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Further Observations on Chain Ladder Bias

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FURTHER OBSERVATIONS ON CHAIN LADDER BIAS

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Summary. The chain ladder forecast (CLF) has previously been shown to be biased upward. The present paper calculates the second order approximation to the magnitude of the bias, ie the Taylor series for the bias truncated at terms involving second order moments of observations. Some order relations between data triangles are obtained with respect to this second order bias. While the prediction error of the CLF, as a predictor of loss reserve, does not have zero mean, it does have zero median under certain circumstances. Some numerical consequences are explored.

Keywords. Chain ladder, loss reserve.

1. Introduction

Taylor (2001) showed that, subject to a couple of technical conditions, the chain ladder estimate of a loss reserve is biased upward. This finding applies separately to each accident year.

The proof involved the Taylor series expansion of bias and the observation that its leading term, that involving second order moments of observations, is positive. The isolation of this term, referred to here as the **second order bias**, provides an approximation to the total bias.

The present paper is concerned with that approximation. Section 5 expresses it in a form that can be readily evaluated. Section 6 takes advantage of this form to obtain some orderings of data triangles with respect to bias on the basis of some simple parameters underlying those triangles.

Section 7 shows that, while the chain ladder provides a biased predictor of loss reserve, its prediction error has **zero median** under certain circumstances.

Section 8 provides a numerical example of the evaluation of second order bias, and Section 9 also examines numerical results.

2. Framework and notation

Consider a square array X of stochastic quantities $X(i,j) \geq 0$, $i = 0,1,\ldots,I$; $j = 0,1,\ldots,I$.

Denote row sums and column sums as follows:

$$R(i,j) = \sum_{h=0}^{j} X(i,h) \tag{2.1}$$

$$C(i,j) = \sum_{g=0}^{i} X(g,j). \tag{2.2}$$

In addition introduce the following notation for the total sum over a rectangular subset of X:

$$\begin{aligned} T(i,j) &= \sum_{g=0}^{i} \sum_{h=0}^{j} X(g,h) \\ &= \sum_{g=0}^{i} R(g,j) \\ &= \sum_{h=0}^{j} C(i,h). \end{aligned} \tag{2.3}$$

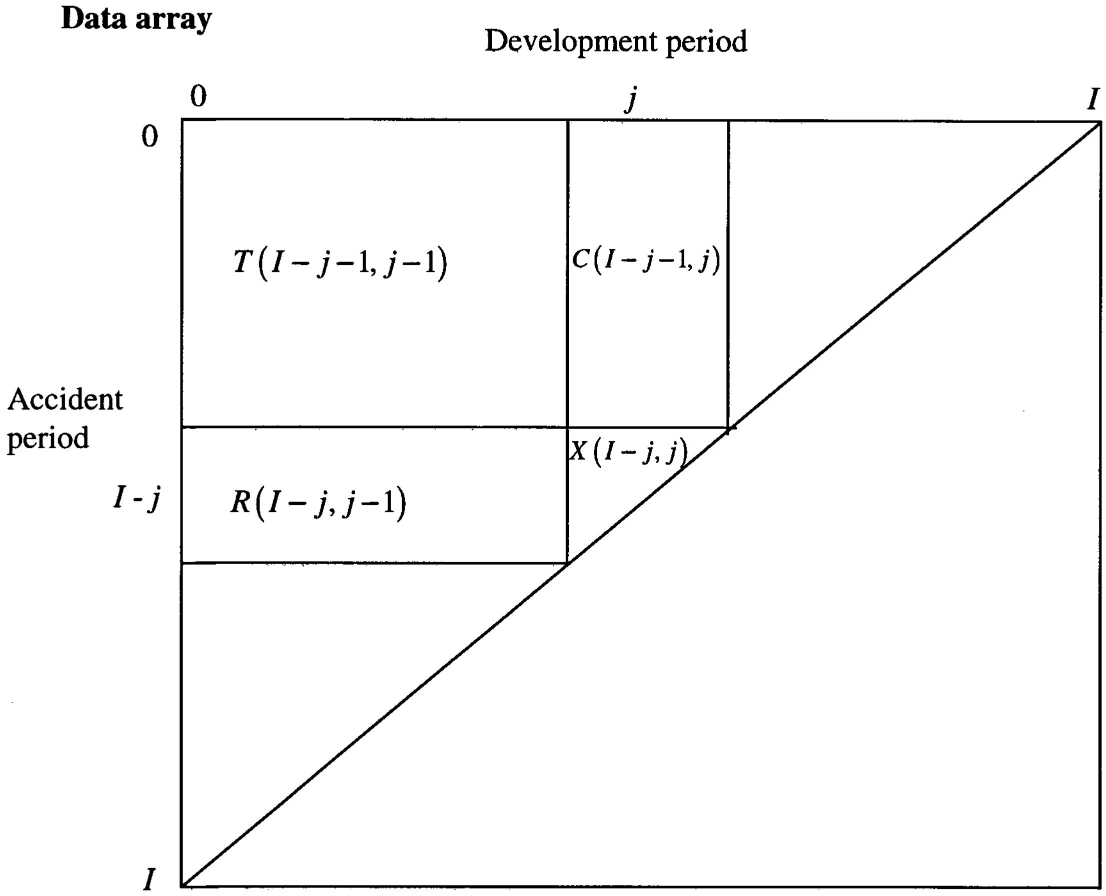
Generally, in the following any summation of the form $\sum_a^b$ with $b < a$ will be taken to be zero.

