A general formula for option prices in a stochastic volatility model

Stephen Chin and Daniel Dufresne*
Centre for Actuarial Studies
University of Melbourne

15 July 2009

Abstract

We consider the pricing of European derivatives in a Black-Scholes model with stochastic volatility. We show how Parseval's theorem may be used to express those prices as Fourier integrals. This is a significant improvement over Monte Carlo simulation in many cases. The main ingredient in our method is the Laplace transform of the ordinary (constant volatility) price of a put or call in the Black-Scholes model, where the transform is taken with respect to maturity (T); this does not appear to have been used before in pricing options under stochastic volatility. We derive these formulas and then apply them to the case where volatility is modelled as a continuous-time Markov chain, the so-called "Markov regime switching model". This model has been used previously in stochastic volatility modelling, but mostly with only $N = 2$ states. We show how to use $N = 3$ states without difficulty, and how larger number of states can be handled. Numerical illustrations are given, including the implied volatility curve in two and three-state models. The curves have the "smile" shape observed in practice.

1 Introduction

It has been known for a long time that geometric Brownian motion does not model stock prices perfectly, at least not in all cases; the conventional lore is that equity returns are not normally distributed, but rather have a density with a higher peak and heavier tails than the normal density. For some purposes this may not be a big problem, but in derivatives pricing it has moreover been noted that put or call prices do not conform to the standard constant-volatility Black-Scholes model; in other words, not only does the

*Corresponding author, Centre for Actuarial Studies, University of Melbourne, Melbourne VIC 3010, Australia, dufresne@unimelb.edu.au

physical measure not agree with Black-Scholes, but the risk-neutral (or "pricing") measure does not either. In particular, observed prices exhibit the now familiar "smile" of implied volatilities against strike prices. To deal with this problem there is one possible approach, going back to the 1980s, that consists in turning the volatility "σ" of the Black-Scholes model into a stochastic process. Specifically, the stock price S is assumed to satisfy

$$dS_t \;=\; \mu S_t\, dt + V_t S_t\, dW_t,$$

where W is a standard Brownian motion. This, however, leads to a difficulty in applying no-arbitrage principles (Harrison and Pliska [5]) to option pricing, since the market, if it consists only of S and a risk-free bond, is then incomplete. There is then an infinite number of martingale (=risk-neutral) measures, and hedging of puts or calls is not possible with certainty. This problem may be solved in theory by assuming that volatility is traded in one way or another. We will not study this problem, as our contribution lies elsewhere. We will assume that

$$dS_t \;=\; rS_t\, dt + V_t S_t\, dW_t,$$

where r is the risk-free rate and W is a standard Brownian motion *under the pricing measure.* We denote the latter $\mathbb{P}$, and the corresponding expectation operator $\mathbb{E}$, for simplicity, as we will only use one probability measure. (Of course, in papers and textbooks on derivative pricing there is almost always a "physical" as well as a "risk-neutral" or "pricing" measure, and two symbols such as $\mathbb{P}$ and $\mathbb{Q}$ are required; we place ourselves in a simpler situation, where we *assume* the pricing measure from the start.) Di Masi *et al.* [2] give a partial justification for our assumption, by showing that the risk-minimizing martingale measure in a model with Markov chain volatilities is precisely the one we describe; we refer the reader to that paper for more details.

In Section 2, the general approach is described, in particular the derivation of the Laplace transform of the Black-Scholes prices for European puts and calls, in terms of the maturity T. The latter is essential to our method. In the same section, some results from Dufresne *et al.* [3] are used to prove a general Fourier integral formula for option prices when volatility is stochastic; this formula applies in all cases where the Black-Scholes "σ" is replaced with a volatility process V, the latter being independent of the Brownian motion driving the stock price (W in the equation above). The Fourier integral for option prices so obtained is an integral over an infinite interval, but we show how to truncate that integral while keeping an upper bound on the error.

