# On the Maximum Severity of Ruin in the Compound Poisson Model with a Threshold Dividend Strategy

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**Abstract**

We study the distribution and moments of the maximum severity of ruin in the compound Poisson risk process with a threshold dividend strategy. The distribution can be analyzed through the probability that the surplus process attains a given level from the initial surplus without first falling below zero. This note extends the results in Picard (1994) and shows that the distribution of the maximum severity of ruin can be expressed explicitly in terms of the ruin probabilities of two classical risk models with different premium rates. The moments of the maximum severity of ruin can be obtained through its distribution function.

**Keywords:** Compound Poisson risk process; Threshold dividend strategy; Time of ruin; Maximum surplus before ruin; Maximum severity of ruin.

## 1 Introduction

Consider the classical risk model in which the surplus process $\{U(t); t \geq 0\}$ is given by

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i \;, \quad t \geq 0 \;, \tag{1}$$

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where $u \geq 0$ is the initial surplus, $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables representing the individual claim amounts, with common probability distribution function (d.f.) $P$, density $p$ and mean $\mu$. The counting process $\{N(t); t \geq 0\}$ denotes the number of claims up to time $t$ and is defined as $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$, where the inter-claim times $W_i$'s are assumed to be i.i.d. random variables with common density function $f_W(t) = \lambda e^{-\lambda t}, t > 0$, that is, $\{N(t); t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$. Further assume that $\{W_i\}_{i \geq 1}$ and $\{X_i\}_{i \geq 1}$ are independent.

Ruin probability and related problems in the classical risk model have been studied extensively. Gerber and Shiu (1998) introduced a discounted penalty function with respect to the time of ruin, the surplus before ruin and the deficit at ruin. Many quantities can be analyzed through this function in a unified manner; see Gerber and Shiu (1998) and Lin and Willmot (1999, 2000).

Dividend strategies for insurance risk models were first proposed by De Finetti (1957). Among all the dividend strategies, two surplus-dependent dividend strategies are of particular interest. The first is the constant dividend barrier strategy under which no dividend is paid when the surplus is below a constant level and the total surplus above this level is paid out as dividends; see Bühlmann (1970), Gerber (1979), and Dickson and Waters (2004) and references therein. For the classical risk model with a constant dividend barrier strategy, Lin et al. (2003) study the expected discounted penalty function and related problems in great detail. The other dividend strategy is the so-called threshold strategy under which no dividends are paid when the surplus is below a constant level and dividends are paid at a constant rate which is less than the premium rate when the surplus is above this level. Gerber and Shiu (2005) show that the threshold strategy is optimal when the dividend rate is bounded from above. Lin and Pavlova (2006) study the expected discounted penalty function and analyze some special cases.

In this paper we consider the surplus process (1) modified by the payment of dividends. Let $\alpha$ $(0 < \alpha < c_1 = c)$ be the dividend rate. When the surplus exceeds the constant barrier $b\,(\geq u)$, dividends are paid continuously at rate $\alpha$ so that the net premium rate after dividend payments is $c_1 - \alpha = c_2$. We set $c_i = (1 + \theta_i)\lambda\,\mu$ with $\theta_i > 0$ for $i = 1, 2$. Let $\{U_b(t); t \geq 0\}$ be the surplus process with initial surplus $U_b(0) = u$ under the threshold dividend strategy above; then it can be expressed as

$$dU_b(t) = \begin{cases} c_1 dt - dS(t), & U_b(t) \leq b, \\ c_2 dt - dS(t), & U_b(t) > b, \end{cases} \tag{2}$$

where $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregate claim amount up to time $t$. The time of ruin is defined as

$$T_b = \inf\{t \geq 0 : U_b(t) < 0\},$$

where $T_b = \infty$ if the surplus never falls below zero. We remark that this model can be viewed as the compound Poisson risk model with a two-step premium rate as defined in Asmussen (2000, Chapter VII, Section 1). In particular, when $b = \infty$, the model with the threshold strategy reduces to the compound Poisson risk model; when $c_2 = 0$, the model simplifies to the compound Poisson risk model with a barrier dividend strategy.

