DICKSON4

ISBN 0 7325 0573 9

THE UNIVERSITY OF MELBOURNE

RUIN PROBLEMS AND DUAL EVENTS

by

David C M Dickson and Alfredo Egidio dos Reis

RESEARCH PAPER NUMBER 4

AUGUST 1993

Centre for Actuarial Studies
Department of Economics
The University of Melbourne
Parkville, Victoria, 3052
Australia.

RUIN PROBLEMS AND DUAL EVENTS

by

David C M Dickson and Alfredo Egidio dos Reis

Abstract

Dickson (1992) uses dual events to explain results relating to the distribution of the surplus immediately prior to ruin in the classical surplus process. In this paper we show that dual events can be used to explain other results in ruin theory. In particular we prove and explain the relationship between the density of the surplus immediately prior to ruin, and the joint density of the surplus immediately prior to ruin and the severity of ruin.

1. Introduction and Notation

In the classical continuous time risk model, the insurer's surplus at time t is denoted Z_t and given by

$$Z_t = Z_0 + ct - X_t$$

where Z_0 is the insurer's initial surplus, c is the premium income per unit time, assumed to be received continuously, and X_t denotes aggregate claims up to time t. The aggregate claims process is a compound Poisson process with Poisson parameter λ. The individual claim amount distribution is denoted P(x), has density function p(x) and mean p_1, and is such that P(0) = 0. We also assume that $c > \lambda p_1$.

We denote by $\psi(u)$ the probability of ultimate ruin from initial surplus u. Let T denote the time of ruin, so that $\psi(u) = P(T < \infty | Z_0 = u)$. We define $\delta(u) = 1 - \psi(u)$ and

$$G(u,y) = P(T < \infty \text{ and } Z_T > -y | Z_0 = u)$$

$$F(u,x) = P(T < \infty \text{ and } Z_{\tilde{T}} < x | Z_0 = u)$$

$$F(u,x,y) = P(T < \infty \text{ and } Z_T > -y \text{ and } Z_{\tilde{T}} < x | Z_0 = u)$$

where $Z_{\tilde{T}}$ denotes the surplus immediately prior to ruin given that ruin occurs. Let g(u,y), f(u,x) and f(u,x,y) respectively denote the (defective) densities associated with G(u,y), F(u,x) and F(u,x,y).

Dufresne and Gerber (1988) show that g(0,y) = f(0,y) and it is well known (see, for example, Bowers *et al* (1986)) that

$$g(0,y) = (\lambda/c)(1 - P(y))$$

Dickson (1992) uses dual events to explain the symmetry of the distributions of the surplus immediately prior to ruin and the severity of ruin when $Z_0 = 0$ and shows (without finding the functional form) that

$$f(0,x,y) = f(0,y,x)$$

Dufresne and Gerber (1988) state that

$$f(u,x,y) = f(u,x)\,\frac{p(x + y)}{1 - P(x)} \tag{1.1}$$

but do not offer a proof of this result. We will prove this result, but our main purpose is to explain this result. It is trivially true that

$$f(u,x,y) = f(u,x)\,\frac{f(u,x,y)}{f(u,x)}$$

so that $f(u,x,y)/f(u,x)$ is the conditional density of the severity of ruin, conditioning on the fact that ruin has occurred and on the surplus level immediately prior to ruin. By comparison with (1.1) this is the same as the conditional density of a claim that exceeds x. However, the conditional density in which we are interested is not that of an ordinary claim, but is that of the claim causing ruin, and the distribution of the claim causing ruin is not $P(x)$. Thus, it would appear that the fact that we are dealing with the claim causing ruin is irrelevant. We will show in Sections 3 and 4 that this is not the case.

2. Results for F(u,x,y)

Let $Z_0 = 0$, and consider the following events:

Event 1: Ruin occurs with severity of ruin between y and y + x.

Event 2: Ruin occurs with a surplus immediately prior to ruin less than x and severity of ruin greater than y.

The probability of Event 1 is $G(0,x + y) - G(0,y)$ and the probability of Event 2 is $F(0,x) - F(0,x,y)$. We can show that these probabilities are equal by considering dual events (see Feller (1966)). We construct a process $\{Z_t^*\}$ as follows. Define τ to be the time of the first upcrossing through -y of a realisation of the surplus process satisfying the conditions of Event 1, and define

$$Z_t^* = -Z_{\tau - t} - y \quad \text{for } 0 \le t \le \tau$$

$$Z_t^* = Z_t \quad \text{for } t > \tau$$

Figure 1 shows a realisation of the surplus process that satisfies the conditions of Event 1 and Figure 2 shows the corresponding realisation of the dual process. To obtain the realisation of the process in Figure 2 from that in Figure 1, shift the graph of Z_t up by y units for $0 \le t \le \tau$, then rotate it through 180 degrees.

