On the ruin time distribution for a Sparre Andersen process with exponential claim sizes

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Abstract

We derive a closed-form (infinite series) representation for the distribution of the ruin time for the Sparre Andersen model with exponentially distributed claims. This extends a recent result of Dickson et al. [7] for such processes with Erlang inter-claim times. We illustrate our result in the cases of gamma and mixed exponential inter-claim time distributions.

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1 Introduction

In the Sparre Anderson model, the (continuous-time) surplus process $\{U(t)\}_{t\geq 0}$ has the form

$$U(t) = u + ct - \sum_{j\leq N(t)} X_j,$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the premium rate, and $\{N(t)\}_{t\geq 0}$ is a delayed renewal process generated by a sequence of inter-claim times $\{T_j\}_{j\geq 0}$:

$$N(t) = \inf\{j \geq 0 : T_0 + \cdots + T_j \geq t\},$$

and $\{X_j\}_{j\geq 1}$ is the sequence of claim sizes (so that a claim of size X_1 is made at time T_0, etc). We assume that the random variables from the above

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sequences are jointly independent, with $\{T_j\}_{j\geq 1}$ and $\{X_j\}_{j\geq 1}$ being i.i.d. sequences. The goal of the present note is to derive an explicit formula for the distribution of the ruin time

$$\tau = \inf\{t > 0 : U(t) < 0\}$$

in the special case when the X_j's follow the exponential distribution.

When claims occur according to a Poisson process and the claim size distribution is exponential, a solution for the distribution of the ruin time τ has been known for many years. See, for example, [2], [8] and [9] for different solutions to this problem. In the recent paper [7], the authors used analytical techniques to obtain an explicit formula for the density of τ in the case when the T_j's have an Erlang distribution. In the present note, we present an alternative probabilistic method, which enables one to derive such an explicit formula in the more general case when the T_j's follow an arbitrary distribution.

2 The main result

We assume that the claim sizes X_j follow the exponential distribution with parameter $\lambda > 0$:

$$\mathbb{P}(X_j > x) = e^{-\lambda x}, \qquad x \geq 0, \tag{1}$$

while the positive random variables T_0 and $T_1 (\stackrel{d}{=} T_j,\ j > 1)$ have densities $f_0(t)$ and $f(t)$, respectively. By $g * h$ we denote the convolution of the functions g, h defined on $(0, \infty)$:

$$(g * h)(t) = \int_0^t g(t - v)h(v)dv,$$

and by $g^{*n} = g^{*(n-1)} * g$, $n \geq 2$, the n-fold convolution of g with itself.

Theorem 1 *Under the above assumptions, the ruin time τ has a (defective) density $p_\tau(t)$ given by*

$$p_\tau(t) = e^{-\lambda(u+ct)}\Bigg\{ f_0(t) + \sum_{n=1}^{\infty} \frac{\lambda^n (u+ct)^{n-1}}{n!} \big[u(f^{*n} * f_0)(t) + c(f^{*n} * f_1)(t)\big]\Bigg\}, \tag{2}$$

where $f_1(t) = tf_0(t)$.

Proof. The idea of the proof is similar to the one used in [4]: first we will translate our problem into the problem of the crossing of a linear boundary by the pure jump process $U^0(t) = U(t) - ct$ and then swap the roles of the time and space coordinates. Then we notice that the generalised inverse of the function $U^0(t)$ is nothing else but the trajectory of a compound Poisson process. Eventually, the original problem proves to be equivalent to finding the distribution of the hitting time of a level by a skip-free Lévy process, of which the solution is well-known and is given by Kendall's identity (see e.g. § 12, Theorem 1 in [3], or [5]).

(i) We will assume in parts (i)-(ii) of the proof that $T_0 \equiv v = \text{const}$ (which is equivalent to conditioning on T_0, but is more convenient from a notation viewpoint).