Section 3 is about the particular application we have in mind, that is, volatility modelled as a continuous-time Markov chain, that has already been studied by others, including Di Masi *et al.* [2] and Fuh *et al.* [4]. We focus on the Laplace transform of

$$U_T \;=\; \int_0^T V_s^2\, ds,$$

or "integrated realized variance". It is apparent that the calculations rapidly become more and more arduous as N (the number of possible states) increases. In the case $N = 2$, the probability distribution of

U_T has been known for a long time, and several proofs are known. In the financial literature, almost all papers are restricted to two states, the cases $N \geq 3$ being seen as too demanding. This is unfortunate, as the Markov chain volatility model is not very realistic when there are only two states. Our method does not require the explicit probability distribution of U_T, but only its Laplace transform $\mathbb{E}e^{-rU_T}$, so we are able to tackle the case $N = 3$ without any difficulty. In theory, our results apply to an arbitrary number of states, but the algebra does become more and more involved as N increases; symbolic mathematics software is essential for $N \geq 3$.

Section 4 gives the results of several numerical experiments, comparing our method with Monte Carlo simulation for the pricing of options (in passing we show that in this and other stochastic volatility models it is not necessary to simulate the stock price process itself, but only the process U_T). It is seen that our method has shorter execution times. On the downside, our method has two shortcomings. First, the general method described in Section 2 usually applies only when V is independent of W. Second, the computing of the Fourier integrals, although rather quick to obtain using Mathematica or C, requires some trial and error to find a good path of integration (as is often the case with Fourier integrals).

2 Option prices as Fourier integrals in stochastic volatility models

Several authors have proposed various Fourier integrals for prices of European options. We will rely on a very well-known idea, introduced in finance by Lewis [8] and others, and systematically studied in Dufresne *et al.* [3], that consists in applying Parseval's theorem with a damping factor. The no-arbitrage price of a European derivative is

$$\mathbb{E}g(X) \;=\; \int g(x)\,d\mu_X(x)$$

for some function $g(\cdot)$ and some random variable X (μ_X is the distribution of X). There are many cases where μ_X is either unknown or too complicated, but where the Fourier transform

$$\hat{\mu}_X(w) \;=\; \mathbb{E}e^{iwX}$$

is known and relatively easy to compute. One way to compute $\mathbb{E}g(X)$ is then to use Parseval's theorem, which says that

$$\int g(x)\,d\mu_X(x) \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}(-w)\hat{\mu}_X(w)\,dw,$$

where $\hat{g}(w) = \int e^{iwx} g(x)\,dx$ is the Fourier transform of g and "PV" means that what follows is a "principal value" integral, that is,

$$PV \int_{-\infty}^{\infty} h(w)\,dw \;=\; \lim_{M\to\infty} \int_{-M}^{M} h(w)\,dw.$$

There are conditions for Parseval's theorem to hold, among them that the function g be Lebesgue integrable. Unfortunately, the functions met in option pricing, such as $x \mapsto (e^x - K)_+$, do not satisfy this integrability condition. Nevertheless, it is often possible to rewrite the problem in such a way that Parseval's theorem may be applied, by introducing a "damping factor", which in effect will replace g with an integrable function. The idea is simple: rewrite the expectation as

$$\int g(x)\,d\mu_X(x) \;=\; \int e^{-\alpha x} g(x) e^{\alpha x}\,d\mu_X(x),$$

and apply Parseval's theorem to $g^{(-\alpha)}(x) = e^{-\alpha x}g(x)$ and $d\mu^{(\alpha)}(x) = e^{\alpha x}d\mu(x)$. In many cases it is possible to find α such that $g^{(-\alpha)}(x)$ is Lebesgue integrable and $\mu^{(\alpha)}$ has finite mass (so its Fourier transform is well-defined). This is discussed in detail in Dufresne *et al.* [3]. In our model, the stock price satisfies

$$dS_t \;=\; rS_t\,dt + V_t S_t\,dW_t,$$

where W is a standard Brownian motion under $\mathbb{P}$, independent of V. (As was said in the Introduction, one would normally write "$\mathbb{Q}$" rather than "$\mathbb{P}$", but in this paper we do all calculations under the pricing measure.) Then