Ruin probability is defined to be $\psi(u; b) = \mathbb{P}(T_b < \infty | U_b(0) = u)$. Since the premium rates are different when $0 \leq u \leq b$ and $u > b$, then $\psi(u; b)$ may be written as

$$\psi(u; b) = \psi_1(u; b)I(0 \leq u \leq b) + \psi_2(u; b)I(u > b), \quad u \geq 0,$$

where $I(\cdot)$ is the indicator function. Let

$$\phi(u; b) = 1 - \psi(u; b) = \phi_1(u; b)I(0 \leq u \leq b) + \phi_2(u; b)I(u > b)$$

be the non-ruin probability, where $\phi_i(u; b) = 1 - \psi_i(u; b)$ for $i = 1, 2$. Detailed analysis of $\psi(u; b)$ can be found in Lin and Pavlova (2006).

Throughout this paper, we let $\Psi_1(u) = 1 - \Phi_1(u)$ be the ruin probability for the surplus process defined in (1) with the premium rate being $c_1$ and $\Psi_2(u) = 1 - \Phi_2(u)$ be the ruin probability for the corresponding model with premium rate $c_2$ without any constraints. As we will see, all the results in the paper can be expressed in terms of these two ruin probabilities.

The aim of this paper is to study the distributions of the maximum surplus before ruin and the maximum severity of ruin in the classical risk process under the threshold dividend strategy. The rest of the paper is organized as follows. Section 2 studies the probability of the surplus attaining a certain level before ruin through which the distribution of the maximum surplus before ruin can be obtained. The distribution and the moments of the maximum severity of ruin are given in Sections 3 and 4, respectively. Section 5 gives the simplified results for exponential claims. An example is given in Section 6 when claim amounts follow a mixture of two exponential distributions.

# 2 The distribution of the maximum surplus before ruin

Define

$$\tau_x = \inf\{t > 0 : U_b(t) \geq x \,|\, U_b(0) = u\}, \qquad u \geq 0, b \geq 0, x \geq u,$$

to be the first time that the surplus process upcrosses the level $x$, and

$$\chi(x; u, b) = \mathbb{P}(T_b > \tau_x \,\big|\, U_b(0) = u)$$

to be the probability that the surplus process attains a given level $x \geq u$ from initial surplus $u$ without first falling below zero. When $x > u$ since $\phi(u; b) = \chi(x; u, b)\phi(x; b)$, then

$$\chi(x; u, b) = \frac{\phi(u; b)}{\phi(x; b)} = \begin{cases} \frac{\phi_1(u;b)}{\phi_1(x;b)}, & u < x \leq b, \\ \frac{\phi_1(u;b)}{\phi_2(x;b)}, & u \leq b < x, \\ \frac{\phi_2(u;b)}{\phi_2(x;b)}, & b < u < x. \end{cases}$$

It follows from Lin and Pavlova (2006) that

$$\phi(u; b) = \begin{cases} \phi_1(u; b) = q(b)\Phi_1(u), & 0 \leq u \leq b\,, \\ \phi_2(u; b) = 1 - \left[h(u) + \frac{1+\theta_2}{\theta_2} \int_0^{u-b} h(u-y)d\Phi_2(y)\right], & u > b, \end{cases}$$

where

$$\begin{aligned} q(b) &= \frac{\theta_2}{(\theta_1 - \theta_2)\Psi_1(b) + \theta_2}, \\ h(u) &= \frac{1}{1+\theta_2}\left[\bar{P}_e(u-b) - q(b)\int_{u-b}^{u} \Phi_1(u-t)dP_e(t)\right], \qquad u > b, \end{aligned}$$

with

$$\bar{P}_e(u) = 1 - P_e(u) = \frac{\int_u^\infty \bar{P}(t)dt}{\mu}$$

being the equilibrium distribution of $P$ and $\bar{P}(t) = 1 - P(t)$ for $t \geq 0$.
**Remark:** For $0 \leq u \leq x \leq b$,

$$\chi(x; u, b) = \frac{\phi(u; b)}{\phi(x; b)} = \frac{\phi_1(u; b)}{\phi_1(x; b)} = \frac{\Phi_1(u)}{\Phi_1(x)}.$$

In particular, when $b = \infty$, the surplus process simplifies to the classical compound Poisson surplus process without any constraints. We denote $\chi(x; u) = \chi(x; u, \infty)$ and then

$$\chi(x; u) = \frac{\Phi_1(u)}{\Phi_1(x)}, \qquad 0 \leq u \leq x,$$

which is given in Dickson and Gray (1984).

Define

$$\xi(x; u,\, b) = \mathbb{P}\Big( \sup_{0\le t\le T_b} U_b(t) < x,\ T_b < \infty \,\Big|\, U_b(0) = u \Big)\,.$$

$\xi(x; u,\, b)$ can be viewed as the defective distribution function of the maximum surplus before ruin. Alternatively, $\xi(x; u,\, b)$ can be viewed as the probability that ruin occurs from initial surplus $u$ without the surplus ever reaching level $x$. Since eventually either ruin occurs without the surplus process attaining $b$ or the surplus attains level $b$, then $\xi(x; u,\, b) = 1 - \chi(x; u\,, b)$.