As we have just said, it is easily seen that, for the pure jump process

$$U^0(t) = U(t) - ct \equiv u - \sum_{j \le N(t)} X_j,$$

one has

$$\tau = \inf\{t > 0 : U^0(t) - (-ct) < 0\}.$$

Next we 'translate' the origin to the point (v, u) and swap the roles of coordinates by introducing the new 'time' $s = u - x$ and 'space' $y = t - v$ (where t and x respectively represent the original time and space). In the new system of coordinates, the trajectory of our process $\{U^0(t)\}$ is again a pure step function, which starts at zero at 'time' $s = 0$ and has jumps of sizes $T_1, T_2, T_3, \ldots,$ at 'times' $X_1, X_1 + X_2, X_1 + X_2 + X_3, \ldots$. Due to our assumption (1), this will be a trajectory of the compound Poisson process

$$Z^0(s) = \sum_{k \le M(s)} T_k,$$

where

$$M(s) = \inf\{k \ge 1 : X_1 + \cdots + X_k > s\} - 1$$

is a Poisson process with rate λ. The distribution of the r.v. $Z^0(s)$ with $s > 0$ has an atom $e^{-\lambda s}$ at zero and a density on $(0, \infty)$ given by

$$p_{Z^0(s)}(y) = e^{-\lambda s} \sum_{n=1}^{\infty} \frac{(\lambda s)^n}{n!} f^{*n}(y), \qquad y > 0. \tag{3}$$

To a crossing of the (lower) linear boundary $x = -ct$ by the process $\{U^0(t)\}$ at time τ (this necessarily is a jump epoch) there corresponds a (continuous) crossing of the (again lower linear) boundary

$$y = s/c - (v + u/c), \qquad s > 0,$$

by the process $\{Z^0(s)\}$ at 'time' $\sigma = u + c\tau$, so that

$$\tau = (\sigma - u)/c. \tag{4}$$

Finally, we notice that σ is the crossing time of the (lower) level $-(v+u/c)$ by the process $Z(s) = Z^0(s) - s/c$, which is clearly a skip-free in the negative direction Lévy process.

Figure 1 illustrates the translation of the original problem. The original surplus process starts at level $u = 1$, and ruin occurs at the fifth claim. Rotating the figure anti-clockwise through 90 degrees we see the corresponding path of the pure jump process $\{Z^0(s)\}$.

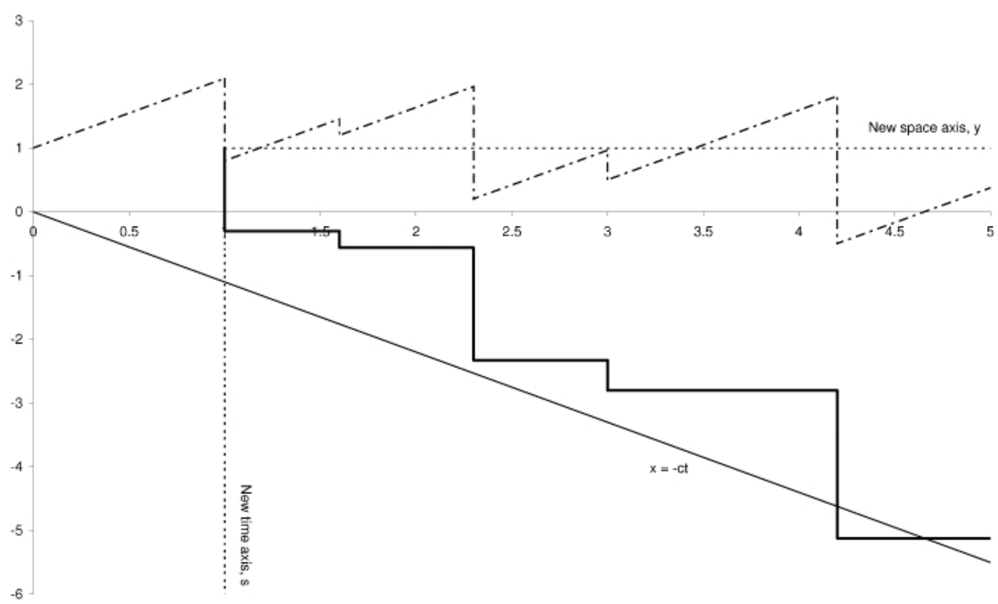

Figure 1: Original and translated processes

(ii) Therefore, provided that $Z(s)$ has a density $p_{Z(s)}(y)$ at the point $y = -(v + u/c)$, the crossing 'time' σ also has a density $p_\sigma(s)$ at the point s, which is given by Kendall's identity (see e.g. § 12, Theorem 1 in [3], or [5]):

$$p_\sigma(s) = \frac{v + u/c}{s} p_{Z(s)}(-(v + u/c)).$$

This together with (3) implies that, for $s > u + cv$, the r.v. σ has the density

$$p_\sigma(s) = \frac{v + u/c}{s}\, e^{-\lambda s} \sum_{n=1}^{\infty} \frac{(\lambda s)^n}{n!} f^{*n}\big((s-u)/c - v\big).$$