Thus for any realisation of the surplus process that satisfies the conditions of Event 1, there is a unique realisation of the dual process $\{Z_t^*\}$ which has the same probability density and which satisfies the conditions of Event 2. Hence

$$G(0,x + y) - G(0,y) = F(0,x) - F(0,x,y)$$

or

$$F(0,x,y) = F(0,x) + G(0,y) - G(0,x + y) \qquad (2.1)$$

Formula (2.1) allows us to find a simple expression for $F(u,x,y)$ when $u \le x$. For a fixed value of $u \le x$, consider the following events:

Event 3: Ruin occurs from initial surplus zero and

(i) there is an upcrossing of the surplus process through u prior to ruin, and

(ii) the surplus immediately prior to ruin is greater than or equal to x and the severity of ruin is less than y.

Event 4: Ruin occurs from initial surplus zero and the surplus immediately prior to ruin is greater than or equal to x and the severity of ruin is less than y.

It is clear that any realisation of the surplus process that satisfies the conditions of Event 3 satisfies the conditions of Event 4, and vice versa. Dickson and Gray (1984) denote the probability of (i) in Event 3 by $\chi(0,u)$ and show that $\chi(0,u) = \delta(0)/\delta(u)$. Hence the probability of Event 3 is

$$\chi(0,u)(G(u,y) - F(u,x,y))$$

The probability of Event 4 is $G(0,y) - F(0,x,y)$, which, by (2.1), is the same as $G(0,x + y) - G(0,x)$ (writing $G(0,x)$ for $F(0,x)$). Hence

$$\chi(0,u)(G(u,y) - F(u,x,y)) = G(0,x + y) - G(0,x)$$

so that for $0 \le u \le x$,

$$F(u,x,y) = G(u,y) - \frac{1 - \psi(u)}{1 - \psi(0)} \Big(G(0,x + y) - G(0,x)\Big) \qquad (2.2)$$

3. The density f(0,x,y)

Differentiating (2.1) with respect to both x and y gives

$$f(0,x,y) = \frac{\lambda}{c}\, p(x + y) = f(0,x)\, \frac{p(x + y)}{1 - P(x)} \qquad (3.1)$$

which proves (1.1) when $u = 0$. To see why the second term on the right hand side of (3.1) depends only on the individual claim amount distribution we use the distribution of the claim causing ruin when $Z_0 = 0$. This distribution has density

$$h(0,z) = (\lambda/c)zp(z)$$

(see Dufresne and Gerber (1988)) and so

$$f(0,x,y) = h(0,x + y)/(x + y)$$

giving

$$\frac{f(0,x,y)}{f(0,x)} = \frac{h(0,x + y)/(x + y)}{(\lambda/c)(1 - P(x))} = \frac{p(x + y)}{1 - P(x)}$$

We can see that although the conditional density of the severity of ruin can be expressed in terms of the density of the claim causing ruin, the form of this latter density is such that the conditional density of the severity of ruin depends only on the individual claim amount distribution.

4. The relationship between f(u,x) and f(u,x,y)

Dickson (1992) shows that

$$f(u,x)/f(0,x) = 1/\chi(0,u) \quad \text{for } u < x \qquad (4.1)$$

and

$$f(u,x)/f(0,x) = [\psi(u - x) - \psi(u)]/[1 - \psi(0)] \quad \text{for } x > u \qquad (4.2)$$

Differentiating (2.2) with respect to both x and y gives

$$f(u,x,y) = f(0,x,y)/\chi(0,u)$$

or

$$f(u,x,y)/f(0,x,y) = 1/\chi(0,u) \quad \text{for } u < x \qquad (4.3)$$

Comparing (4.1) and (4.3) we conclude that $f(u,x,y)/f(u,x)$ is independent of u for $u < x$ and equal to $p(x + y)/(1 - P(x))$. To explain why $f(u,x,y)/f(u,x)$ is independent of u, we consider the following events.

Event 5: Ruin occurs from initial surplus zero and

(i) there is an upcrossing of the surplus process through u prior to ruin, and

(ii) the surplus immediately prior to ruin is less than x, where $x > u$, and the severity of ruin is less than y.

Event 6: Ruin occurs from initial surplus zero and

(i) the surplus immediately prior to ruin is less than y and the severity of ruin is less than x, and

(ii) there is an upcrossing of the surplus process through -u prior to the first upcrossing through zero. (This is guaranteed if the severity of ruin exceeds u.)