# 3 The maximum severity of ruin

In this section, we allow the surplus process to continue if ruin occurs, and consider the insurer's maximum severity of ruin from the time of ruin until the time that the surplus next returns to level zero. When surplus is below zero, no dividends are paid so that the premium rate is $c_1$. Since we assume that $c_1 > \lambda\,\mu$, it is certain that the surplus process attains this level. As presented in Gerber (1990), sometimes, the event ruin has a very small probability and the portfolio is just one out of many existing in the company. The company has enough funds available to support the negative surplus for some time, in the hope that the portfolio will recovery in the future, allowing the company to keep this business alive. The maximum severity of ruin describes the worst situation the company would experience before reaching the time of recovery. See Picard (1994) for more interpretation of the maximum severity of ruin.

Define $T'$ to be the first time of the surplus process upcrossing level 0 after ruin occurs, i.e.,

$$T' = \inf\{t : \, t > T_b, U_b(t) \ge 0\},$$

and define

$$M_{u,b} = \sup\{|U_b(t)|,\, T_b \le t \le T' \,\big|\, U_b(0) = u\}, \qquad u \ge 0,$$

to be the maximum severity of ruin. For $z, u, b \ge 0$, Define

$$J(z;\, u, b) = \mathbb{P}(M_{u,b} \le z \,\big|\, T_b < \infty)$$

to be the distribution function of the maximum severity of ruin given that ruin occurs. $J(z;\, u, b)$ may be written as

$$J(z;\, u, b) = J_1(z; u, b) I(0 \le u \le b) + J_2(z; u, b) I(u > b).$$

It follows from the reasoning in Dickson (2005, p. 164) that if the surplus process starts from an initial surplus $u$, then the maximum severity of ruin will be no more than $z$ if ruin occurs with a deficit $y \leq z$ and if the surplus does not fall below $-z$ from level $-y$. The probability of the latter event is equal to $\chi(z; z-y)$, which is the probability that the surplus reaches $z$ from the initial surplus $z-y$ in classical risk model with premium rate being $c_1$, since attaining level 0 from level $-y$ without falling below $-z$ is equivalent to attaining level $z$ from level $z-y$ without falling below 0 for the surplus with a premium rate $c_1$. Thus

$$J(z;\, u, b) = \frac{1}{\psi(u; b)} \int_0^z g(y\big|u, b)\chi(z; z-y)dy = \frac{\int_0^z g(y\big|u, b)\Phi_1(z-y)dy}{\psi(u; b)\Phi_1(z)},$$

where $g(y\big|u, b) = \partial G(y\big|u, b)/\partial y$, with

$$\begin{aligned} G(y\big|u, b) &= \mathbb{P}(T_b < \infty,\, U_b(T_b) \geq -y) \\ &= G_1(y|u, b)I(0 \leq u \leq b) + G_2(y|u, b)I(u > b) \end{aligned}$$

being the probability that ruin occurs and that the deficit at ruin is at most $y$.

Let $G_1(y|u) = \mathbb{P}(T_\infty < \infty, U_\infty(T_\infty) \geq -y)$ be the marginal distribution function of the deficit at ruin for the classical risk model with premium rate $c_1$, and $g_1(y|u) = \partial G_1(y|u)/\partial y$ be the corresponding marginal density function of the deficit at ruin for the classical risk model with premium rate $c_1$.

The Corollary 8.3. of Lin and Pavlova (2006) gives an expression for $G_i(y|u, b)$ for $i = 1, 2$, in terms of $G_1(y|u)$, $\Psi_1(u)$, and $\Psi_2(u)$, but there are some mistakes. Here are the correct expressions with our new notation.

$$\begin{aligned} G_1(y|u, b) &= G_1(y|u) + \kappa(b)\Phi_1(u), \qquad 0 \leq u \leq b, \\ G_2(y|u, b) &= \frac{1+\theta_2}{\theta_2} \int_0^{u-b} h(u-x)d\Phi_2(x) + h(u), \qquad u > b, \end{aligned}$$

where

$$\begin{aligned} \kappa(b) &= \frac{(\theta_1 - \theta_2)G_1(y|b)}{(\theta_1 - \theta_2)\Psi_1(b) + \theta_2}, \\ h(u) &= \frac{1}{1+\theta_2} \left[ \int_{u-b}^{u} G_1(y|u-x, b)dP_e(x) + \bar{P}_e(u) - \bar{P}_e(u+y) \right], \qquad u > b. \end{aligned}$$