$$S_t \;=\; S_0 \exp\left(rt - \frac{U_t}{2} + \int_0^t V_s\,dW_s\right).$$

Next, from the independence of V and W,

$$\begin{aligned} e^{-rT}\mathbb{E}(K - S_T)_+ \;&=\; e^{-rT}\mathbb{E}\,\mathbb{E}[(K - S_T)_+\,|\,V] \;=\; e^{-rT}\mathbb{E}[K - S_0\exp(rT - \tfrac{1}{2}U_T + \sqrt{U_T}W_1)]_+ \;=\; \mathbb{E}g(U_T) \\ g(u) \;&=\; \mathbb{E}[e^{-rT}K - S_0\exp(-\tfrac{u}{2} + \sqrt{u}W_1)]_+. \end{aligned}$$

The function g will immediately be recognized as the price of a European put in the Black-Scholes model. This leads us to the problem of finding the simplified expression for the Laplace transform, in the time variable, of the price of a European call or put in the Black-Scholes model.

Theorem 2.1. *Suppose $r \in \mathbb{R}, \sigma, K, S_0 > 0$, and let*

$$\beta = \frac{\gamma + r}{\sigma^2}, \quad \rho \;=\; \frac{r}{\sigma^2} - \frac{1}{2}, \quad \mu_1 \;=\; \rho + \sqrt{\rho^2 + 2\beta}, \quad \mu_2 \;=\; -\rho + \sqrt{\rho^2 + 2\beta}, \quad \overline{K} \;=\; \frac{K}{S_0}.$$

(a) If $\gamma > -r$, then

$$\int_0^\infty e^{-\gamma t} e^{-rt}\mathbb{E}(K - S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W_t})_+\,dt \;=\; \begin{cases} \frac{S_0}{\sigma^2\sqrt{\rho^2+2\beta}} \frac{\overline{K}^{1+\mu_1}}{\mu_1(1+\mu_1)} & \text{if } K \le S_0 \\ \frac{S_0}{\beta\sigma^2}\left[\overline{K} - \frac{2\beta}{2\beta - 2\rho - 1} + \frac{\beta\overline{K}^{1-\mu_2}}{\sqrt{\rho^2+2\beta}} \frac{1}{\mu_2(\mu_2 - 1)}\right] & \text{if } K > S_0. \end{cases}$$

(b) If $\gamma > \max(0, -r)$, then

$$\int_0^\infty e^{-\gamma t} e^{-rt}\mathbb{E}(S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W_t} - K)_+\,dt \;=\; \begin{cases} \frac{S_0}{\sigma^2\sqrt{\rho^2+2\beta}} \frac{\overline{K}^{1+\mu_1}}{\mu_1(1+\mu_1)} + \frac{S_0}{\gamma} - \frac{K}{\gamma + r} & \text{if } K \le S_0 \\ \frac{S_0}{\beta\sigma^2}\left[\overline{K} - \frac{2\beta}{2\beta - 2\rho - 1} + \frac{\beta\overline{K}^{1-\mu_2}}{\sqrt{\rho^2+2\beta}} \frac{1}{\mu_2(\mu_2 - 1)}\right] + \frac{S_0}{\gamma} - \frac{K}{\gamma + r} & \text{if } K > S_0. \end{cases}$$

Proof. (a) Rewrite the integral as

$$\begin{aligned} S_0 \int_0^\infty e^{-(\gamma+r)t}\mathbb{E}(\overline{K} - e^{(r-\frac{\sigma^2}{2})t+W_{\sigma^2 t}})_+ \, dt &= \frac{S_0}{\sigma^2}\int_0^\infty e^{-(\frac{\gamma+r}{\sigma^2})s}\, \mathbb{E}(\overline{K} - e^{(\frac{r}{\sigma^2}-\frac{1}{2})s+W_s})_+ \, ds \\ &= \frac{S_0}{\beta\sigma^2}\int_0^\infty \beta e^{-\beta s}\, \mathbb{E}(\overline{K} - e^{\rho s+W_s})_+ \, ds \\ &= \frac{S_0}{\beta\sigma^2}\mathbb{E}(\overline{K} - e^{\rho\tau+W_\tau})_+, \end{aligned}$$