If we define τ to be the time of the first upcrossing through zero of a realisation of the surplus process which satisfies the conditions of Event 5, then we can construct a dual process $\{Z_t^*\}$ defined by

$$Z_t^* = -Z_{\tau-t} \quad \text{for } 0 \le t \le \tau$$

$$Z_t^* = Z_t \quad \text{for } t > \tau$$

Then for any realisation of the surplus process satisfying the conditions of Event 5, there is a unique realisation of the dual process which satisfies the conditions of Event 6, and which has the same probability density. Figure 3 shows a realisation of the surplus process that satisfies the conditions of Event 5 and Figure 4 shows the corresponding realisation of the dual process.

Event 5 has probability $\chi(0,u)F(u,x,y)$ and by considering the severity of ruin we can write the probability of Event 6 as

$$\int_0^y\int_0^u f(0,w,z)\xi(u - z,u)dzdw + \int_0^y\int_u^x f(0,w,z)dzdw$$

where $\xi(u - z,u) = 1 - \chi(u - z,u)$. (This term gives the probability that the surplus falls below -u before attaining zero, so that condition (ii) of Event 6 is satisfied, as illustrated in Figure 6.) Differentiating the probability of each event with respect to both x and y we have

$$\chi(0,u)f(u,x,y) = f(0,y,x) \quad \text{for } u < x$$

and integrating out y gives

$$\chi(0,u)f(u,x) = g(0,x)$$

Hence the conditional distribution of the severity of ruin in the process $\{Z_t\}$ is identical to the conditional distribution of the severity of ruin in the process $\{Z_t^*\}$ because of the symmetry of $f(0,x,y)$. Thus, considering $f(u,x,y)/f(u,x)$ for the process $\{Z_t\}$ is the same as considering $f(0,x,y)/f(0,x)$ for the process $\{Z_t^*\}$ and this explains why $f(u,x,y)/f(u,x)$ is independent of u.

The corresponding result when u > x is found by considering the following events:

Event 7: Ruin occurs from initial surplus zero and

(i) there is an upcrossing of the surplus process through u, where $u > x$, prior to ruin, and

(ii) the surplus immediately prior to ruin is less than x and the severity of ruin is less than y.

Event 8: Ruin occurs from initial surplus zero and

(i) the surplus immediately prior to ruin is less than y and the severity of ruin is less than x, and

(ii) the surplus falls below -u before there is an upcrossing of the surplus process through zero.

For any realisation of the surplus process satisfying the conditions of Event 7 we can use the dual construction described earlier in this section to construct a unique realisation of a dual process which has the same probability density and which satisfies the conditions of Event 8. Figure 5 shows a realisation of the surplus process that satisfies the conditions of Event 7 and Figure 6 shows the corresponding realisation of the dual process. Event 7 has probability $\chi(0,u)F(u,x,y)$ and Event 8 has probability

$$\int_0^y\int_0^x f(0,w,z)\xi(u - z,u)dzdw$$

Hence

$$F(u,x,y) = \frac{1}{1 - \psi(0)} \int_0^y\int_0^x f(0,w,z)[\psi(u - z) - \psi(u)]dzdw \qquad (4.4)$$

and differentiation with respect to both x and y yields

$$f(u,x,y) = f(0,y,x) \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)} \qquad \text{for } u > x$$

Integrating out y we have

$$f(u,x) = g(0,x) \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)}$$

Since $f(0,y,x)$ and $g(0,x)$ can be replaced by $f(0,x,y)$ and $f(0,x)$ we have

$$\frac{f(u,x,y)}{f(0,x,y)} = \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)} = \frac{f(u,x)}{f(0,x)} \qquad \text{for } u > x$$

(This follows from (4.2).) Hence $f(u,x,y)/f(u,x)$ is also independent of u for $u > x$, for the same reasons as when $u < x$, and so for all values of u

$$f(u,x,y) = f(u,x)\,\frac{p(x + y)}{1 - P(x)}$$

For completeness we note that (4.4) can be shown to yield

$$F(u,x,y) = G(u,y) - G(u-x,x+y) + G(u-x,x) - \frac{G(0,x+y) - G(0,x)}{1 - \psi(0)}\Big(\psi(u-x) - \psi(u)\Big)$$

for $u \geq x$. To prove this result we replace $f(0,w,z)$ by $(\lambda/c)p(w + z)$ in (4.4) and integrate to get

$$F(u,x,y) = \frac{1}{\delta(0)}\Big(\int_0^x \psi(u - z)[g(0,z) - g(0,z + y)]dz - \psi(u)F(0,y,x)\Big)$$

We can then integrate by parts using the results

$$\frac{1 - G(0,x)}{\delta(0)}\frac{d}{dx}\psi(u - x) = \frac{d}{dx}G(u - x,x)$$

(see Dickson(1992)) and

$$\frac{1 - G(0,x + y)}{\delta(0)}\frac{d}{dx}\psi(u - x) = \frac{d}{dx}G(u - x,x + y)$$

which is proved by the same method.