Differentiating $G_1(y|u, b)$ and $G_2(y|u, b)$ with respect to $y$ gives, for $0 \leq u \leq b$,

$$g(y|u, b) = g_1(y|u, b) = g_1(y|u) + \frac{(\theta_1 - \theta_2)g_1(y|b)}{(\theta_1 - \theta_2)\Psi_1(b) + \theta_2}\Phi_1(u),$$

and for $u > b$,

$$
\begin{aligned}
g(y|u,b) &= g_2(y|u,b) = \frac{1}{1+\theta_2}\left[\int_{u-b}^{u} g_1(y|u-t,b)dP_e(t) + p_e(u+y)\right] \\
&\quad + \frac{1}{\theta_2}\left[\int_0^{u-b}\int_{u-b-t}^{u-t} g_1(y|u-x-t)dP_e(x)d\Phi_2(t)\right. \\
&\quad \left. + \int_0^{u-b} p_e(u-t+y)d\Phi_2(t)\right].
\end{aligned}
$$

It follows from Dickson (2005, p. 164) that

$$
\Psi_1(u+z) = \int_z^{\infty} g_1(y|u)dy + \int_0^z g_1(y|u)\Psi_1(z-y)dy
$$

and

$$
\begin{aligned}
\int_0^z g_1(y|u)\Phi_1(z-y)dy &= \int_z^{\infty} g_1(y|u)dy + \int_0^z g_1(y|u)dy - \Psi_1(u+z) \\
&= \Psi_1(u) - \Psi_1(u+z).
\end{aligned}
$$

Now for $0 \le u \le b$, we have

$$
\begin{aligned}
J_1(z;\, u,b) &= \frac{\int_0^z g_1(y|u,b)\Phi_1(z-y)dy}{\psi_1(u;b)\Phi_1(z)} \\
&= \frac{1}{\psi_1(u;b)}\left\{\frac{\Psi_1(u)-\Psi_1(u+z)}{1-\Psi_1(z)}\right. \\
&\quad \left. + \frac{(\theta_1-\theta_2)[1-\Psi_1(u)]}{(\theta_1-\theta_2)\Psi_1(b)+\theta_2}\left[\frac{\Psi_1(b)-\Psi_1(b+z)}{1-\Psi_1(z)}\right]\right\}.
\end{aligned} \tag{3}
$$

Let

$$
R(z;u) = \frac{\Psi_1(u)-\Psi_1(u+z)}{\Psi_1(u)[1-\Psi_1(z)]}, \qquad u \ge 0, z \ge 0.
$$

Note that $R(z;u)$ is a proper distribution function in $z$. Moreover, Picard (1994) shows that $R(z;u)$ is the distribution of the maximum severity of ruin in the classical surplus process defined in (1) with premium rate $c_1$. Then Eq. (3) can be expressed as

$$
J_1(z;\, u,b) = \alpha(u,b)R(z;u) + \beta(u,b)R(z;b), \qquad 0 \le u \le b, \tag{4}
$$

where

$$\alpha(u,b) = \frac{\Psi_1(u)}{\psi_1(u;b)} = \frac{\Psi_1(u)}{[1-q(b)+q(b)\Psi_1(u)]},$$
$$\beta(u,b) = \frac{(\theta_1-\theta_2)\Psi_1(b)[1-\Psi_1(u)]}{[1-q(b)+q(b)\Psi_1(u)][(\theta_1-\theta_2)\Psi_1(b)+\theta_2]},$$

with $0 \le \alpha(u,b) \le 1, 0 \le \beta(u,b) \le 1$, and $\alpha(u,b)+\beta(u,b)=1$. This shows that $J_1(z;u,b)$ is a weighted average of $R(z;u)$ and $R(z;b)$. In particular, when $u=b$, $J_1(z;b,b)=R(z;b)$, that is to say, when the surplus process starts from $b$, the distribution of the maximum severity of ruin given ruin occured in the classical risk model under a threshold dividend strategy is the same as that in the classical risk model with premium rate $c_1$ without dividend payments.