Therefore, it follows now from (4) that, given $T_0 = v$, for $t > v$ the stopping time τ has a conditional density given by

$$p_\tau(t|v) = cp_\sigma(u + ct) = \frac{u + cv}{u + ct}\, e^{-\lambda(u+ct)} \sum_{n=1}^{\infty} \frac{(\lambda(u + ct))^n}{n!} f^{*n}(t - v). \qquad (5)$$

(iii) To obtain the density of τ in the general case, we observe that $\tau \geq T_0$ always and so, using (1),

$$\begin{aligned} \mathbb{P}(\tau \leq t) &= \mathbb{P}(T_0 = \tau \leq t) + \mathbb{P}(T_0 < \tau \leq t) \\ &= \int_0^t \mathbb{P}(u + cv - X_1 < 0) f_0(v)dv + \int_0^t \mathbb{P}(v < \tau \leq t|\, T_0 = v) f_0(v)dv \\ &= \int_0^t e^{-\lambda(u+cv)} f_0(v)dv + \int_0^t \left[\int_v^t p_\tau(r|v)dr\right] f_0(v)dv. \end{aligned}$$

Differentiating both sides and substituting the representation for $p_\tau(t|v)$ from (5) yields the density of τ:

$$\begin{aligned} p_\tau(t) &= e^{-\lambda(u+ct)} f_0(t) + \int_0^t p_\tau(t|v) f_0(v)dv \\ &= e^{-\lambda(u+ct)}\left[f_0(t) + \frac{1}{u+ct} \sum_{n=1}^{\infty} \frac{(\lambda(u+ct))^n}{n!} \int_0^t (u + cv) f^{*n}(t - v) f_0(v)dv\right] \end{aligned}$$

(the change of the order of integration/summation is justified as the integrand is a non-negative function). As the last expression is equivalent to the RHS of (2), the theorem is proved. □

3 Examples

3.1 Gamma inter-claim times

Let us first consider the situation where claims occur according to an ordinary renewal process, so that each T_j, $j = 0, 1, 2, ...$ has density function

$$f(t) = f_0(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{\Gamma(n)},$$

where $n > 0$ and $\beta > 0$. It is well known that

$$f^{*(m+1)}(t) = \frac{\beta^{n(m+1)} t^{n(m+1)-1} e^{-\beta t}}{\Gamma(n(m+1))}$$

and it is straightforward to show that

$$f^{*m} * f_1(t) = \frac{n}{\beta} \frac{\beta^{n(m+1)+1} t^{n(m+1)} e^{-\beta t}}{\Gamma(n(m+1)+1)}.$$

Then formula (2) gives

$$\begin{aligned} p_\tau(t) &= (\beta t)^{n-1} \frac{u\beta e^{-\lambda(u+ct)-\beta t}}{u+ct} \sum_{m=0}^{\infty} \frac{\lambda^m (u+ct)^m}{m!} \frac{(\beta t)^{nm}}{\Gamma(n(m+1))} \\ &\quad + (\beta t)^n \frac{cne^{-\lambda(u+ct)-\beta t}}{u+ct} \sum_{m=0}^{\infty} \frac{\lambda^m (u+ct)^m}{m!} \frac{(\beta t)^{nm}}{\Gamma(n(m+1)+1)}. \end{aligned}$$

In the special case when n is a positive integer, we can compute this as

$$\begin{aligned} p_\tau(t) = \frac{\beta e^{-\lambda(u+ct)-\beta t}}{u+ct} \frac{(\beta t)^{n-1}}{\Gamma(n)} \Bigg(u \; {}_0F_n \left(1, 1+\frac{1}{n}, ..., 1+\frac{n-1}{n}; \frac{\lambda(u+ct)(\beta t)^n}{n^n} \right) \\ + ct \; {}_0F_n \left(1+\frac{1}{n}, 1+\frac{2}{n}, ..., 1+\frac{n}{n}; \frac{\lambda(u+ct)(\beta t)^n}{n^n} \right) \Bigg), \quad (6) \end{aligned}$$

where

$${}_pF_q(B_1, B_2, ..., B_p, C_1, C_2, \ldots C_q; Z) = \sum_{m=0}^{\infty} \frac{(B_1)_m (B_2)_m ... (B_p)_m}{(C_1)_m (C_2)_m ... (C_q)_m} \frac{Z^m}{m!}$$

is the generalised hypergeometric function (and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol). Formula (6) follows from the identity

$$\frac{\Gamma(n+1)}{\Gamma((n(m+1)+1)} = \frac{1}{n^{nm}} \prod_{k=0}^{n-1} \frac{\Gamma(1+\frac{k+1}{n})}{\Gamma(m+1+\frac{k+1}{n})},$$

which can be derived by applying the multiplication formula of Gauss as described in [7].