In a typical loss reserving framework, i denotes accident period, j development period, and available data will consists of observations on the triangular subset Δ of X:

$$\Delta = \{X(i,j), i = 0,1,...,I; j = 0,1,...,I-i\} \tag{2.4}$$

Figure 2.1 illustrates the situation.

Figure 2.1
Data array



Still in a loss reserving context, Δ would represent some form of claims experience, eg claim counts or claim amounts. The loss reserving problem consists of forecasting the lower triangle in Figure 2.1, conditional on Δ. There is particular interest in forecasting $R(i,I)\,|\,\Delta,\ i = 1,...,I$.

3. Chain ladder forecast

Define

$$\begin{aligned}\hat{v}(j) &= T(I-j-1, j+1)/T(I-j-1, j) \\ &= \sum_{g=0}^{I-j-1} R(g, j+1) / \sum_{g=0}^{I-j-1} R(g, j) \\ &= 1 + C(I-j-1, j+1)/T(I-j-1, j) \end{aligned} \quad (3.1)$$

and

$$\hat{R}(i,m) = R(i, I-i) \prod_{k=I-i}^{m-1} \hat{v}(k). \quad (3.2)$$

The value of $\hat{R}(i,m)$ calculated in this way will be referred to as the **chain ladder forecast** (CLF) of $R(i,m)$.

A special case of (3.2) is that for which $m = I$, where the CLF $\hat{R}(i,I)$ is the forecast of the **ultimate total** of row i.

4. Previous results

Taylor (2001) considered the CLF under the following assumptions.

Assumption 1. $E[R(i,j+1)]/E[R(i,j)] = \eta(j)$. (4.1)

Assumption 2. $X(i_1, j_1)$ and $X(i_2, j_2)$ are stochastically independent for $(i_1, j_1) \neq (i_2, j_2)$.

Define the set

$$D_i = \{(g,h) : g \leq I-k-1, h \leq k+1, \ k = I-i, \ldots, I-1\}. \quad (4.2)$$

Assumption 3. $T(g,h) > 0$ for $(g,h) \in D_i$.

Remark 1. It is implicit in Assumption 1 that $E[R(i,j)] \neq 0$. It then follows from the assumed non-negativity of the $X(i,j)$ that $E[R(i,j)] > 0$ for each i, j.

Remark 2. By Assumption 3, applied to (3.1), all $\hat{v}(k)$ appearing in (3.2) are defined and strictly positive.

Taylor obtained the following result.

Theorem 1. Define

$$Y=\prod_{k=I-i}^{I-1}\hat{v}(k). \tag{4.3}$$

Then

$$\frac{\partial^2 Y}{\partial X^2(g,h)}=0 \text{ for } (g,h)\notin D_i; \tag{4.4}$$

for $(g,h)\in D_i$ and $h\le I-i$,

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)}=2\sum_{k=I-i}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\text{x}$$
$$\left[\frac{1}{T(i-1,I-i)}+\sum_{l=I-i+1}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}\right] \tag{4.5}$$

for $(g,h)\in D_i$ and $h>I-i$,

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)}=2\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\sum_{l=h}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}. \tag{4.6}$$

These results do **not** depend on the Assumptions 1, 2 and 3. □

5. Magnitude of bias

Bias in the CLF can be estimated by the usual Taylor series method. Write Y as a function of the vector X whose components are the $X(g,h)\in\Delta$:

$$Y=f(X)=f(\mu)+\sum_{g,h}\left[X(g,h)-\mu(g,h)\right]\frac{\partial f(\mu)}{\partial X(g,h)}$$
$$+\tfrac{1}{2}\sum_{g,h}\sum_{k,l}\left[X(g,h)-\mu(g,h)\right]\left[X(k,l)-\mu(k,l)\right]\frac{\partial^2 f(\mu)}{\partial X(g,h)\partial X(k,l)}$$
$$+\ldots \tag{5.1}$$

where the summations run over Δ, $\mu(g,h)=E\left[X(g,h)\right]$, and μ is the vector with components $\mu(g,h)$.

By (5.1),

$$E[Y]=f(\mu)+\tfrac{1}{2}\sum_{g,h}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}V\left[X(g,h)\right]+\ldots \tag{5.2}$$

where Assumption 2 has been used to eliminate the quadratic cross terms.