where $\tau \sim \mathbf{Exp}(\beta)$ is independent of W. For simplicity, let $S_0 = 1$ and write K for $\overline{K}$ in the rest of this proof. We will use the distribution of $\rho\tau + W_\tau$ (when $\rho = 0$ this is of course a symmetric double exponential). We find

$$\mathbb{E}e^{q(\rho\tau+W_\tau)} = \mathbb{E}e^{(\rho q+\frac{q^2}{2})\tau} = \frac{2\beta}{2\beta - 2\rho q - q^2} = \frac{2\beta}{(\mu_1+q)(\mu_2-q)},$$

where $-\mu_1$ and μ_2 are the roots of $2\beta - 2\rho q - q^2$. Partial fractions show that the distribution of $\rho\tau + W_\tau$ is an asymmetric double exponential:

$$\frac{\beta}{\sqrt{\rho^2+2\beta}}\cdot\frac{1}{\mu_1+q} + \frac{\beta}{\sqrt{\rho^2+2\beta}}\cdot\frac{1}{\mu_2-q}$$

(μ_1 is the parameter of the exponential on $(-\infty, 0)$, and μ_2 is the parameter of the one on $(0,\infty)$). The density of this distribution is

$$\frac{\beta}{\sqrt{\rho^2+2\beta}}e^{\mu_1 x}\mathbb{1}_{\{x<0\}} + \frac{\beta}{\sqrt{\rho^2+2\beta}}e^{-\mu_2 x}\mathbb{1}_{\{x>0\}}.$$

Then

$$\mathbb{E}(K - e^{\rho\tau+W_\tau})_+ = \frac{\beta}{\sqrt{\rho^2+2\beta}}\left[\int_{-\infty}^0 (K-e^x)_+\, e^{\mu_1 x}\, dx + \int_0^\infty (K-e^x)_+\, e^{-\mu_2 x}\, dx\right] = \frac{\beta}{\sqrt{\rho^2+2\beta}}(I_1+I_2).$$

The cases $K \le 1$ and $K > 1$ lead to different formulas. If $K \le 1$ then $I_2 = 0$ and

$$I_1 = \int_{-\log K}^\infty (K - e^{-x})_+\, e^{-\mu_1 x}\, dx = \frac{K^{1+\mu_1}}{\mu_1} - \frac{K^{1+\mu_1}}{1+\mu_1} = \frac{K^{1+\mu_1}}{\mu_1(1+\mu_1)}.$$

Hence, when $K \le 1$,

$$\int_0^\infty e^{-(\gamma+r)t}\mathbb{E}(K - e^{(r-\frac{\sigma^2}{2})t+W_{\sigma^2 t}})_+ \, dt = \frac{1}{\sigma^2\sqrt{\rho^2+2\beta}}\frac{K^{1+\mu_1}}{\mu_1(1+\mu_1)}.$$

If $K > 1$, then

$$\mathbb{E}(K - e^{\rho\tau+W_\tau})_+ = K - \mathbb{E}e^{\rho\tau+W_\tau} + \mathbb{E}(e^{\rho\tau+W_\tau} - K)_+ = K - \frac{2\beta}{2\beta-2\rho-1} + \mathbb{E}(e^{\rho\tau+W_\tau} - K)_+.$$

The last term reduces to

$$\frac{\beta}{\sqrt{\rho^2+2\beta}}\int_{\log K}^\infty (e^x - K)\, e^{-\mu_2 x}\, dx = \frac{\beta}{\sqrt{\rho^2+2\beta}}\frac{K^{1-\mu_2}}{\mu_2(\mu_2-1)}.$$

(b) Use put call parity to express the price of the call as the price of the put plus

$$S_0 - Ke^{-rt},$$

and then multiply by $e^{-\gamma t}$ before integrating between 0 and ∞. □

Theorem 2.2. *Let U_T and S_T be as above, and let ν be the distribution of U_T, so that*

$$\widehat{\nu^{(\alpha)}}(u) \;=\; \mathbb{E}e^{(\alpha+iu)U_T}.$$

(a) Suppose that $\mathbb{E}e^{\alpha^* U_T} < \infty$ for some $\alpha^* > 0$. Then, for any $0 < \alpha < \alpha^*$,

$$e^{-rT}\mathbb{E}(K - S_T)_+ \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g_1^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\, du, \tag{2.1}$$

where, if $\overline{k} = Ke^{-rT}/S_0$,

$$\widehat{g_1^{(-\alpha)}}(-u) \;=\; \begin{cases} \frac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} < S_0 \\ \frac{S_0(\overline{k}-1)}{\alpha+iu} + \frac{S_0\overline{k}^{(1-\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} \geq S_0. \end{cases}$$

