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RUIN PROBLEMS AND DUAL EVENTS

by

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Abstract
Dickson (1992) uses dual events to explain results relating to the distribution of the surplus immediately prior to ruin in the classical surplus process. In this paper we show that dual events can be used to explain other results in ruin theory. In particular we prove and explain the relationship between the density of the surplus immediately prior to ruin, and the joint density of the surplus immediately prior to ruin and the severity of ruin.

1. Introduction and Notation
In the classical continuous time risk model, the insurer's surplus at time $t$ is denoted $Z_t$ and given by

$$Z_t = Z_0 + ct - X_t$$

where $Z_0$ is the insurer's initial surplus, $c$ is the premium income per unit time, assumed to be received continuously, and $X_t$ denotes aggregate claims up to time $t$. The aggregate claims process is a compound Poisson process with Poisson parameter $\lambda$. The individual claim amount distribution is denoted $P(x)$, has density function $p(x)$ and mean $p_1$, and is such that $P(0) = 0$. We also assume that $c > \lambda p_1$.

We denote by $\psi(u)$ the probability of ultimate ruin from initial surplus $u$. Let $T$ denote the time of ruin, so that $\psi(u) = P(T < \infty | Z_0 = u)$. We define $\delta(u) = 1 - \psi(u)$ and

$$G(u, y) = P(T < \infty \text{ and } Z_T > -y | Z_0 = u)$$

$$F(u, x) = P(T < \infty \text{ and } Z_T < x | Z_0 = u)$$

$$F(u, x, y) = P(T < \infty \text{ and } Z_T > -y \text{ and } Z_T < x | Z_0 = u)$$

where $Z_T$ denotes the surplus immediately prior to ruin given that ruin occurs. Let $g(u, y)$, $f(u, x)$ and $f(u, x, y)$ respectively denote the (defective) densities associated with $G(u, y)$, $F(u, x)$ and $F(u, x, y)$.

Dufresne and Gerber (1988) show that $g(0, y) = f(0, y)$ and it is well known (see, for example, Bowers et al (1986)) that

$$g(0, y) = (\lambda/c)(1 - P(y))$$

Dickson (1992) uses dual events to explain the symmetry of the distributions
of the surplus immediately prior to ruin and the severity of ruin when \( Z_0 = 0 \) and shows (without finding the functional form) that

\[
f(0,x,y) = f(0,y,x)
\]

Dufresne and Gerber (1988) state that

\[
f(u,x,y) = f(u,x) \frac{p(x+y)}{1 - p(x)} \tag{1.1}
\]

but do not offer a proof of this result. We will prove this result, but our main purpose is to explain this result. It is trivially true that

\[
f(u,x,y) = f(u,x) \frac{f(u,x,y)}{f(u,x)}
\]

so that \( f(u,x,y)/f(u,x) \) is the conditional density of the severity of ruin, conditioning on the fact that ruin has occurred and on the surplus level immediately prior to ruin. By comparison with (1.1) this is the same as the conditional density of a claim that exceeds \( x \). However, the conditional density in which we are interested is not that of an ordinary claim, but is that of the claim causing ruin, and the distribution of the claim causing ruin is not \( p(x) \). Thus, it would appear that the fact that we are dealing with the claim causing ruin is irrelevant. We will show in Sections 3 and 4 that this is not the case.

2. Results for \( F(u,x,y) \)

Let \( Z_0 = 0 \), and consider the following events:

**Event 1:** Ruin occurs with severity of ruin between \( y \) and \( y + x \).

**Event 2:** Ruin occurs with a surplus immediately prior to ruin less than \( x \) and severity of ruin greater than \( y \).

The probability of Event 1 is \( G(0,x+y) - G(0,y) \) and the probability of Event 2 is \( F(0,x) - F(0,x,y) \). We can show that these probabilities are equal by considering dual events (see Feller (1966)). We construct a process \( (Z^*_t) \) as follows. Define \( \tau \) to be the time of the first upcrossing through \(-y\) of a realisation of the surplus process satisfying the conditions of Event 1, and define

\[
Z^*_t = Z^*_{\tau - t} - y \quad \text{for} \quad 0 \leq t \leq \tau
\]
\[
Z^*_t = Z^*_{\tau} \quad \text{for} \quad t > \tau
\]

Figure 1 shows a realisation of the surplus process that satisfies the conditions of Event 1 and Figure 2 shows the corresponding realisation of the
dual process. To obtain the realisation of the process in Figure 2 from that in Figure 1, shift the graph of \( Z_t \) up by \( y \) units for \( 0 \leq t \leq r \), then rotate it through 180 degrees.