By (3.1), (4.1) and (4.3),

$$f(\mu)=\prod_{k=I-i}^{I-1}\eta(k)=E\left[R(i,I)\right]/E\left[R(i,I-i)\right]. \tag{5.3}$$

By (3.2) and (4.3),

$$E\left[\hat{R}(i,I)\right]=E\left[R(i,I-i)Y\right]. \tag{5.4}$$

It may be checked from (3.1) and (5.3) that Y depends only on rows 0 to $i-1$ of Δ, and so, by Assumption 2, Y and $R(i,I-i)$ are stochastically independent. Hence

$$\begin{aligned}E\left[\hat{R}(i,I)\right]&=E\left[R(i,I-i)\right]E\left[Y\right]\\&=\left[f(\mu)+Q(i,I-i)+\ldots\right]E\left[R(i,I-i)\right]\end{aligned} \tag{5.5}$$

by (5.2) with

$$Q(i,I-i)=\tfrac{1}{2}\sum_{(g,h)\in D_i}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}\sigma^2(g,h) \tag{5.6}$$

with

$$\sigma^2(g,h)=V\left[X(g,h)\right]. \tag{5.7}$$

The range of summation over (g,h) has been reduced from Δ to D_i by virtue of (4.4).

Substitute (5.3) in (5.5):

$$E\left[\hat{R}(i,I)\right]=E\left[R(i,I)\right]+Q(i,I-i)E\left[R(i,I-i)\right] \tag{5.8}$$

omitting terms of third and higher order. Thus, the **second order bias** in $\hat{R}(i,I)$ as a predictor of $R(i,I)$ is $Q(i,I-i)E\left[R(i,I-i)\right]$.

The following results consider two triangles of the form (2.4). Denote them Δ_1 and Δ_2. All quantities associated with Δ_r will be subscripted by r, eg $X_r(g,h)$ denotes the observation $X(g,h)$ in Δ_r.

Theorem 2. Suppose that:

$$\mu_2(g,h)=\mu_1(g,h) \tag{5.9}$$

$$\sigma_2^2(g,h)=K\sigma_1^2(g,h) \tag{5.10}$$

for $(g,h)\in D_i$. Then

$$Q_2(i,I-i)=KQ_1(i,I-i). \tag{5.11}$$

That is, the second order bias in $\hat{R}_2(i,I)$ is K times that in $\hat{R}_1(i,I)$.

Proof. Follows directly from (5.6). □

In other words, if the variances of all observations in a triangle are changed by a common factor, then all second order biases of CLFs are changed by the same factor.
Recall Assumption 1, and note that an equivalent form to (4.1) is:

$$\mu(i,j)=E\left[X(i,j)\right]=a(i)b(j) \tag{5.12}$$

for some functions $a(.)b(.)$.

By (2.1) – (2.3),

$$E\left[R(i,j)\right]=a(i)B(j) \tag{5.13}$$

$$E\left[C(i,j)\right]=A(i)b(j) \tag{5.14}$$

$$E\left[T(i,j)\right]=A(i)B(j) \tag{5.15}$$

where

$$A(i)=\sum_{g=0}^{i}a(g) \tag{5.16}$$

$$B(j)=\sum_{h=0}^{j}b(j). \tag{5.17}$$

Remark 3. Note that Remark 1 and (5.13) imply that

$$a(i)>0 \text{ for each } i=0,1,\ldots,I \tag{5.18}$$

$B(j) > 0$ for each $j = 0,1,...,I$. (5.19)

Theorem 3. The quantities $\partial^2 f(\mu)/\partial X^2(g,h)$ appearing in the bias term (5.6) are evaluated as follows:

For $(g,h) \in D_i$ and $h \leq I-i$,

$$\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)} = 2\sum_{k=I-i}^{I-g-1}\frac{b(k+1)}{A(I-k-1)B(k)B(k+1)} \times \left[\frac{1}{A(i-1)B(I-i)} + \sum_{l=I-i+1}^{k}\frac{a(I-l)}{A(I-l-1)A(I-l)B(l)}\right] \tag{5.20}$$

For $(g,h) \in D_i$ and $h > I-i$,

$$\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)} = 2\sum_{k=h}^{I-g-1}\frac{b(k+1)}{A(I-k-1)B(k)B(k+1)} \times \sum_{l=h}^{k}\frac{a(I-l)}{A(I-l-1)A(I-l)B(l)}. \tag{5.21}$$

Proof. Note that $\partial^2 f(\mu)/\partial X^2(g,h)$ is just $\partial^2 Y/\partial X^2(g,h)$ evaluated at $X = \mu$ and the second derivative is given by (4.5) and (4.6). Substitute $X = \mu$ in those equations and then apply (5.13) – (5.15). □

6. Ordering of triangles with respect to bias

Write (5.8) in the alternative form:

$$E\left[\hat{R}(i,I)\right] = E\left[R(i,I)\right] + q(i,I-i)f(\mu)E\left[R(i,I-i)\right] \tag{6.1}$$

where

$$\begin{aligned} q(i,I-i) &= Q(i,I-i)/f(\mu) \\ &= \tfrac{1}{2}\sum_{(g,h)\in D_i}\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}\sigma^2(g,h) \end{aligned} \tag{6.2}$$

by (5.6).

Substitute (5.3) into (6.1):

$$E\left[\hat{R}(i,I)\right] = \left[1 + q(i,I-i)\right]E\left[R(i,I)\right]. \tag{6.3}$$

On the basis of (6.3), $q(i,I-i)$ may reasonably be referred to as the **relative second order bias** in $\hat{R}(i,I)$ as a predictor of $R(i,I)$. Note that this quantity, as given by (6.2), is well adapted to evaluation, since the quantities $[1/f(\mu)]\partial^2 f(\mu)/\partial X^2(g,h)$ are given by (5.20) and (5.21). These last relations may be used to obtain a partial ordering of the $q(i,I-i)$ for different triangles.