(b) Suppose that $\mathbb{E}e^{\alpha^* U_T} < \infty$ for some $\alpha^* > 0$. Then, for any $0 < \alpha < \alpha^*$,

$$e^{-rT}\mathbb{E}(S_T - K)_+ \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g_2^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\, du, \tag{2.2}$$

where

$$\widehat{g_2^{(-\alpha)}}(-u) \;=\; \begin{cases} \frac{S_0(1-\overline{k})}{\alpha+iu} + \frac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} < S_0 \\ \frac{S_0\overline{k}^{(1-\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} \geq S_0. \end{cases}$$

Proof. (a) Define

$$g_1(x) \;=\; \mathbb{E}(Ke^{-rT} - S_0 e^{-\frac{x}{2}+\sqrt{x}W_1})_+ \mathbb{1}_{\{x\geq 0\}}.$$

Then

$$e^{-rT}\mathbb{E}(K - S_T)_+ \;=\; \int g_1 d\nu \;=\; \int g_1^{(-\alpha)} d\nu^{(\alpha)}.$$

Apply Theorem 1 in Dufresne *et al.* [3] with $r(x) = e^{-\alpha x}$, $X = U_T$, and $0 < \alpha < \alpha^*$. That theorem lists three conditions under which

$$\int g_1^{(-\alpha)} d\nu^{(\alpha)} \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\, du$$

(those conditions are labelled (a), (b), (c) in Dufresne *et al.* [3], but we refer to them as (i), (ii), (iii) in what follows.) The μ_X in that paper is our ν, and so $d\mu_X^{(\alpha)}(x) = d\nu^{(\alpha)}(x) = e^{\alpha x} d\nu(x)$. The first condition is that (i) the total mass $|\nu^{(\alpha)}|$ be finite; this is true because of the assumption $\mathbb{E}e^{\alpha^* U_T} < \infty$. The second condition is (ii) that

$$\int_{-\infty}^{\infty} |g_1^{(-\alpha)}(x)|\, dx \;=\; \int_{-\infty}^{\infty} e^{-\alpha x}|g_1(x)|\, dx \;<\; \infty;$$

this is true because $\alpha > 0$ and $0 \leq g_1(x) \leq e^{-rT}K$. The third condition (iii) is in two parts. The first part is that the function

$$G_1(y) \;=\; \int_{\mathbb{R}} g_1^{(-\alpha)}(x-y)\,d\nu^{(\alpha)}(x)$$

be continuous at $y = 0$; this is correct by dominated convergence (see Lemma 1 of Dufresne *et al.* [3]). The second part of condition (iii) is verified because g_1 has a derivative that exists everywhere except at the origin and is integrable elsewhere (see Lemma 2 of Dufresne *et al.* [3]). The last step is to identify $\widehat{g_1^{(-\alpha)}}(-u)$, from the definition of $g_1(x)$ above. Proposition 2.1 with $\sigma = 1$ and $r = 0$ yields:

$$\int_0^\infty e^{-zx}\mathbb{E}(k - S_0 e^{-\frac{x}{2}+\sqrt{x}W_1})_+\,dx \;=\; \begin{cases} \frac{S_0\overline{k}^{(1+\sqrt{1+8z})/2}}{z\sqrt{1+8z}} & \text{if } k < S_0 \\ \frac{S_0}{z}\left(\overline{k} - 1 + \frac{\overline{k}^{(1-\sqrt{1+8z})/2}}{\sqrt{1+8z}}\right) & \text{if } k \geq S_0 \end{cases}$$

for $k, \Re(z) > 0$, if $\overline{k} = k/S_0$. Finally, replace k with Ke^{-rT} to obtain the formula for $\widehat{g_1^{(-\alpha)}}(-u)$. Observe that the "$PV$" is not needed in front of the integral when $Ke^{-rT} \leq S_0$ because

$$\int_{-\infty}^{\infty}\left|\widehat{g_1^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\right|du \;<\; \infty.$$

