repullo, 1990, "Nash Implementation: A Full Characterization," Econometrica, 58, pp. 1083–1099.

[20] Matsushima Hitoshi, 1998, "A New Approach to the Implementation Problem," Journal of Economic Theory, 45, pp. 128-144.

[21] Saijo Tatsuyoshi, 1987, "On Constant Maskin Monotonic Social Choice Functions," Journal of Economic Theory, 42, pp. 382-386.

[22] Sanver M. Remzi, 2006, "Nash Implementing Non-Monotonic Social Choice Rules by Awards," Economic Theory, 28, pp. 453-460.

[23] Serrano Roberto and Rajiv Vohra, 2010, "Multiplicity of Mixed Equilibria in Mechanisms: a Unified Approach to Exact and Approximate Implementation," Journal of Mathematical Economics, 46, pp. 775-785.

[24] Sjöström Tomas, 1991, "On the Necessary and Sufficient Conditions for Nash Implementation," Social Choice and Welfare, 8, pp. 333-340.

[25] Vartiainen Hannu, 2007, "Subgame Perfect Implementation of Voting Rules via Randomized Mechanisms," Social Choice and Welfare, 29, pp. 353-367.