In the following, let subscripts have the same meaning as in Section 5.

Theorem 4. Relative second order bias $q(i,I-i)$ is scale independent. That is, if $X_2(i,j)=KX_1(i,j)$ for all *i,j* and for *K* const. > 0, then $q_2(i,I-i)=q_1(i,I-i)$.

Proof. Note that $\sigma_2^2(g,h)=K^2\sigma_1^2(g,h)$. Note also that the condition of the theorem can be achieved by setting $a_2(i)=Ka_1(i)$, $b_2(j)=b_1(j)$. Then a change in the $X(i,j)$ by a factor of *K* causes the quantities (5.20) and (5.21) to change by a factor of $1/K^2$. This offsets the effect of the change in the $\sigma^2(g,h)$ in (6.2).

□

Theorem 5. Suppose that the following three conditions hold for all $(g,h)\in D_i$:

$$a_2(g)/A_2(g)\geq a_1(g)/A_1(g) \tag{6.4}$$

$$b_2(h)/B_2(h)\geq b_1(h)/B_1(h) \tag{6.5}$$

$$\sigma_2^2(g,h)/E^2[T_2(I,I)]\geq\sigma_1^2(g,h)/E^2[T_1(I,I)] \tag{6.6}$$

Then

$$q_2(i,I-i)\geq q_1(i,I-i). \tag{6.7}$$

If strict inequality holds:

- in (6.4) for at least one $g, 1\leq g\leq i-1$;
- in (6.5) for at least one $h, I-i+1\leq h\leq I$; and
- $\sigma_1^2(g,h)>0$ for this choice of (g,h); (6.8)

then strict inequality holds in (6.7).

If strict inequality holds in (6.6) for at least one $(g,h) \in D_i$, then strict inequality holds in (6.7).

Proof. See Appendix A. □

7. Median estimation

7.1 Theoretical

Since Taylor showed that the CLF of $R(i,I)$ is not unbiased for $E\left[R(i,I)\right]$, it is interesting to consider other estimation properties of the CLF.

Consider the chain ladder ratio

$$Z(i,j) = R(i,j+1)/R(i,j)$$

and define

$$Y(i,j) = \log Z(i,j) = \log R(i,j+1) - \log R(i,j). \tag{7.1}$$

In Theorems 8 and 9, observations $Z(i,j)$ will be regarded as predictors of ratios $Z(i',j)$ (different i, same j), as yet unobserved. Similarly, $\hat{v}(j)$ will be regarded as a predictor of the $Z(i',j)$, $Y(i,j)$ a predictor of $Y(i',j)$ and $\hat{R}(i,m)$ as a predictor of $R(i,m)$.

Subsequently, $Z(i',j)$ will be referred to as a **predictand** of $Z(i,j)$ and $\hat{v}(j)$. Similarly, $Y(i',j)$ and $R(i,m)$ will be referred to as predictands of $Y(i,j)$ and $\hat{R}(i,m)$ respectively.

Generally, if a variable $\hat{U}$ predicts U, then $\hat{U} - U$ will be referred to as its **prediction error** (with respect to U).

Definition. A random n-tuple $(U_1,\ldots,U_n)$ will be said to be **symmetrically distributed** about $(\mu_1,\ldots,\mu_n)$ if

$$\text{Prob}\left[U_i \le \mu_i - w_i, i = 1,2,\ldots,n\right] = \text{Prob}\left[U_i \ge \mu_i + w_i, i = 1,2,\ldots,n\right] \textit{ for all } w_1,\ldots,w_n \ge 0. \tag{7.2}$$

Only the cases n = 1, 2 will be required below.

Remark 4. If U and V are both symmetrically distributed, then $U - V$ is symmetrically distributed. If $\hat{U}$ is an unbiased predictor of U, then $\hat{U} - U$ is symmetrically distributed about zero, and hence the prediction error of $\hat{U}$ has zero mean and median.

Remark 5. If the prediction bias of $\hat{U}$ with respect to U has zero median, then so does the prediction error of $f\left(\hat{U}\right)$ with respect to $f(U)$ for any one-one transformation f.

Definition. Random variables U and V with joint d.f. F will be said to be **exchangeable** if $F(u,v) = F(v,u)$.

Theorem 6. Let $R(i,j+1)$ and $k\,R(i,j)$ be exchangeable for some constant k. Then $Y(i,j)$ is symmetrically distributed about $\log k$.

Proof. See Appendix A. □

Remark 6. There is no assumption here about stochastic independence.

Theorem 7. Let the ordered pair $\left[\log R(i,j+1), \log R(i,j)\right]$ be symmetrically distributed. Then $Y(i,j)$ is symmetrically distributed.

Proof. See Appendix A. □

Theorem 8. For given j, let each pair $R(i,j+1)$ and $k_j R(i,j)$, $i = 0,1,\ldots,I-j-1$, be exchangeable for some constant k_j. Then each $Y(i,j)$ is symmetrically distributed. Moreover, the prediction error of each $Y(i,j)$, $Z(i,j)$ and $\hat{v}(j)$ with respect to its predictand has zero median. If the hypotheses hold for each $i = 0, 1, \ldots, I$, $j = 0, 1, \ldots, I-1$, then the prediction error of each $\hat{R}(i,m)$, $i = 1,2,\ldots,I$; $m = I-i+1,\ldots,I$ with respect to $R(i,m)$ has zero median.