(b) If

$$g_2(x) \;=\; \mathbb{E}(S_0 e^{-\frac{x}{2}+\sqrt{x}W_1} - Ke^{-rT})_+\,\mathbb{1}_{\{x\geq 0\}}$$

then

$$e^{-rT}\mathbb{E}(S_T - K)_+ \;=\; \int g_2 d\nu \;=\; \int g_2^{(-\alpha)} d\nu^{(\alpha)}$$

and conditions (i), (ii), (iii) are verified. Since

$$g_2(x) \;=\; g_1(x) + (S_0 - Ke^{-rT})\mathbb{1}_{\{x\geq 0\}},$$

we have

$$\widehat{g_2^{(-\alpha)}}(-u) \;=\; \int_0^\infty e^{-(\alpha+iu)}g_1(x)\,dx + \frac{1}{\alpha+iu}(S_0 - Ke^{-rT}) \;=\; \widehat{g_1^{(-\alpha)}}(-u) + \frac{S_0(1-\overline{k})}{\alpha+iu}.$$

□

2.1 Numerical analysis

The integrals in (2.1) and (2.2) have an infinite range, and it is useful to know how to restrict the range while ensuring an upper bound for the truncation error (ϵ in what follows). Observe that the function $\widehat{\nu^{(\alpha)}}(u)$ is uniformly bounded by $\mathbb{E}e^{\alpha U_T}$, which is finite by assumption. As to the functions $\widehat{g_1^{(-\alpha)}}(-u)$ and $\widehat{g_2^{(-\alpha)}}(-u)$, one notices that the first one is $\mathcal{O}(|u|^{-\frac{3}{2}})$ when $Ke^{-rT} < S_0$, while it is only $\mathcal{O}(|u|^{-1})$ when $Ke^{-rT} > S_0$; the reverse holds for $\widehat{g_2^{(-\alpha)}}(-u)$. Because of put-call parity, one may use either formula for a put or a call, and one would naturally choose to use the formula that tends to 0 more quickly.

Let us consider the formula for the put price in part (a) of the theorem, when $Ke^{-rT} < S_0$. The function $\widehat{g_1^{(-\alpha)}}(-u)$ may be rewritten as

$$\frac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} \;=\; \frac{S_0 e^{-\frac{\ell}{2}-\frac{\ell}{2}\sqrt{1+8\alpha+iu}}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}},$$

where $\ell = -\log\overline{k} > 0$. It is obvious that

$$\left|\frac{S_0 e^{-\frac{\ell}{2}}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}}\right| \;\le\; \frac{S_0 e^{-\frac{\ell}{2}}}{2\sqrt{2}|u|^{\frac{3}{2}}}.$$

Next, if $u > 0$ then $\theta = \arg(1+8\alpha+8iu) \in (0, \frac{\pi}{2})$, and so

$$\sqrt{1+8\alpha+8iu} \;=\; [(1+8\alpha)^2 + 64u^2]^{\frac{1}{4}} e^{\frac{i\theta}{2}},$$

with $\frac{\theta}{2} \in (0, \frac{\pi}{4})$. Then $\Re(e^{\frac{i\theta}{2}}) \in (\frac{1}{\sqrt{2}}, 1)$, implying that

$$\Re(\sqrt{1+8\alpha+8iu}) \;\ge\; [(1+8\alpha)^2+64u^2]^{\frac{1}{4}}\frac{1}{\sqrt{2}} \;\ge\; 2\sqrt{|u|}.$$

We then have

$$\left|\frac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}}\right| \;\le\; \frac{S_0 e^{-\frac{\ell}{2}-\ell\sqrt{u}}}{2\sqrt{2}|u|^{\frac{3}{2}}},$$

and are ready to find $M > 0$ such that

$$\frac{1}{2\pi}\left|\int_M^\infty \widehat{g_1^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\,du\right| \;\le\; \frac{\epsilon}{2}.$$

