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An Affine Property Of The Reciprocal Asian Option Process

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Abstract

This note describes a property the Asian option process, which neatly links the process at μ to the one at $-\mu$, where μ is the drift of the geometric Brownian motion. The proof is based on (i) a known result due to Yor, on the law of the Asian option process taken at exponential times, and (ii) a recent result on beta and gamma distributions.

OPTION PRICING; ASIAN OPTIONS, BETA DISTRIBUTION, GAMMA DISTRIBUTION

1. Relation between the Asian processes at μ and $-\mu$

Suppose W is one-dimensional standard Brownian motion, and define what this author calls the *Asian option process*, for want of a better name:

$$A_t^{(\mu)} = \int_0^t e^{2\mu s+2W_s} ds, \qquad t \geq 0, \quad \mu \in \mathbb{R}. \tag{1}$$

Theorem 1 (Yor, 1992). *Let T_λ be an exponentially distributed random variable, independent of W, with mean $1/\lambda$. Then*

$$A_{T_\lambda}^{(\mu)} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta},$$

where $B_{1,\alpha} \sim$ Beta$(1,\alpha)$ and $G_\beta \sim (\beta, 1)$ are independent, $\alpha = \frac{\mu}{2} + \frac{1}{2}\sqrt{2\lambda + \mu^2}$, $\beta = \alpha - \mu$.

Theorem 2 (Dufresne, 1998) *For any $a, b, c > 0$,*

$$\frac{G_a}{B_{b,a+c}} + G'_c \stackrel{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b,a}}. \tag{11}$$

where $G_a \sim$ Gamma$(a,1)$, $G'_c \sim$ Gamma$(c,1)$, $B_{b,a+c} \sim$ Beta$(b, a+c)$, $B_{b,a} \sim$ Beta(b,a) and all variables are independent.

Theorem 3 *For any $\mu, t > 0$,*

$$\frac{1}{2A_t^{(\mu)}} + G_\mu \stackrel{\mathcal{L}}{=} \frac{1}{2A_t^{(-\mu)}},$$

where $G_\mu \sim$ Gamma$(\mu, 1)$ is independent of W.

The last result follows directly from the two previous ones, upon setting $a = \beta$, $b = 1$, $c = \mu$, and then inverting the Laplace transform represented by the exponential time T_λ.

Theorem 3 allows us to recover the following well-known formula.

Corollary 4 *For any $\mu > 0$,* $\dfrac{1}{2A_\infty^{(-\mu)}} \sim \mathrm{Gamma}(\mu, 1)$.

Theorem 5 *Let $\{U_k;\ k \geq 1\}$ be independent variables with the same distribution as*

$$U \stackrel{\mathcal{L}}{=} \frac{B_1}{B_2}, \qquad B_1 \sim \mathrm{Beta}(\beta, \mu), \qquad B_2 \sim \mathrm{Beta}(1+\beta, \mu)$$

where B_1 and B_2 are independent, $\mu > 0$ and β is the same as in Theorem 1. Then

$$(a) \qquad \frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} U\left(\frac{1}{A_{T_\lambda}^{(\mu)}} + G_\mu\right)$$

$$(b) \qquad \frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\infty} U_1 \cdots U_k G_\mu^{(k)}$$

$$(c) \quad \mathsf{E}\, e^{-s/2A_{T_\lambda}^{(\mu)}} = \mathsf{E}\left(\prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k}\right)^{\mu}$$

$$(d) \ \mathsf{E}\, e^{-s/2A_{T_\lambda}^{(-\mu)}} = \mathsf{E}\left(\frac{1}{1+s}\prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k}\right)^{\mu} = \left(\frac{1}{1+s}\right)^{\mu} \mathsf{E}\, e^{-s/2A_{T_\lambda}^{(\mu)}}.$$

In (a), $A_{T_\lambda}^{(\mu)}$ and G_μ are independent; moreover, given G_μ and U with the given distributions, the solution $1/2A_{T_\lambda}^{(\mu)}$ is unique (in distribution). In (b), $\{G_\mu^{(k)};\ k \geq 1\}$ is a sequence of independent variables with a common Gamma$(\mu, 1)$ distribution, indepedent of $\{U_k;\ k \geq 1\}$.

Part (a) follows from computing the Mellin transform of each side, which from Theorem 1 is

$$\frac{\Gamma(1-s)\Gamma(\beta+s)\Gamma(1+\beta+\mu)}{\Gamma(1+\beta+\mu-s)\Gamma(\beta)} = \frac{\Gamma(1-s)\Gamma(\beta+s)\Gamma(1+\alpha)}{\Gamma(1+\alpha-s)\Gamma(\beta)}.$$

Part (b) results from iterating (a) (see also Theorems 3 and 4 of Dufresne (1998)). Parts (c) and (d) result from conditioning on $\{U_k;\ k \geq 1\}$ in (b).

Theorem 5 (b) is another instance of the relationship between perpetuities and the Asian option process, observed in Dufresne (1990).

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