Proof. See Appendix A. □

Remark 7. The conditions of the proposition require only pairwise exchangeability of the $R(i,j+1)$ and $k_j R(i,j)$. For different i, the distributions involved may be quite different.

Theorem 9. Suppose that each ordered pair $\left[\log R(i,j+1), \log R(i,j)\right]$, $i = 0,1,\ldots,I$; $j = 0,1,\ldots,I-1$ (that is both past and future) is symmetrically distributed. Then the prediction error of each $Y(i,j)$, $Z(i,j)$ and $\hat{v}(j)$ and $\hat{R}(i,m)$ with respect to its predictand has zero median.

Proof. See Appendix A. □

Remark 8. Again the distributions of the $R(i,j)$ may differ for different i, j.

It follows from the symmetry of the distribution of $Y(i,j)$ in Theorems 8 and 9 that log $\hat{R}(i,m)$ is an unbiased estimator of log $R(i,m)$. That is, the prediction error of log $\hat{R}(i,m)$ has zero mean. It also has zero median. However, the prediction error of $\hat{R}(i,m)$ does not necessarily have zero mean, but it does have zero median.

Examples

Example 1. Consider the case in which

$$R \sim N(\mu_R, \sigma_R^2), \qquad X \sim N(\mu_X, \sigma_X^2)$$

and R, X are stochastically independent. Here, for brevity R and X denote $R(i,j)$ and $X(i,j+1)$. Define

$$k = (\mu_R + \mu_X)/\mu_R.$$

Then

$$kR \sim N(\mu_R + \mu_X, k^2\sigma_R^2)$$
$$R + X \sim N(\mu_R + \mu_X, \sigma_R^2 + \sigma_X^2).$$

If $k = \left[(\sigma_R^2 + \sigma_X^2)/\sigma_R^2\right]^{\frac{1}{2}}$, then

$$kR \sim N(\mu_R + \mu_X, \sigma_R^2 + \sigma_X^2)$$
$$R + X \sim N(\mu_R + \mu_X, \sigma_R^2 + \sigma_X^2)$$

and $kR, R+X$ are exchangeable, and Theorem 8 applies.

The assumption of stochastic independence made here is compatible with Assumption 2, but not with the assumptions under which the CLF is unbiased (see Taylor, 2001). In this latter case, $X = VR$ with V, R stochastically independent. Then

$$\log(R+X) = \log R + \log(1+V)$$
$$\log kR = \log R + \log k.$$

These two variables will have different variances, and therefore cannot be exchangeable, unless V has a point distribution. Thus, except for this degenerate case, $R + X$ and kR cannot be exchangeable, and Theorem 8 does **not** apply.

Example 2. Consider the case in which R and $R + X$ are jointly log normally distributed, but with R and X stochastically independent, as in Assumption 2. Then Theorem 9 applies.

7.2 Practical

It is possible to produce examples that satisfy the conditions of Theorem 8. For example,

$$X(i,j) \sim N\left(\mu_j, \sigma^2{}_j\right) \tag{7.3}$$

with

$$\mu_{j+1} = \left(k_j - 1\right)\sum_{m=0}^{j} \mu_m \tag{7.4}$$

$$\sigma^2_{j+1} = \left(k_j^2 - 1\right)\sum_{m=0}^{j} \sigma^2_m \tag{7.5}$$

and subject to Assumption 2.

This example (which is in fact a re-statement of Example 1 above) is rather contrived, however. Equations (7.4) and (7.5) are restrictive in the relations they allow between the μ_j and σ^2_j.

In practice, the conditions of Theorem 8 are unlikely to hold precisely. It is likely, however, that they will hold approximately, in which case its conclusion will hold approximately.

Similar remarks apply to Theorem 9. In this case it is difficult even to produce theoretical examples. This has to do with the fact that the quantities involved in the proposition are logged sums of random variables. Families of variables that are closed under addition typically do not yield tractable log forms.

Once again, however, the proposition may apply approximately. Typically, the $R(i,j)$ tend to be right skewed. The log transformation is right tail reducing, and so log $R(i,j)$ will be less skewed to the right, possibly approximately symmetrical.

8. Numerical example

Appendix B.1 reproduces incremental loss payment data from Table 3.4 of Taylor (2000). The data relate to a Motor Bodily Injury portfolio. They have been adjusted for inflation, ie brought to constant dollar values. The portfolio can be seen to be moderately long tailed.

The standard chain ladder has been applied to these data, producing the parameter estimates set out in Appendix B.2. The $b(j)$ have been estimated directly from the model, as described in Appendix B.2.

The $a(i)$ are estimated from (5.13) as the quantity $R(i,I-i)/B(I-i)$.