5. Other results

We can interpret other known ruin theory results via dual events. For example, if we let $x \to \infty$ in Events 5 and 6 we have

$$\chi(0,u)G(u,y) = \int_0^y\int_0^u f(0,w,z)\xi(u - z,u)dzdw + \int_0^y\int_u^\infty f(0,w,z)dzdw \qquad (5.1)$$

and differentiation with respect to y gives

$$\chi(0,u)g(u,y) = \int_0^u f(0,y,z)\xi(u - z,u)dz + \int_u^\infty f(0,y,z)dz$$

$$\Rightarrow g(u,y) = \frac{1}{\delta(0)}\frac{\lambda}{c}\Big(\int_0^u p(y+z)\psi(u-z)dz - \psi(u)\int_0^u p(y+z)dz + \delta(u)\int_u^\infty p(y+z)dz\Big)$$

$$= \frac{1}{\delta(0)}\frac{\lambda}{c}\Big(\int_0^u p(y+z)\psi(u-z)dz - \psi(u)(P(u+y) - P(y)) + \delta(u)(1 - P(u+y))\Big)$$

$$= \frac{1}{\delta(0)}\Big(\frac{\lambda}{c}\int_0^u p(y+z)\psi(u-z)dz + g(0,u+y) - \psi(u)g(0,y)\Big)$$

This expression can also be obtained via Laplace transforms (see Panjer and Willmot (1992)), but whilst that approach provides the result, it does not explain the result's origin.

Similarly, if we let $y \to \infty$ in equation (5.1) we have

$$\chi(0,u)\psi(u) = \int_0^\infty\int_0^u f(0,w,z)\xi(u - z,u)dzdw + \int_0^\infty\int_u^\infty f(0,w,z)dzdw$$

$$\Rightarrow \psi(u) = \frac{1}{\delta(0)}\frac{\lambda}{c}\left(\int_0^\infty\int_0^u p(w + z)\psi(u - z)dzdw - \psi(u)\int_0^\infty\int_0^u p(w + z)dzdw + \delta(u)\int_0^\infty\int_u^\infty p(w + z)dzdw\right)$$

$$= \frac{1}{\delta(0)}\frac{\lambda}{c}\left(\int_0^u \psi(u - z)[1 - P(z)]dz - \psi(u)\int_0^u [1 - P(z)]dz + \delta(u)\int_u^\infty [1 - P(z)]dz\right)$$

Hence

$$\psi(u)\left(1 + \frac{\psi(0)}{\delta(0)}\right) = \frac{1}{\delta(0)}\frac{\lambda}{c}\left(\int_0^u \psi(u - z)[1 - P(z)]dz + \int_u^\infty [1 - P(z)]dz\right)$$

which leads to the familiar formula

$$\psi(u) = \frac{\lambda}{c}\int_0^u \psi(u - z)[1 - P(z)]dz + \frac{\lambda}{c}\int_u^\infty [1 - P(z)]dz$$

References

Bowers, N.L., H.U. Gerber, J.C. Hickman, D.A. Jones and C.J. Nesbitt (1987). *Actuarial Mathematics*. Society of Actuaries, Itasca, IL.

Dickson, D.C.M. and J.R. Gray (1984). Approximations to ruin probability in the presence of an upper absorbing barrier. *Scandinavian Actuarial Journal, 105-115*.

Dickson, D.C.M. (1992). On the distribution of the surplus prior to ruin. *Insurance: Mathematics and Economics 11, 191-207.*

Dufresne, F. and H.U. Gerber (1988). The surpluses immediately before and at ruin, and the amount of the claim causing ruin. *Insurance: Mathematics and Economics 7, 193-199*.

Feller, W. (1966). *An Introduction to probability theory and its applications, Volume 2*. Wiley, New York.

Gerber, H.U., M.J. Goovaerts and R. Kaas (1987). On the probability and severity of ruin. *ASTIN Bulletin 17, 151-163*.

Panjer, H.H. and G.E. Willmot (1992). *Insurance Risk Models*. Society of Actuaries, Schaumberg, IL.