Formula (6) is in a different form to the formula for $p_\tau(t)$ derived in [7]. A comparison of these two formulae for $p_\tau(t)$ yields the identity

$$\begin{aligned} & {}_0F_n\left(1, 1+\frac{1}{n}, ..., 1+\frac{n-1}{n}; \frac{\lambda(u+ct)(\beta t)^n}{n^n}\right) \\ - & {}_0F_n\left(1+\frac{1}{n}, 1+\frac{2}{n}, ..., 1+\frac{n}{n}; \frac{\lambda(u+ct)(\beta t)^n}{n^n}\right) \\ = & \lambda(u+ct)(\beta t)^n \frac{n!}{(2n)!}\, {}_0F_n\left(2+\frac{1}{n}, 2+\frac{2}{n}, ..., 2+\frac{n}{n}; \frac{\lambda(u+ct)(\beta t)^n}{n^n}\right). \end{aligned}$$

In the special case $n = 1$, by writing $z = \sqrt{4\lambda\beta t(u+ct)}$ this identity reduces to the well-known result (e.g. [1])

$$I_0(z) - \frac{2}{z} I_1(z) = I_2(z),$$

where I_v is the modified Bessel function of order v.

Next, let us consider the special case when $n = 2$, and let us further assume that claims occur according to a stationary renewal process, so that the distribution of T_0 is the equilibrium distribution of T_1. Then we find that

$$f_0(t) = \frac{\beta}{2} e^{-\beta t}(1+\beta t) = \frac{1}{2}(\beta e^{-\beta t} + \beta^2 t e^{-\beta t}),$$

giving

$$f^{*m} * f_0(t) = \frac{1}{2}\left(\frac{\beta^{2m+1} t^{2m} e^{-\beta t}}{\Gamma(2m+1)} + \frac{\beta^{2m+2} t^{2m+1} e^{-\beta t}}{\Gamma(2m+2)}\right).$$

Further,

$$f_1(t) = t f_0(t) = \frac{1}{2}(\beta t e^{-\beta t} + \beta^2 t^2 e^{-\beta t}),$$

giving

$$f^{*m} * f_1(t) = \frac{1}{2}\frac{\beta^{2m+1}t^{2m+1}e^{-\beta t}}{\Gamma(2m+2)} + \frac{\beta^{2m+2}t^{2m+2}e^{-\beta t}}{\Gamma(2m+3)}.$$

Then formula (2) gives

$$\begin{aligned} p_\tau(t) &= e^{-\lambda(u+ct)}\left(f_0(t) + \frac{u}{u+ct}\sum_{m=1}^{\infty}\frac{\lambda^m(u+ct)^m}{m!}\frac{1}{2}\left(\frac{\beta^{2m+1}t^{2m}e^{-\beta t}}{\Gamma(2m+1)} + \frac{\beta^{2m+2}t^{2m+1}e^{-\beta t}}{\Gamma(2m+2)}\right)\right. \\ &\quad \left. + \frac{c}{u+ct}\sum_{m=1}^{\infty}\frac{\lambda^m(u+ct)^m}{m!}\left(\frac{1}{2}\frac{\beta^{2m+1}t^{2m+1}e^{-\beta t}}{\Gamma(2m+2)} + \frac{\beta^{2m+2}t^{2m+2}e^{-\beta t}}{\Gamma(2m+3)}\right)\right), \end{aligned}$$

and we can incorporate $f_0(t)$ into the sums so that both start at $m = 0$. For computational purposes we can write this in terms of generalised hypergeometric functions as

$$\begin{aligned} p_\tau(t) &= \frac{\beta e^{-\lambda(u+ct)-\beta t}}{2(u+ct)}\left(u\ {}_0F_2\left(\frac{1}{2}, 1; \frac{\lambda(u+ct)(\beta t)^2}{4}\right)\right. \\ &+ t(\beta u + c)\ {}_0F_2\left(1, \frac{3}{2}; \frac{\lambda(u+ct)(\beta t)^2}{4}\right) + \left. c\beta t^2\ {}_0F_2\left(\frac{3}{2}, 2; \frac{\lambda(u+ct)(\beta t)^2}{4}\right)\right). \end{aligned} \tag{7}$$