It has been assumed that the coefficient of variation of $X(g,h)$ depends on just h, ie

$$V\left[X(g,h)\right]/E^2\left[X(g,h)\right]=\tau^2(h). \qquad (8.1)$$

Substitution of (5.7) and (5.12) into (8.1) shows that an equivalent form of this assumption is:

$$\sigma^2(g,h)=\left[a(g)b(h)\right]^2\tau^2(h) \qquad (8.2)$$

An initial estimate of the quantity $\tau(h)$ has been taken as the sample standard deviation of the set $\{X(g,h)/a(g)b(h),\ g=0,1,...,I-h\}$.

These estimates are then smoothed to give

$$\tau(h)=0.24\,\mathrm{x}\,(1.1)^h. \qquad (8.3)$$

The bias term (6.2) may now be calculated by substitution of (5.20), (5.21), (8.2) and (8.3), with the results set out in Table 8.1.

Table 8.1
Second order bias in forecasts

Accident year i	Paid losses to date R(i,I-i)	Age to ultimate factor (4.3)	Ultimate losses Estimate $\hat{R}(i,I)$	Ultimate losses Relative bias	Loss reserve Estimate	Loss reserve Relative bias
	$000		$000	%	$000	%
1978	55,081	1	55,081	0	0	0
1979	42,039	1.0003	42,050	0.001	11	2.9
1980	58,610	1.0010	58,671	0.002	60	1.8
1981	65,791	1.0018	65,911	0.003	121	1.4
1982	61,872	1.0068	62,294	0.006	422	0.9
1983	53,683	1.0115	54,299	0.009	616	0.8
1984	66,093	1.0170	67,220	0.012	1,126	0.7
1985	51,674	1.0266	53,049	0.016	1,375	0.6
1986	51,277	1.0465	53,659	0.023	2,382	0.5
1987	47,416	1.0739	50,921	0.033	3,505	0.5
1988	38,677	1.1245	43,491	0.048	4,814	0.4
1989	41,900	1.2048	50,479	0.070	8,580	0.4
1990	39,133	1.3582	53,150	0.105	14,016	0.4
1991	31,999	1.6062	51,396	0.152	19,397	0.4
1992	30,123	2.0777	62,587	0.220	32,464	0.4
1993	16,090	3.1940	51,393	0.308	35,302	0.4
1994	8,330	5.7516	47,909	0.449	39,579	0.5
1995	2,827	18.1679	51,369	0.706	48,542	0.7
Total					212,313	0.53

The chain ladder loss reserve is seen to be biased upward by about ½%.

9. Other numerical results

Stanard (1985) simulated chain ladder bias. His model was one in which, for a particular accident year:

- The ultimate number of claims was simulated as a Poisson variate; and then
- Its distribution over reporting years was simulated according to a multinomial distribution.

Distinct accident years were stochastically independent. It can be shown that the resulting array of claim counts then satisfies Assumption 2. The CLF applied to the triangle of claim counts is upward biased, as predicted by Taylor (2001).

Now associate a claim size with each claim, with all claim sizes stochastically independent.

In this framework, let

$\lambda =$ expected claim count for the accident year
$p_j =$ expected proportion of claims reported in development year j (=0,1,...,I)
$\alpha_n =$ n-th uncentralised moment of individual claim size
$X(i,j) =$ total cost of claims reported in cell (i,j).

It may be shown that $X(i,j)$ is compound Poisson distributed with Poisson parameter λp_j. Hence

$$\begin{aligned}
\mu(i,j) &= \lambda p_j \text{ for claim counts} \\
&= \lambda p_j \alpha_1 \text{ for claim amounts} \\
\sigma^2(i,j) &= \lambda p_j \text{ for claim counts} \\
&= \lambda p_j \alpha_2 \text{ for claim amounts.}
\end{aligned}$$

Then Theorem 2 shows that the relative bias in the CLF forecast of loss reserve (claim amounts) exceeds that in the forecast of IBNR count by a factor of α_2 / α_1 (to second order at least).

In Stanard's example,

$$\begin{aligned}
\alpha_2 / \alpha_1 &= \left(34{,}800^2 + 10{,}400^2\right) / 10{,}400 \\
&= 126{,}846.
\end{aligned}$$

This is in fact close to the factor of increase found empirically by Stanard (compare his Exhibits I and II).

Appendix A
Proofs of theorems

Proof of Theorem 5. Write (6.2) in the alternative form:

$$q(i,I-i)=\tfrac{1}{2}\sum_{(g,h)\in D_i} E^2\left[T(I,I)\right]\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}\frac{\sigma^2(g,h)}{E^2\left[T(I,I)\right]} \tag{A.1}$$

where, by (5.20), (5.21) and (5.15),

$$E^2\left[T(I,I)\right]\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}=2\sum_{k=I-i}^{I-g-1}\frac{b(k+1)}{B(k+1)}\frac{A(I)}{A(I-k-1)}\frac{B(I)}{B(k)}$$
$$\left[\frac{A(I)}{A(i-1)}\frac{B(I)}{B(I-i)}+\sum_{l=I-i+1}^{k}\frac{a(I-l)}{A(I-l)}\frac{A(I)}{A(I-l-1)}\frac{B(I)}{B(l)}\right] \tag{A.2}$$

for $(g,h)\in D_i$ and $h\le I-i$, and

$$E^2\left[T(I,I)\right]\frac{1}{f(\mu)}\frac{\partial^2 f(\mu)}{\partial X^2(g,h)}=2\sum_{k=h}^{I-g-1}\frac{b(k+1)}{B(k+1)}\frac{A(I)}{A(I-k-1)}\frac{B(I)}{B(k)}$$
$$\sum_{l=h}^{k}\frac{a(I-l)}{A(I-l)}\frac{A(I)}{A(I-l-1)}\frac{B(I)}{B(l)} \tag{A.3}$$

for $(g,h)\in D_i$ and $h>I-i$.