For $u>b$, we have

$$\begin{aligned} J_2(z;u,b) &= \frac{\int_0^z g_2(y|u,b)\Phi_1(z-y)dy}{\psi_2(u;b)\Phi_1(z)} \\ &= \frac{1}{(1+\theta_2)\psi_2(u;b)}\left[\int_{u-b}^{u} J_1(z;u-x,b)\psi_1(u-x;b)dP_e(x) \right. \\ &\quad \left. + \frac{\int_0^z \Phi_1(z-y)dP_e(u+y)}{\Phi_1(z)}\right] \\ &\quad + \frac{1}{\theta_2\psi_2(u;b)}\left[\int_0^{u-b}\int_{u-b-t}^{u-t} J_1(z;u-x-t)\psi_1(u-x-t;b)dP_e(x)d\Phi_2(t) \right. \\ &\quad \left. + \frac{\int_0^{u-b}\int_0^z \Phi_1(z-y)dP_e(u+y-t)d\Phi_2(t)}{\Phi_1(z)}\right]. \end{aligned} \tag{5}$$

For simplicity, let

$$Q(y;u) = \frac{P_e(u+y)-P_e(u)}{\bar{P}_e(u)}, \qquad u \ge 0, y \ge 0,$$

and define

$$V(z;u) = \frac{\int_0^z \Phi_1(z-y)Q(dy;u)}{\Phi_1(z)}, \qquad z \ge 0.$$

Note that $Q(y;u)$ is a distribution function in $y$ and $V(z;u)$ is a distribution function in $z$. Further for $u>b$ write

$$w(z;u,b) = \int_{u-b}^{u} J_1(z;u-x,b)\psi_1(u-x;b)dP_e(x) + \bar{P}_e(u)V(z;u),$$

then (5) can be expressed as

$$J_2(z;u,b) \quad = \quad \frac{w(z;u,b)}{(1+\theta_2)\psi_2(u;b)} + \frac{\int_0^{u-b} w(z;u-t,b)d\Phi_2(t)}{\theta_2\psi_2(u;b)}. \tag{6}$$

**Remarks:**

1. For $0 \le u \le b$, the distribution function of the maximum severity of ruin given that ruin occurs only depends on the ruin probability $\Psi_1(u)$ in the classical risk model with premium rate $c_1$ and the loading factor $\theta_2$.

2. When $b = \infty$, $q(\infty) = 1, \theta_1 = \theta_2$, and $\alpha(u,b) = 1$, then $J_1(z;u,b)$ in (4) simplifies to

$$J_1(z;u,\infty) = R(z;u) = \frac{\Psi_1(u) - \Psi_1(u+z)}{\Psi_1(u)[1-\Psi_1(z)]},$$

which was first given in Picard (1994).

3. When $0 \le u \le b$ and $c_2 = \lambda\mu$, so $\theta_2 = 0, q(b) = 0$, and the result obtained in (4) simplifies to

$$J_1(z;u,b) = \Psi_1(u)R(z;u) + [1-\Psi_1(u)]R(z;b)$$

and the weights reduce to $\Psi_1(u)$ and $\Phi_1(u)$, respectively.

4. For $u > b$, the distribution of the maximum severity of ruin in (5) is much complicated, however, $J_2(z;u,b)$ only depends on ruin probabilities $\Psi_1(u)$, $\Psi_2(u)$ and $J_1(z;s,b)$ for $s \le b$.

# 4 The moments of the maximum severity of ruin

Define $H^{(n)}(u,b) = \mathbb{E}\left[M_{u,b}^n \middle| T_b < \infty\right]$ to be the $n$-th moment of $M_{u,b}$ given that ruin occured. Denote $H_i^{(n)}(u,b) = \int_0^\infty z^n J_i(dz;u,b)$, for $i = 1,2$. Then

$$H^{(n)}(u,b) = H_1^{(n)}(u,b)I(0 \le u \le b) + H_2^{(n)}(u,b)I(u > b).$$

Further define $M_1^{(n)}(u)$ to be the $n$-th moment of the maximum severity of ruin given that ruin occurs in the classical risk model with premium rate $c_1$, i.e.,

$$M_1^{(n)}(u) = \int_0^\infty z^n R(dz;u) = n\int_0^\infty z^{n-1}[1-R(z;u)]dz.$$

Then we have

$$\begin{aligned} H_1^{(n)}(u,b) &= \alpha(u,b)M_1^{(n)}(u) + \beta(u,b)M_1^{(n)}(b) \\ &= \frac{\Psi_1(u)}{\psi_1(u;b)}M_1^{(n)}(u) + \frac{\psi_1(u;b)-\Psi_1(u)}{\psi_1(u;b)}M_1^{(n)}(b), \quad 0 \le u \le b. \end{aligned} \tag{7}$$