Thus for any realisation of the surplus process that satisfies the conditions of Event 1, there is a unique realisation of the dual process \( \left( Z_t^* \right) \) which has the same probability density and which satisfies the conditions of Event 2. Hence

\[
G(0,x + y) - G(0,y) = F(0,x) - F(0,x,y)
\]
or
\[
F(0,x,y) = F(0,x) + G(0,y) - G(0,x + y) \tag{2.1}
\]

Formula (2.1) allows us to find a simple expression for \( F(u,x,y) \) when \( u \leq x \).

For a fixed value of \( u \leq x \), consider the following events:

**Event 3:** Ruin occurs from initial surplus zero and

(i) there is an upcrossing of the surplus process through \( u \) prior to ruin,

and

(ii) the surplus immediately prior to ruin is greater than or equal to \( x \) and

the severity of ruin is less than \( y \).

**Event 4:** Ruin occurs from initial surplus zero and the surplus immediately prior to ruin is greater than or equal to \( x \) and the severity of ruin is less than \( y \).

It is clear that any realisation of the surplus process that satisfies the conditions of Event 3 satisfies the conditions of Event 4, and vice versa. Dickson and Gray (1984) denote the probability of (i) in Event 3 by \( \chi(0,u) \) and show that \( \chi(0,u) = \delta(0)/\delta(u) \). Hence the probability of Event 3 is

\[
\chi(0,u)(G(u,y) - F(u,x,y))
\]

The probability of Event 4 is \( G(0,y) - F(0,x,y) \), which, by (2.1), is the same as \( G(0,x + y) - G(0,x) \) (writing \( G(0,x) \) for \( F(0,x) \)). Hence

\[
\chi(0,u)(G(u,y) - F(u,x,y)) = G(0,x + y) - G(0,x)
\]

so that for \( 0 \leq u \leq x \),

\[
F(u,x,y) = G(u,y) - \frac{\psi(u)}{1 - \psi(0)} \left( G(0,x + y) - G(0,x) \right) \tag{2.2}
\]

3. The density \( f(0,x,y) \)

Differentiating (2.1) with respect to both \( x \) and \( y \) gives
\[ f(0,x,y) = \frac{\lambda}{c} \frac{p(x + y)}{1 - P(x)} = f(0,x) \frac{p(x + y)}{1 - P(x)} \quad (3.1) \]

which proves (1.1) when \( u = 0 \). To see why the second term on the right hand side of (3.1) depends only on the individual claim amount distribution we use the distribution of the claim causing ruin when \( Z_0 = 0 \). This distribution has density

\[ h(0,z) = (\lambda/c)zp(z) \]

(see Dufresne and Gerber (1988)) and so

\[ f(0,x,y) = h(0,x + y)/(x + y) \]

giving

\[ \frac{f(0,x,y)}{f(0,x)} = \frac{h(0,x + y)/(x + y)}{h(0,z)/(z)} = \frac{p(x + y)}{(\lambda/c)(1 - P(x))} = \frac{1 - F(x)}{1 - F(u)} \]

We can see that although the conditional density of the severity of ruin can be expressed in terms of the density of the claim causing ruin, the form of this latter density is such that the conditional density of the severity of ruin depends only on the individual claim amount distribution.

4. The relationship between \( f(u,x) \) and \( f(u,x,y) \)

Dickson (1992) shows that

\[ f(u,x)/f(0,x) = 1/\chi(0,u) \quad \text{for} \quad u < x \quad (4.1) \]

and

\[ f(u,x)/f(0,x) = [\psi(u - x) - \psi(u)]/[1 - \psi(0)] \quad \text{for} \quad x > u \quad (4.2) \]

Differentiating (2.2) with respect to both \( x \) and \( y \) gives

\[ f(u,x,y) = f(0,x,y)/\chi(0,u) \]

or

\[ f(u,x,y)/f(0,x,y) = 1/\chi(0,u) \quad \text{for} \quad u < x \quad (4.3) \]

Comparing (4.1) and (4.3) we conclude that \( f(u,x,y)/f(u,x) \) is independent of \( u \) for \( u < x \) and equal to \( p(x + y)/(1 - P(x)) \). To explain why \( f(u,x,y)/f(u,x) \) is independent of \( u \), we consider the following events.