Note that

$$\frac{A(I)}{A(i)}=\prod_{g=i}^{I-1}\frac{A(g+1)}{A(g)} \tag{A.4}$$

and

$$\frac{A(g+1)}{A(g)}=\left[1-\frac{a(g+1)}{A(g+1)}\right]^{-1}. \tag{A.5}$$

It follows that, if (6.4) holds, then

$$\frac{A_2(g+1)}{A_2(g)}\ge\frac{A_1(g+1)}{A_1(g)} \tag{A.6}$$

and

$$\frac{A_2(I)}{A_2(i)}\ge\frac{A_1(I)}{A_1(i)}. \tag{A.7}$$

Similarly (6.5) implies

$$\frac{B_2(I)}{B_2(j)} \geq \frac{B_1(I)}{B_1(j)}. \tag{A.8}$$

Substitution of (6.4), (6.5), (A.7) and (A.8) in (A.2) and (A.3) yields

$$E^2\left[T_2(I,I)\right]\frac{1}{f_2(\mu_2)}\frac{\partial^2 f_2(\mu_2)}{\partial X_2^2(g,h)} \geq E^2\left[T_1(I,I)\right]\frac{1}{f_1(\mu_1)}\frac{\partial^2 f_1(\mu_1)}{\partial X_1^2(g,h)} \tag{A.9}$$

for all $g,h \in D_i$.

Substitution of (A.9) and (6.6) in (A.1) yields (6.7).

It is evident from (5.20) that $\partial^2 f(\mu)/\partial X^2(g,h)$ for $h \leq I-i$ involves the quantity $b(k+1)/B(k+1)$ for $I-i+1 \leq k+1 \leq I-g$. Similarly, in (5.20) $\partial^2 f(\mu)/\partial X^2(g,h)$ involves the quantity $a(I-l)/A(I-l)$ for $g+1 \leq I-l \leq i-1$.

Hence, if strict inequality holds in (6.4) for $1 \leq g \leq i-1$, and in (6.5) for $I-i+1 \leq h \leq I$, then strict inequality holds (A.9) for that (g,h). Take this and (6.8) into account in the reasoning that led from (A.9) to (6.7) to see that strict inequality holds in (6.7).

To prove the final statement of the theorem, note that (5.20) and (5.21) imply that (A.2) and (A.3) are strictly positive for each (g,h). If strict inequality holds in (6.6) for this choice of (g,h), then (A.9) yields

$$E^2\left[T_1(I,I)\right]\frac{1}{f_1(\mu_1)}\frac{\partial^2 f_1(\mu_1)}{\partial X_1^2(g,h)}\sigma_1^2(g,h) >$$
$$E^2\left[T_2(I,I)\right]\frac{1}{f_2(\mu_2)}\frac{\partial^2 f_2(\mu_2)}{\partial X_2^2(g,h)}\sigma_2^2(g,h). \tag{A.10}$$

Substitute this result in (A.1) to obtain strict inequality in (6.7).

Proof of Theorem 6. It follows from the definition of exchangeability that, for exchangeable U and V, U-V is symmetrically distributed about zero. Apply this result to the case $U = \log R(i,j+1)$, $V = \log\left[kR(i,j)\right]$. This gives $Y(i,j) - \log k$ as symmetrically distributed about zero. The result follows.

□

Proof of Theorem 7. Suppose initially that (U_1, U_2) is symmetrically distributed about (0,0). Let $V = U_1 - U_2$. Then

$$\text{Prob}\left[V \leq -v\right] = \int_S d\ \text{Prob}\left[U_1 \leq u_1, U_2 \leq u_2\right] \tag{A.11}$$

where

$$S=\{(u_1,u_2):-u_1-(-u_2)\le v\}=\{(u_1,u_2):u_1-u_2\ge v\}. \qquad \text{(A.12)}$$

By (A.11) and the symmetry of (U_1,U_2),

$$\text{Prob}[V\le -v]=\int_S d\ \text{Prob}[U_1\ge u_1,U_2\ge u_2]=\text{Prob}[V\ge v]$$

by (A.12).

Thus, V is symmetrically distributed about zero. A simple modification adapts this proof to the case where (U_1,U_2) are symmetrically distributed about a point other than (0,0). □

Proof of Theorem 8. By the exchangeability hypothesis and Theorem 6, $Y(i,j)$ is symmetrically distributed about k_j. As this latter quantity is independent of i, $Y(i,j)$ is an unbiased predictor of $Y(i',j)$.

The result of the theorem for $Y(i,j)$ follows immediately from the symmetry of its distribution and that of its predictand, together with Remark 4. By (7.1), $Z(i,j)$ is related one-one to $Y(i,j)$, and so the result follows from Remark 5 for $Z(i,j)$.