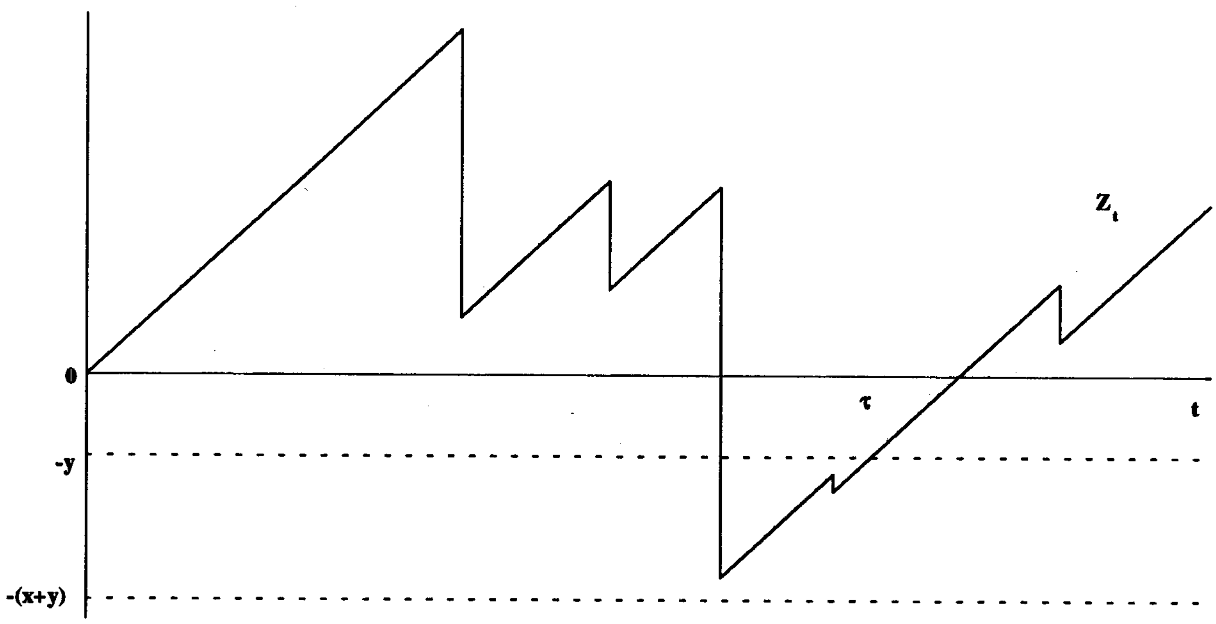

Figure 1: A realisation of the surplus process satisfying the conditions of Event 1