Event 5: Ruin occurs from initial surplus zero and

(i) there is an upcrossing of the surplus process through \( u \) prior to ruin, and

(ii) the surplus immediately prior to ruin is less than \( x \), where \( x > u \), and the severity of ruin is less than \( y \).
Event 6: Ruin occurs from initial surplus zero and
(1) the surplus immediately prior to ruin is less than \( y \) and the severity of
ruin is less than \( x \), and
(2) there is an upcrossing of the surplus process through \(-u\) prior to the
first upcrossing through zero. (This is guaranteed if the severity of
ruin exceeds \( u \).)

If we define \( \tau \) to be the time of the first upcrossing through zero of a
realisation of the surplus process which satisfies the conditions of Event 5,
then we can construct a dual process \( \{Z^*_t\} \) defined by
\[
Z^*_t = -Z_{\tau - t} \quad \text{for} \quad 0 \leq t \leq \tau \\
Z^*_t = Z_t \quad \text{for} \quad t > \tau
\]

Then for any realisation of the surplus process satisfying the conditions of
Event 5, there is a unique realisation of the dual process which satisfies
the conditions of Event 6, and which has the same probability density. Figure
3 shows a realisation of the surplus process that satisfies the conditions of
Event 5 and Figure 4 shows the corresponding realisation of the dual process.

Event 5 has probability \( \chi(0,u)f(u,x,y) \) and by considering the severity of
ruin we can write the probability of Event 6 as
\[
\int_0^y \int_0^z f(0,w,z)\xi(u - z,u)dzdw + \int_0^u \int_0^x f(0,w,z)dzdw
\]
where \( \xi(u - z,u) = 1 - \chi(u - z,u) \). (This term gives the probability that the
surplus falls below \(-u\) before attaining zero, so that condition (2) of
Event 6 is satisfied, as illustrated in Figure 6.) Differentiating the
probability of each event with respect to both \( x \) and \( y \) we have
\[
\chi(0,u)f(u,x,y) = f(0,y,x) \quad \text{for} \quad u < x
\]
and integrating out \( y \) gives
\[
\chi(0,u)f(u,x) = g(0,x)
\]
Hence the conditional distribution of the severity of ruin in the process
\( \{Z_t\} \) is identical to the conditional distribution of the severity of ruin in
the process \( \{Z^*_t\} \) because of the symmetry of \( f(0,x,y) \). Thus, considering
\( f(u,x,y)/f(u,x) \) for the process \( \{Z_t\} \) is the same as considering
\( f(0,x,y)/f(0,x) \) for the process \( \{Z^*_t\} \) and this explains why \( f(u,x,y)/f(u,x) \) is
independent of \( u \).

The corresponding result when \( u > x \) is found by considering the following
events:

Event 7: Ruin occurs from initial surplus zero and
(i) there is an upcrossing of the surplus process through $u$, where $u > x$,
prior to ruin, and
(ii) the surplus immediately prior to ruin is less than $x$ and the severity of
ruin is less than $y$.

Event 8: Ruin occurs from initial surplus zero and
(i) the surplus immediately prior to ruin is less than $y$ and the severity of
ruin is less than $x$, and
(ii) the surplus falls below $-u$ before there is an upcrossing of the surplus
process through zero.

For any realisation of the surplus process satisfying the conditions of Event 7 we can use the dual construction described earlier in this section to construct a unique realisation of a dual process which has the same probability density and which satisfies the conditions of Event 8. Figure 5 shows a realisation of the surplus process that satisfies the conditions of Event 7 and Figure 6 shows the corresponding realisation of the dual process. Event 7 has probability $\chi(0,u)F(u,x,y)$ and Event 8 has probability

$$\int_0^x \int_0^w f(0,w,z)\xi(u - z,u)dzdw$$

Hence

$$F(u,x,y) = \frac{1}{1 - \psi(0)} \int_0^w \int_0^y f(0,w,z)[\psi(u - z) - \psi(u)]dzdw$$

(4.4)

and differentiation with respect to both $x$ and $y$ yields

$$f(u,x,y) = f(0,y,x) \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)}$$

for $u > x$

Integrating out $y$ we have

$$f(u,x) = g(0,x) \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)}$$

Since $f(0,y,x)$ and $g(0,x)$ can be replaced by $f(0,x,y)$ and $f(0,x)$ we have

$$\frac{f(u,x,y)}{f(0,x,y)} = \frac{\psi(u - x) - \psi(u)}{f(0,x,y)}$$

for $u > x$.