To calculate $H_2^{(n)}(u,b)$, let

$$L^{(n)}(u) = \int_0^\infty z^n V(dz;u), \qquad n \in \mathbb{N}^+,$$

be the $n$-th moment of the distribution $V$ and

$$w^{(n)}(u;b) = \int_0^\infty z^n w(dz;u,b) = \int_{u-b}^u H_1^{(n)}(u-x,b)\psi_1(u-x;b)dP_e(x) + \bar{P}_e(u)L^{(n)}(u)\,.$$

It follows from (6) that

$$H_2^{(n)}(u,b) = \frac{w^{(n)}(u;b)}{(1+\theta_2)\psi_2(u;b)} + \frac{\int_0^{u-b} w^{(n)}(u-t;b)d\Phi_2(t)}{\theta_2\,\psi_2(u;b)}\,. \tag{8}$$

Substituting (7) into (8) and noting that $\psi_1(u;b) - \Psi_1(u) = [1-q(b)]\Phi_1(u)$, we have

$$H_2^{(n)}(u,b) = \frac{m^{(n)}(u;b)}{(1+\theta_2)\psi_2(u;b)} + \frac{\int_0^{u-b} m^{(n)}(u-t;b)d\Phi_2(t)}{\theta_2\,\psi_2(u;b)}\,, \tag{9}$$

where for $u > b$,

$$\begin{aligned} m^{(n)}(u;b) &= \int_{u-b}^u M_1^{(n)}(u-x)\Psi_1(u-x)dP_e(x) \\ &\quad +[1-q(b)]M_1^{(n)}(b)\int_{u-b}^u \Phi_1(u-x)dP_e(x) + \bar{P}_e(u)L^{(n)}(u)\,. \end{aligned}$$

We remark that both $H_1^{(n)}(u,b)$ and $H_2^{(n)}(u,b)$ depend on $M_1^{(n)}(u)$, the $n$-th moment of the maximum severity of ruin in the classical risk model with premium rate $c_1$. $M_1^{(n)}(u)$ has been studied in Dickson (2002) for some claim size distributions.

# 5 Explicit results for exponential claims

In this section, we aim at calculating the distribution and moments of $M_{u,b}$ when claim amounts are exponentially distributed.

Let us suppose that $P(x) = 1 - e^{-\eta x}, x > 0$, and $c_i = (1+\theta_i)\lambda/\eta$ with $\theta_i > 0$ being the relative safety loading factor for $i = 1, 2$. It is well known that

$$\Psi_i(u) = \frac{1}{1+\theta_i} e^{-R_i u}, \qquad u \geq 0,\ i = 1, 2,$$

where $R_i = \theta_i \eta/(1+\theta_i), i = 1, 2$, is the adjustment coefficient of the classical risk model with premium rate $c_i$. We have

$$\begin{aligned} R(z; u) &= \frac{\Psi_1(u) - \Psi_1(u+z)}{\Psi_1(u)[1 - \Psi_1(z)]} = \frac{1 - e^{-R_1 z}}{1 - \frac{1}{1+\theta_1} e^{-R_1 z}} \\ &= [1 - e^{-R_1 z}] \sum_{k=0}^{\infty} \left( \frac{1}{1+\theta_1} \right)^k e^{-kR_1 z} = \sum_{k=1}^{\infty} \omega_k (1 - e^{-kR_1 z}) = K(z), \end{aligned}$$

where $\omega_k = \frac{\theta_1}{(1+\theta_1)^k}$ so that $\sum_{k=1}^{\infty} \omega_k = 1$. $K(z)$ is an infinite mixture of exponential distributions. Then, for the case $0 \leq u \leq b$, Eq. (4) simplifies to

$$J_1(z;\, u, b) = \alpha(u, b) R(z; u) + \beta(u, b) R(z; b) = \sum_{k=1}^{\infty} \omega_k (1 - e^{-kR_1 z}) = K(z),$$

which is independent of $u$ and $b$. Furthermore, for $0 \leq u \leq b$, we have

$$\begin{aligned} H_1^{(n)}(u, b) &= \int_0^{\infty} z^n J_1(dz; u, b) = \int_0^{\infty} z^n K(dz) \\ &= \sum_{k=1}^{\infty} \omega_k \frac{n!}{(kR_1)^n} = \sum_{k=1}^{\infty} \frac{\theta_1}{(1+\theta_1)^k} \frac{(1+\theta_1)^n n!}{(k\,\eta\,\theta_1)^n}, \end{aligned}$$