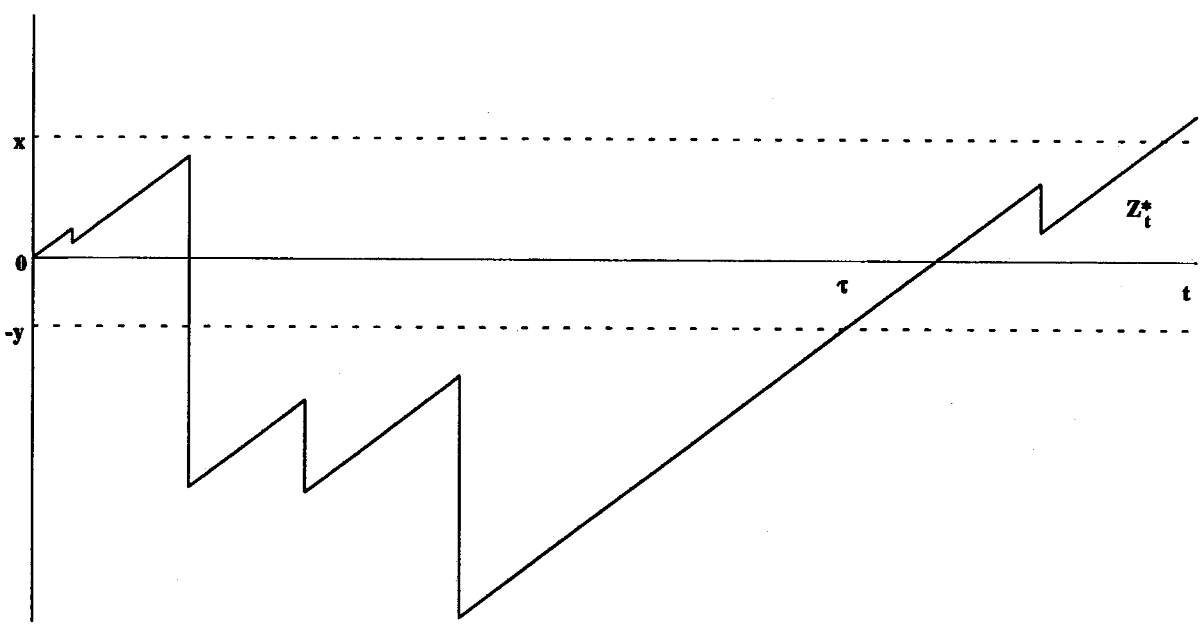

Figure 2: The dual of the surplus process in Figure 1

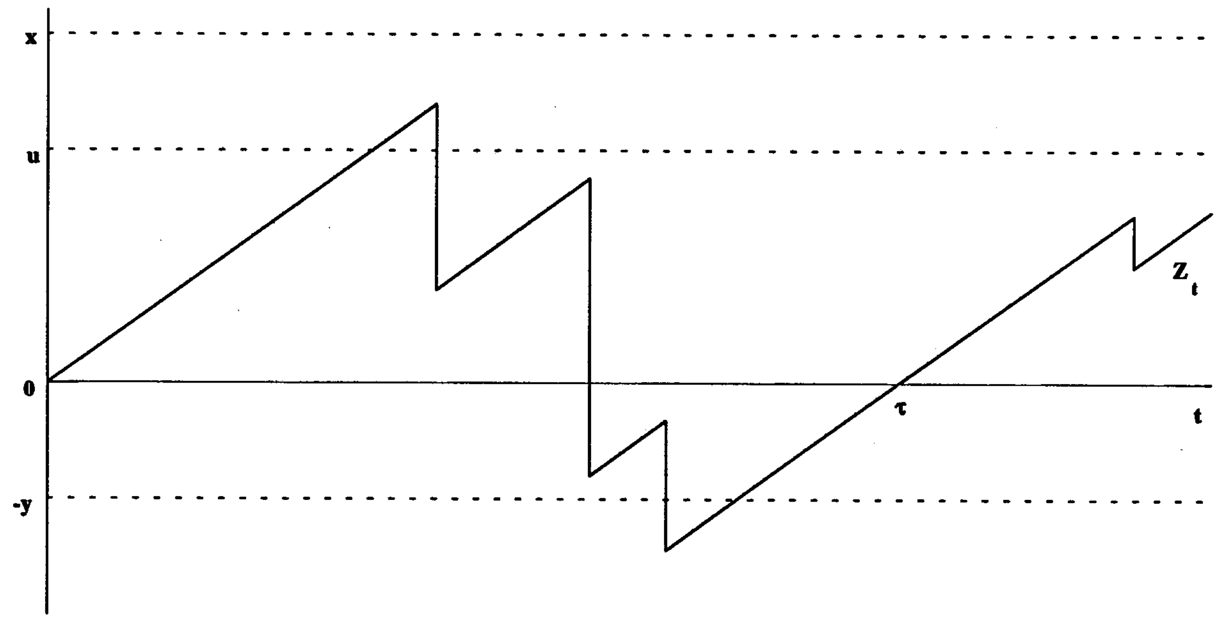

Figure 3: A realisation of the surplus process satisfying the conditions of Event 5