By (3.1),

$$\log\hat{v}(j)-\log k_j=\log\sum_{i=0}^{I-j-1}R(i,j+1)-\log\sum_{i=0}^{I-j-1}k_jR(i,j). \qquad \text{(A.13)}$$

By hypothesis, and by stochastic independence with respect to i, the two members on the right side of (A.13) are exchangeable. Hence $\log\hat{v}(j)$ is symmetrically distributed about $\log k_j$, the proof of this parallel to that of Theorem 6. Now log $\hat{v}(j)$ is a predictor of $Y(i',j)$, just as was $Y(i,j)$. Therefore, the required result for $\hat{v}(j)$ may be established by the same argument as for $Z(i,j)$, $Y(i,j)$ replaced by $\hat{v}(j)$, log $\hat{v}(j)$.

By (3.2),

$$\log\hat{R}(i,m)=\log R(i,I-i)+\log\hat{v}(I-i)+...+\log\hat{v}(m-1). \qquad \text{(A.14)}$$

Since the $\log \hat{v}(j)$ have just been shown symmetrically distributed, so must be log $\hat{R}(i,m)$, and so the prediction error of $\log \hat{R}(i,m)$ with respect to log *R(i,m)* has zero median. By Remark 5, the prediction error of $\hat{R}(i,m)$ with respect to *R(i,m)* also has zero median. □

Proof of Theorem 9. The result for $Y(i,j)$ follows immediately from the symmetry of its distribution, established by Theorem 7. The result for $Z(i,j)$ then follows just as in Theorem 8.

By (3.1),

$$\log \hat{v}(j) = \log \sum_{i=0}^{I-j-1} R(i,j+1) - \log \sum_{i=0}^{I-j-1} R(i,j). \tag{A.15}$$

By the symmetry hypothesis, and stochastic independence with respect to *i*, the two members of the right side of (A.15) form a symmetrically distributed ordered pair, and so $\log \hat{v}(j)$ is symmetrically distributed. The proof of this is parallel to that of Theorem 7.

The remainder of the proof follows that of Theorem 8.

Appendix B
Numerical example

B.1 Data

Accident year	Incremental paid losses in development year 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	3,323	8,532	9,372	10,172	7,630	3,856	3,252	4,434	2,188	332	200	692	311	0	604	94	76	14
1979	3,785	10,341	8,331	7,849	2,838	3,577	1,405	1,721	1,065	155	36	259	250	419	8	0	0	
1980	4,677	9,989	8,746	10,228	8,572	5,786	3,855	1,445	1,612	626	1,172	589	438	473	370	31		
1981	5,288	8,089	12,839	11,830	7,760	6,182	4,118	3,016	1,775	1,785	2,645	266	38	45	114			
1982	2,294	9,869	10,242	13,808	8,785	5,409	2,425	1,597	2,149	3,296	917	295	428	359				
1983	3,600	7,514	8,247	9,327	8,584	4,245	4,096	3,216	2,014	592	1,188	691	367					
1984	3,642	7,394	9,838	9,734	6,377	4,884	11,920	4,189	4,492	1,760	944	922						
1985	2,463	5,033	6,980	7,722	6,702	7,834	5,579	3,622	1,300	3,069	1,370							
1986	2,267	5,959	6,175	7,051	8,102	6,339	6,978	4,396	3,107	903								
1987	2,009	3,701	5,297	6,886	6,496	7,550	5,855	5,751	3,871									
1988	1,860	5,282	3,650	7,528	5,156	5,766	6,862	2,573										
1989	2,331	3,517	5,310	6,066	10,149	9,265	5,262											
1990	2,314	4,486	4,113	6,999	11,163	10,058												
1991	2,607	3,952	8,228	7,905	9,307													
1992	2,595	5,404	6,578	15,546														
1993	3,155	4,975	7,961															
1994	2,626	5,703																
1995	2,827																	

B.2 Parameter estimates

Development year j	Age to age factor $\hat{v}(j)$	b(j)
0	3.1588	0.0550
1	1.8007	0.1188
2	1.5373	0.1392
3	1.2936	0.1682
4	1.1826	0.1413
5	1.1273	0.1137
6	1.0714	0.0938
7	1.0471	0.0593
8	1.0262	0.0419
9	1.0193	0.0244
10	1.0094	0.0185
11	1.0055	0.0092
12	1.0046	0.0054
13	1.0050	0.0046
14	1.0008	0.0049
15	1.0008	0.0008
16	1.0003	0.0008
17		0.0003

Accident year i	a(i)
1978	55,081
1979	42,050
1980	58,671
1981	65,911
1982	62,294
1983	54,299
1984	67,220
1985	53,049
1986	53,659
1987	50,921
1988	43,491
1989	50,479
1990	53,150
1991	51,396
1992	62,587
1993	51,393
1994	47,909
1995	51,369

In this table, $\hat{v}(j)$ has been calculated according to (3.1). Then $b(j)$ has been calculated as

$$b(j)=B(j)-B(j-1) \qquad \text{(B.1)}$$

with

$$B(j)=1/\prod_{k=j}^{I-1}\hat{v}(k). \qquad \text{(B.2)}$$

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