which is again independent of $u$ and $b$. In particular, when $n = 1$,

$$H_1^{(1)}(u, b) = \frac{(1+\theta_1)}{\eta} \ln(1 + \theta_1^{-1}).$$

For the case $u > b$, since $Q(y; u) = P_e(y) = P(y) = 1 - e^{-\eta y}$, and

$$V(z; u) = \frac{\int_0^z \Phi_1(z-y) dP_e(y)}{\Phi_1(z)} = \frac{1 - e^{-R_1 z}}{1 - \frac{1}{1+\theta_1} e^{-R_1 z}} = \sum_{k=1}^{\infty} \omega_k (1 - e^{-kR_1 z}) = K(z),$$

which is independent of $u$, then Eq. (6) simplifies to

$$\begin{aligned} J_2(z;u,b) &= \frac{K(z)}{(1+\theta_2)\psi_2(u;b)}\left[\int_{u-b}^{u}\psi_1(u-x)dP_e(x)+\bar{P}_e(u)\right] \\ &\quad+\frac{K(z)}{\theta_2\psi_2(u;b)}\left[\int_0^{u-b}\int_{u-b-t}^{u-t}\psi_1(u-x-t)dP_e(x)d\Phi_2(t)\right. \\ &\quad\left.+\int_0^{u-b}\bar{P}_e(u-t)d\Phi_2(t)\right]=K(z)\,, \end{aligned} \tag{10}$$

where the last step follows from (6.1) of Lin and Pavlova (2006), i.e.,

$$\psi_2(u;b)=\frac{1+\theta_2}{\theta_2}\int_0^{u-b}h(u-y)d\Phi_2(y)+h(u),$$

where

$$h(u)=\frac{1}{1+\theta_2}\left[\int_{u-b}^{u}\psi_1(u-t)dP_e(t)+\bar{P}_e(u)\right].$$

Then

$$H_2^{(n)}(u,b)=\int_0^\infty z^n J_2(dz;u,b)=\int_0^\infty z^n K(dz)=H_1^{(n)}(u,b).$$

We remark that when the claim amounts are exponentially distributed, the distribution and moments of the maximum severity of ruin in the classical risk model with a threshold strategy are the same as those in the corresponding classical risk process without any constraints, in other words, the threshold dividend strategy does not affect the maximum severity of ruin for the model with the exponential claims. This can be explained as follows: due to the memoryless property of exponential distribution, the surplus immediately before ruin and the deficit at ruin are independent and so the behavior of the surplus process after the time of ruin is not affected by the surplus process before the time of ruin.

# 6 An example

In this section, we consider the mean and the standard deviation of the maximum severity of ruin when the claim amounts have a mixture of two exponential distributions, i.e.,

$$\bar{P}(y)=\pi_1 e^{-\eta_1 y}+\pi_2 e^{-\eta_2 y}, \qquad y>0,$$

with $\pi_2 = 1 - \pi_1$ and $\eta_j > 0$. Then $\bar{P}_e(y) = \sum_{j=1}^{2} \pi_j^* e^{-\eta_j y}, y > 0$, with $\pi_j^* = (\pi_j/\eta_j)/(\sum_{j=1}^{2} \pi_j/\eta_j), j = 1, 2$. Gerber et al. (1987) show that

$$\Psi_i(u) = \sum_{j=1}^{2} d_{i,j} e^{-R_{i,j} u}, \qquad u \geq 0, i = 1, 2,$$

where $0 < R_{i,1} < R_{i,2}$ are the roots of the equation $\sum_{j=1}^{2} [\pi_j^* \eta_j/(\eta_j - R)] = 1 + \theta_i$, for $i = 1, 2$, and

$$d_{i,j} = \frac{\sum_{k=1}^{2} \frac{\pi_k^*}{\eta_k - R_{i,j}}}{\sum_{k=1}^{2} \frac{\pi_k^* \eta_k}{(\eta_k - R_{i,j})^2}}, \quad i, j = 1, 2.$$