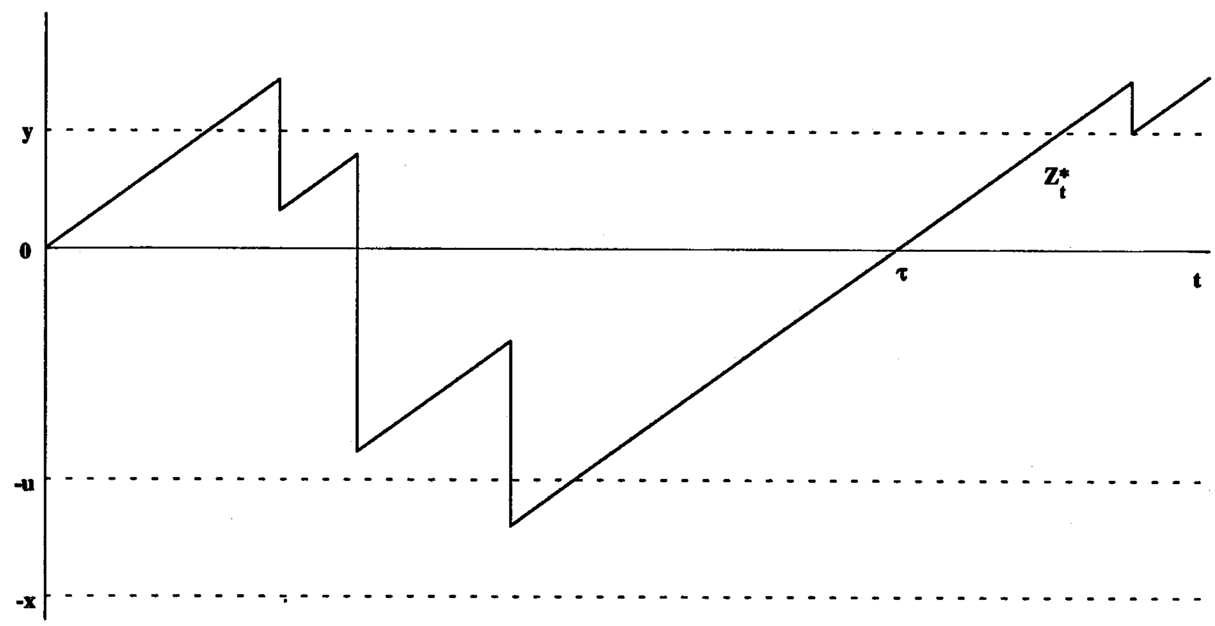

Figure 4: The dual of the surplus process in Figure 3

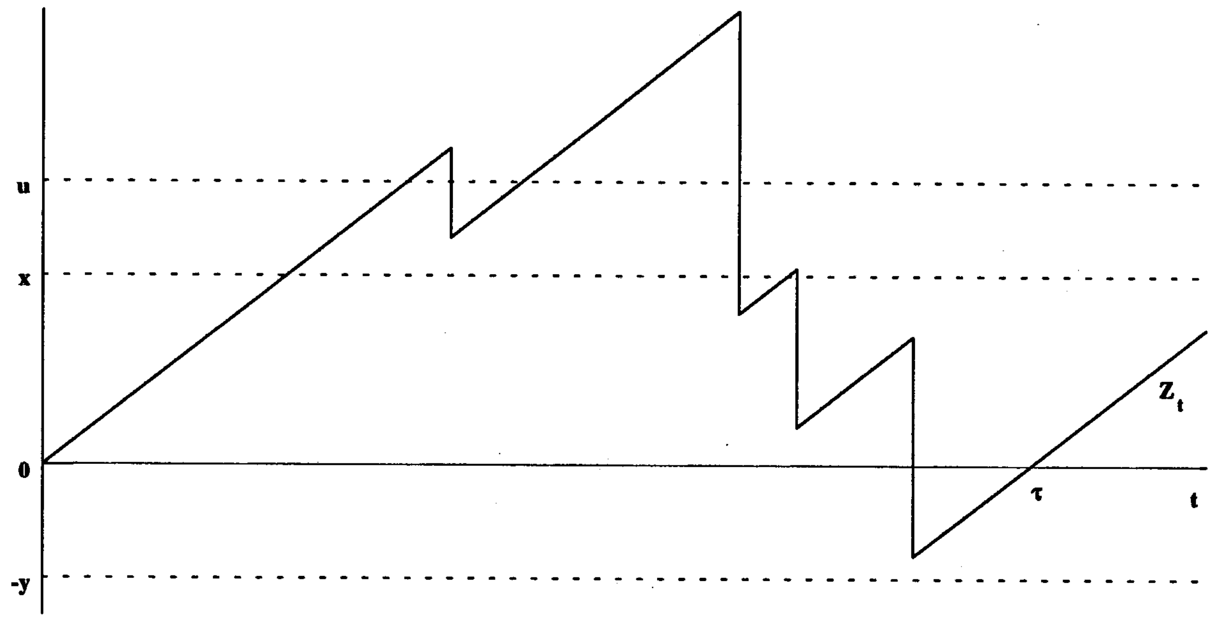

Figure 5: A realisation of the surplus process satisfying the conditions of Event 7