(This follows from (4.2).) Hence $f(u,x,y)/f(u,x)$ is also independent of $u$ for
$u > x$, for the same reasons as when $u < x$, and so for all values of $u$.
\[ f(u,x,y) = f(u,x) \frac{p(x+y)}{1 - P(x)} \]

For completeness we note that (4.4) can be shown to yield

\[ F(u,x,y) = G(u,y) - G(u-x,x+y) + G(u-x,x) - \frac{G(0,x+y) - G(0,x)}{1 - \psi(0)} \left( \psi(u-x) - \psi(u) \right) \]

for \( u \geq x \). To prove this result we replace \( f(0,w,z) \) by \((\lambda/c)p(w+z)\) in (4.4) and integrate to get

\[ F(u,x,y) = \frac{1}{\delta(0)} \left( \int_0^u \psi(u-z)[g(0,z) - g(0,z+y)]dz \right) - \psi(u)F(0,y,x) \]

We can then integrate by parts using the results

\[ \frac{1 - G(0,x+y)}{\delta(0)} \frac{d}{dx} \psi(u-x) = \frac{d}{dx} G(u-x,x) \]

(see Dickson(1992)) and

\[ \frac{1 - G(0,x+y)}{\delta(0)} \frac{d}{dx} \psi(u-x) = \frac{d}{dx} G(u-x,x+y) \]

which is proved by the same method.

5. Other results

We can interpret other known ruin theory results via dual events. For example, if we let \( x \to \infty \) in Events 5 and 6 we have

\[ \chi(0,u)G(u,y) = \int_0^\infty f(0,w,z)\xi(u-z,u)dw + \int_u^\infty f(0,w,z)dw \]

(5.1)

and differentiation with respect to \( y \) gives

\[ \chi(0,u)g(u,y) = \int_0^u f(0,y,z)\xi(u-z,u)dz + \int_u^\infty f(0,y,z)dz \]

\[ \Rightarrow g(u,y) = \frac{1}{\delta(0)} \frac{\lambda}{c} \left[ \int_0^u p(y+z)\psi(u-z)dz - \psi(u)\int_0^u p(y+z)dz + \delta(u)\int_0^\infty p(y+z)dz \right] \]

\[ = \frac{1}{\delta(0)} \frac{\lambda}{c} \left[ \int_0^u p(y+z)\psi(u-z)dz - \psi(u)(P(u+y) - P(y)) + \delta(u)(1 - P(u+y)) \right] \]

\[ - \frac{1}{\delta(0)} \frac{\lambda}{c} \left[ \int_0^\infty p(y+z)\psi(u-z)dz + g(0,u+y) - \psi(u)g(0,y) \right] \]

This expression can also be obtained via Laplace transforms (see Panjer and Willmot (1992)), but whilst that approach provides the result, it does not explain the result's origin.

Similarly, if we let \( y \to \infty \) in equation (5.1) we have
\[ \chi(0,u)\psi(u) = \int_0^\infty f(0,w,z)\xi(u-z)du + \int_0^\infty f(0,\omega,z)dz + \delta(u) \int_0^\infty p(w+z)dz \]

\[ \psi(u) = \frac{1}{\delta(0)} \frac{\lambda}{c} \left( \int_0^u p(u-z)\psi(u-z)dz - \psi(u) \int_0^u [1 - P(z)]dz + \delta(u) \int_0^\infty [1 - P(z)]dz \right) \]

Hence

\[ \psi(u) \left( 1 + \frac{\Psi(0)}{\delta(0)} \right) = \frac{1}{\delta(0)} \frac{\lambda}{c} \left( \int_0^u \psi(u-z)[1-P(z)]dz + \int_0^\infty [1-P(z)]dz \right) \]

which leads to the familiar formula

\[ \psi(u) = \frac{\lambda}{c} \int_0^u \psi(u-z)[1-P(z)]dz + \frac{\lambda}{c} \int_0^\infty [1-P(z)]dz \]
References


Figure 1: A realisation of the surplus process satisfying the conditions of Event 1

Figure 2: The dual of the surplus process in Figure 1
Figure 3: A realisation of the surplus process satisfying the conditions of Event 5

Figure 4: The dual of the surplus process in Figure 3
Figure 5: A realisation of the surplus process satisfying the conditions of Event 7

Figure 6: The dual of the surplus process in Figure 5
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