Table 1 shows some values of finite time ruin probabilities when $\lambda = 1$, $\beta = 2$ and $c = 1.1$. We use the notation $\psi(u,t)$ to denote the probability of ruin by time t from initial surplus u when the density of τ is given by formula (6) with $n = 2$, and $\psi_e(u,t)$ denotes the corresponding probability when the density of τ is given by formula (7). These values have been found by integrating the density functions numerically using Mathematica. We can observe from this table that for each combination of u and t, the finite time ruin probability is greater when the distribution of T_0 is the equilibrium distribution of T_1. This arises because both the mean and variance of T_0 are smaller than the corresponding values for T_1.

3.2 Mixed exponential inter-claim times

Let us now consider the situation when the distribution of each T_j, $j = 0, 1, 2, \ldots$, is mixed exponential with density function

$$f(t) = f_0(t) = p\alpha e^{-\alpha t} + q\beta e^{-\beta t},$$

t	$\psi(0,t)$	$\psi_e(0,t)$	$\psi(10,t)$	$\psi_e(10,t)$	$\psi(20,t)$	$\psi_e(20,t)$
20	0.7973	0.8463	0.0457	0.0509	0.0009	0.0010
40	0.8332	0.8735	0.1008	0.1082	0.0060	0.0066
60	0.8481	0.8848	0.1387	0.1469	0.0138	0.0148
80	0.8564	0.8912	0.1651	0.1737	0.0218	0.0232
100	0.8618	0.8952	0.1842	0.1930	0.0292	0.0309

Table 1: Finite time ruin probabilities.

where $0 < p < 1$, $q = 1 - p$, and $\beta > \alpha > 0$. Following ideas in [10], it is shown in [6] that the m-fold convolution of f with itself as can be written as

$$f^{*m}(t) = \sum_{j=0}^{\infty} \gamma_{m,j}\ e(m+j, \beta; t),$$

where $e(m, \beta; t)$ denotes the Erlang(m) density with scale parameter β and

$$\gamma_{m,j} = q^m (1 - \alpha/\beta)^j \sum_{r=0}^{m} \binom{m}{r} \frac{(r)_j}{j!} \left(\frac{\alpha p}{\beta q}\right)^r .$$

We can find a similar type of expression for $f^{*m} * f_1(t)$ by using Laplace transforms. For a function w, let

$$\tilde{w}(s) = \int_0^{\infty} e^{-st} w(t) dt.$$

Then

$$\tilde{f}(s) = \frac{p\alpha}{\alpha + s} + \frac{q\beta}{\beta + s}$$

and

$$\tilde{f}_1(s) = \frac{p\alpha}{(\alpha + s)^2} + \frac{q\beta}{(\beta + s)^2},$$

leading to

$$\begin{aligned} \left[\tilde{f}(s)\right]^m \ \tilde{f}_1(s) \quad = \quad & \frac{p}{\alpha} \sum_{r=0}^{m} \binom{m}{r} p^r q^{m-r} \left(\frac{\alpha}{\alpha + s}\right)^{r+2} \left(\frac{\beta}{\beta + s}\right)^{m-r} \\ & + \frac{q}{\beta} \sum_{r=0}^{m} \binom{m}{r} p^r q^{m-r} \left(\frac{\alpha}{\alpha + s}\right)^{r} \left(\frac{\beta}{\beta + s}\right)^{m-r+2} . \end{aligned}$$