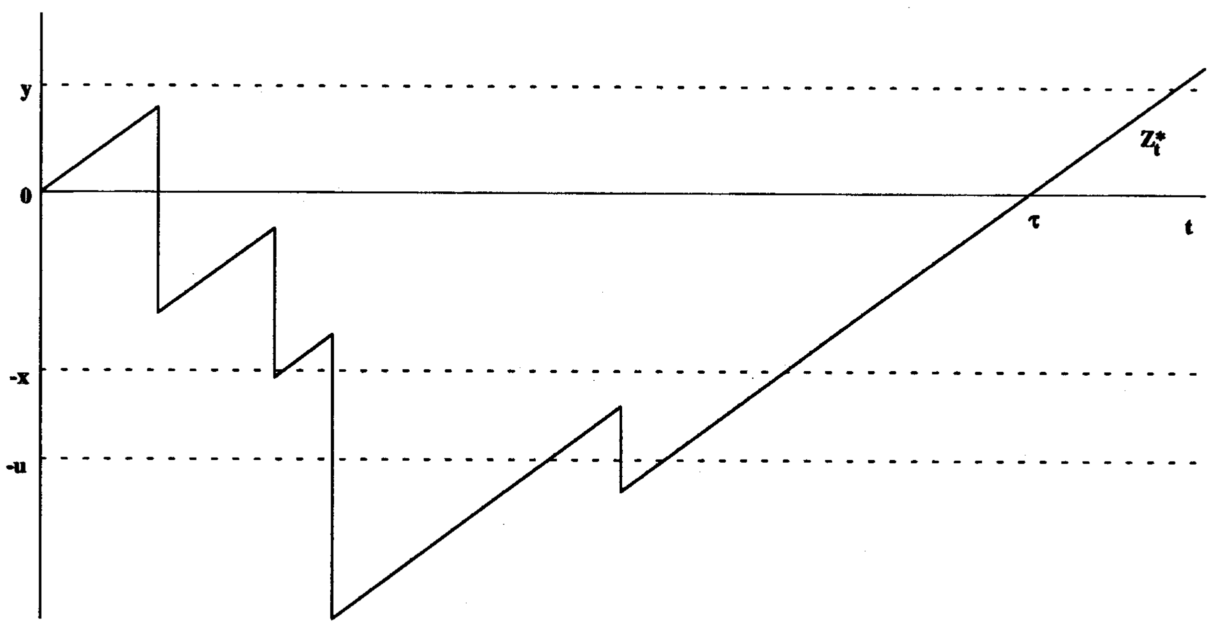

Figure 6: The dual of the surplus process in Figure 5

RESEARCH PAPER SERIES

No.	Date	Subject	Author
1	MAR 93	AUSTRALIAN SUPERANNUATION : THE FACTS, THE FICTION, THE FUTURE	David M Knox
2	APR 93	AN EXPONENTIAL BOUND FOR RUIN PROBABILITIES	David C M Dickson
3	APR 93	SOME COMMENTS ON THE COMPOUND BINOMIAL MODEL	David C M Dickson
4	AUG 93	RUIN PROBLEMS AND DUAL EVENTS	David CM Dickson Alfredo E dos Reis
5	SEP 93	CONTEMPORARY ISSUES IN AUSTRALIAN SUPERANNUATION - A CONFERENCE SUMMARY	David M Knox John Piggott
6	SEP 93	AN ANALYSIS OF THE EQUITY INVESTMENTS OF AUSTRALIAN SUPERANNUATION FUNDS	David M Knox
7	OCT 93	A CRITIQUE OF DEFINED CONTRIBUTION USING A SIMULATION APPROACH	David M Knox
8	JAN 94	REINSURANCE AND RUIN	David C M Dickson Howard R Waters
9	MAR 94	LIFETIME INCOME, TAXATION, EXPENDITURE AND SUPERANNUATION (LITES): A LIFE-CYCLE SIMULATION MODEL	Margaret E Atkinson John Creedy David M Knox
10	FEB 94	SUPERANNUATION FUNDS AND THE PROVISION OF DEVELOPMENT/VENTURE CAPITAL: THE PERFECT MATCH? YES OR NO	David M Knox
11	JUNE 94	RUIN PROBLEMS: SIMULATION OR CALCULATION?	David C M Dickson Howard R Waters
12	JUNE 94	THE RELATIONSHIP BETWEEN THE AGE PENSION AND SUPERANNUATION BENEFITS, PARTICULARLY FOR WOMEN	David M Knox
13	JUNE 94	THE COST AND EQUITY IMPLICATIONS OF THE INSTITUTE OF ACTUARIES OF AUSTRALIA PROPOSED RETIREMENT INCOMES STRATEGY	Margaret E Atkinson John Creedy David M Knox Chris Haberecht
14	SEPT 94	PROBLEMS AND PROSPECTS FOR THE LIFE INSURANCE AND PENSIONS SECTOR IN INDONESIA	Catherine Prime David M Knox
15	OCT 94	PRESENT PROBLEMS AND PROSPECTIVE PRESSURES IN AUSTRALIA'S SUPERANNUATION SYSTEM	David M Knox

16	DEC 94	PLANNING RETIREMENT INCOME IN AUSTRALIA: ROUTES THROUGH THE MAZE	Margaret E Atkinson John Creedy David M Knox
17	JAN 95	ON THE DISTRIBUTION OF THE DURATION OF NEGATIVE SURPLUS	David C M Dickson Alfredo E dos Reis
18	FEB 95	OUTSTANDING CLAIM LIABILITIES: ARE THEY PREDICTABLE?	Ben Zehnwirth
19	MAY 95	SOME STABLE ALGORITHMS IN RUIN THEORY AND THEIR APPLICATIONS	David C M Dickson Alfredo E dos Ries Howard R Waters
20	JUN 95	SOME FINANCIAL CONSEQUENCES OF THE SIZE OF AUSTRALIA'S SUPERANNUATION INDUSTRY IN THE NEXT THREE DECADES	David M Knox
21	JUN 95	MODELLING OPTIMAL RETIREMENT IN DECISIONS IN AUSTRALIA	Margaret E Atkinson John Creedy
22	JUN 95	AN EQUITY ANALYSIS OF SOME RADICAL SUGGESTIONS FOR AUSTRALIA'S RETIREMENT INCOME SYSTEM	Margaret E Atkinson John Creedy David M Knox
23	SEP 95	EARLY RETIREMENT AND THE OPTIMAL RETIREMENT AGE	Angela Ryan
24	OCT 95	APPROXIMATE CALCULATION OF MOMENTS OF RUIN RELATED DISTRIBUTIONS	David C M Dickson