We remark that if $\pi_1 > 0$ and $\pi_2 > 0$, $d_{i,j} > 0$ for $i, j = 1, 2$. Lin and Pavlova (2006) show that

$$\psi(u; b) = \begin{cases} \psi_1(u; b) = 1 - q(b) + q(b) \sum_{j=1}^{2} d_{1,j} e^{-R_{1,j} u}, & 0 \leq u \leq b, \\ \psi_2(u; b) = \frac{1}{\theta_2} \sum_{j=1}^{2} R_{2,j} d_{2,j} \left[ \sum_{k=1}^{2} \frac{\pi_k^* Q_k(b)}{\eta_k - R_{2,j}} \right] e^{-R_{2,j}(u-b)}, & u > b, \end{cases}$$

where

$$\begin{aligned} q(b) &= \frac{\theta_2}{(\theta_1 - \theta_2) \sum_{j=1}^{2} d_{1,j} e^{-R_{1,j} b} + \theta_2}, \\ Q_j(b) &= 1 - q(b) + q(b) \sum_{k=1}^{2} \frac{\eta_j d_{1,k}}{\eta_j - R_{1,k}} e^{-R_{1,k} b}, \quad j = 1, 2. \end{aligned}$$

Dickson (2002) shows that for $n = 1, 2$,

$$\begin{aligned} M_1^{(n)}(u) &= n[a(u) - d_{1,1}] \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{\binom{r}{j} (d_{1,1})^j (d_{1,2})^{r-j}}{[(j+1) R_{1,1} + (r-j) R_{1,2}]^n} \\ &\quad + n[b(u) - d_{1,2}] \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{\binom{r}{j} (d_{1,1})^j (d_{1,2})^{r-j}}{[j R_{1,1} + (r-j+1) R_{1,2}]^n}, \qquad u > 0, \end{aligned}$$

where $a(u) = d_{1,1} e^{-R_{1,1} u}/\Psi_1(u)$ and $b(u) = d_{1,2} e^{-R_{1,2} u}/\Psi_1(u)$. Using the results in (7) and (9), we can calculate the moments of $M_{u,b}$.

Now we show some numerical results. Set $\lambda = 1$, $\pi_1 = 1/3, \pi_2 = 2/3, \eta_1 = 2/3, \eta_2 = 4/3$ so $\mu = \int_0^\infty \bar{P}(y) dy = 1$. Table 1 gives the means and the standard deviations of $M_{u,b}$ for $b = 5$ and $\infty$ and $u = 0, 1, \ldots, 10$ and for different combinations of $(\theta_1, \theta_2)$. From the table, we can see for each combination of $(\theta_1, \theta_2)$, the means and the standard deviations for the maximum severity of ruin given ruin occured are increasing in $u$. For a fixed $\theta_1$, the means and standard deviations are decreasing in $\theta_2$ when $u < b$, are unchanged with $\theta_2$ when $u = b$, and are increasing in $\theta_2$ when $u > b$.

Table 1: The values for $E(M_{u,b})$ and $s.d.(M_{u,b})$ for $b = 5$ and $\infty$

| $u$ | ($\theta_1 = 0.3$, $E(M_{u,5})$ | $\theta_2 = 0.10$) $s.d.(M_{u,5})$ | ($\theta_1 = 0.3$, $E(M_{u,5})$ | $\theta_2 = 0.2$) $s.d.(M_{u,5})$ | ($\theta_1 = 0.3$, $E(M_{u,\infty})$ | $\theta_2 = 0.3$) $s.d.(M_{u,\infty})$ |
|---|---|---|---|---|---|---|
| 0 | 2.1302 | 3.2168 | 2.1152 | 3.2046 | 2.1066 | 3.1976 |
| 1 | 2.2648 | 3.3212 | 2.2530 | 3.3124 | 2.2455 | 3.3068 |
| 2 | 2.3196 | 3.3612 | 2.3127 | 3.3563 | 2.3079 | 3.3528 |
| 3 | 2.3412 | 3.3766 | 2.3379 | 3.3743 | 2.3354 | 3.3725 |
| 4 | 2.3495 | 3.3825 | 2.3484 | 3.3817 | 2.3474 | 3.3810 |
| **5** | 2.3526 | 3.3847 | 2.3526 | 3.3847 | 2.3526 | 3.3847 |
| 6 | 2.3539 | 3.3856 | 2.3544 | 3.3860 | 2.3549 | 3.3863 |
| 7 | 2.3545 | 3.3860 | 2.3552 | 3.3865 | 2.3559 | 3.3870 |
| 8 | 2.3547 | 3.3862 | 2.3555 | 3.3867 | 2.3563 | 3.3873 |
| 9 | 2.3548 | 3.3862 | 2.3557 | 3.3868 | 2.3565 | 3.3874 |
| 10 | 2.3548 | 3.3862 | 2.3557 | 3.3869 | 2.3566 | 3.3875 |

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