Hence

$$\begin{aligned}
f^{*m} * f_1(t) &= \frac{p}{\alpha}\sum_{r=0}^{m}\binom{m}{r}p^r q^{m-r}\int_0^t \frac{\alpha^{r+2}y^{r+1}e^{-\alpha y}}{\Gamma(r+2)}\frac{\beta^{m-r}(t-y)^{m-r-1}e^{-\beta(t-y)}}{\Gamma(m-r)}dy \\
&\quad +\frac{q}{\beta}\sum_{r=0}^{m}\binom{m}{r}p^r q^{m-r}\int_0^t \frac{\alpha^{r}y^{r-1}e^{-\alpha y}}{\Gamma(r)}\frac{\beta^{m-r+2}(t-y)^{m-r+1}e^{-\beta(t-y)}}{\Gamma(m-r+2)}dy \\
&= p\alpha\sum_{r=0}^{m}\binom{m}{r}(\alpha p)^r(\beta q)^{m-r}\frac{e^{-\beta t}t^{m+1}}{\Gamma(m+2)}\,{}_1F_1(r+2,m+2,(\beta-\alpha)t) \\
&\quad +q\beta\sum_{r=0}^{m}\binom{m}{r}(\alpha p)^r(\beta q)^{m-r}\frac{e^{-\beta t}t^{m+1}}{\Gamma(m+2)}\,{}_1F_1(r,m+2,(\beta-\alpha)t).
\end{aligned}$$

If we now replace the ${}_1F_1$ functions by their series representations, we find after a small amount of manipulation, that

$$f^{*m} * f_1(t) = \sum_{i=0}^{\infty}\eta_{i,m}e(m+i+2,\beta;t),$$

where

$$\eta_{i,m} = \frac{q^m}{\beta^2}(1-\alpha/\beta)^i\sum_{r=0}^{m}\binom{m}{r}\frac{(r)_i}{i!}\left(\frac{\alpha p}{\beta q}\right)^r((r+2)(r+1)\alpha p+\beta q).$$

Thus, we have formulae for all the ingredients in formula (2).

Figure 2 shows three plots of the density of τ when $u = 10$, $c = 1.1$ and the parameters of the mixed exponential distribution are as in Table 2. In Figure 2, Case A is illustrated by the dotted line, Case B by the solid line, and Case C by the bold line. We observe that the ordering of these three plots leads to highest finite time ruin probabilities for Case A and lowest for Case C, consistent with the ordering of the three values of $\mathbb{V}[T_0]$.

Case	α	β	p	$\mathbb{E}[T_0]$	$\mathbb{V}[T_0]$
A	2/5	2	1/4	1	5/2
B	1/2	2	1/3	1	2
C	3/5	2	3/7	1	5/3

Table 2: Parameters of mixed exponential distributions.

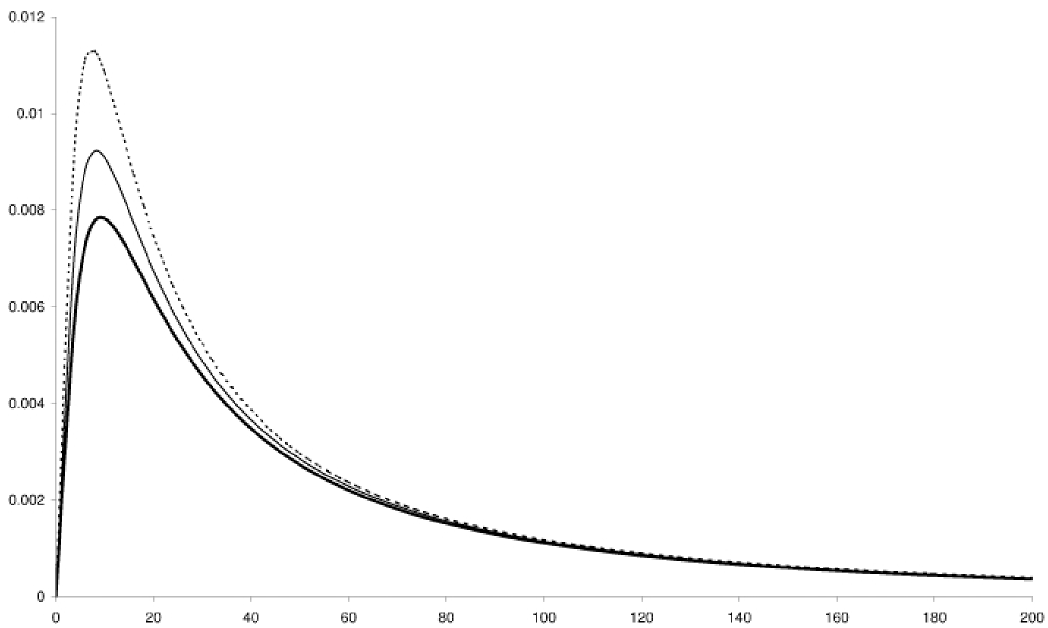

Figure 2: Densities of time to ruin for mixed exponential inter-claim times.

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