From the above it is sufficient that

$$\int_M^\infty \frac{e^{-\ell\sqrt{u}}}{u^{\frac{3}{2}}}\,du \;\le\; \frac{2\sqrt{2}\epsilon\pi e^{\frac{\ell}{2}}}{S_0\mathbb{E}e^{\alpha U_T}} \;=\; \delta.$$

Now

$$\int_M^\infty \frac{e^{-\ell\sqrt{u}}}{u^{\frac{3}{2}}}\,du \;=\; 2\ell\int_{\ell\sqrt{M}}^\infty \frac{e^{-y}}{y^2}\,dy \;\le\; \frac{2}{\ell M}\int_{\ell\sqrt{M}}^\infty e^{-y}\,dy \;=\; \frac{2e^{-\ell\sqrt{M}}}{\ell M}.$$

The last expression is a decreasing function of M, and thus we look for M such that

$$\frac{2e^{-\ell\sqrt{M}}}{\ell M} \;=\; \delta, \qquad \text{or} \qquad \ell M e^{\ell\sqrt{M}} \;=\; \frac{2}{\delta}.$$

Letting $x = \ell\sqrt{M}/2$, this is the same as

$$xe^x \;=\; \sqrt{\frac{\ell}{2\delta}}.$$

The solution of this equation is an instance of the function $\mathcal{W}(z)$, defined implicitly by

$$\mathcal{W}(z)e^{\mathcal{W}(z)} \;=\; z,$$

and called Lambert's function (or product logarithm). Hence, we find that if

$$M \;\geq\; M_\epsilon \;=\; \left(\frac{2}{\ell}\right)^2 \mathcal{W}\left(\sqrt{\frac{\ell}{2\delta}}\right)^2,$$

then

$$\left| e^{-rT}\mathbb{E}(K-S_T)_+ - \frac{1}{2\pi}\int_{-M}^{M} \widehat{g_1^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\,du \right| \;\leq\; \epsilon$$

(ignoring the error made in the numerical computation of $\int_{-M}^{M}$). In the case of the call price formula (part (b) of the theorem), a similar calculation tells us that

$$\left| e^{-rT}\mathbb{E}(S_T-K)_+ - \frac{1}{2\pi}PV\int_{-M}^{M} \widehat{g_2^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\,du \right| \;\leq\; \epsilon$$

if $Ke^{-rT} > S_0$ and

$$M \;\geq\; M'_\epsilon \;=\; \left(\frac{2}{\ell'}\right)^2 \mathcal{W}\left(\sqrt{\frac{\ell'}{2\delta'}}\right)^2, \qquad \ell' \;=\; \log\frac{Ke^{-rT}}{S_0}, \qquad \delta' \;=\; \frac{2\sqrt{2}\epsilon\pi e^{-\frac{\ell'}{2}}}{S_0\mathbb{E}e^{\alpha U_T}}.$$

For instance, if $S_0 = 100, r = .05, T = 1, K = 90, \epsilon = .01$, $\mathbb{E}e^{\alpha U_T} = 2$, then $M_\epsilon = 647$. For $K = 100$, $M_\epsilon = 4382$. The formulas for M_ϵ or M'_ϵ yield smaller values when the strike K is further away from $e^{rT}S_0$. When $K = e^{rT}S_0$ the exponential disappears from $\widehat{g_1^{(-\alpha)}}(-u)$ and $\widehat{g_2^{(-\alpha)}}(-u)$, and the above formulas do not apply; in that case, an upper bound for the norm of

$$\int_M^\infty \frac{S_0\widehat{\nu^{(\alpha)}}(u)}{(\alpha+iu)\sqrt{1+8\alpha+8iu}}\,du$$

is

$$\frac{S_0\mathbb{E}e^{\alpha U_T}}{\sqrt{2M}},$$

and thus one may choose

$$M \;\geq\; M''_\epsilon \;=\; \frac{1}{2}\left(\frac{S_0\mathbb{E}e^{\alpha U_T}}{\epsilon}\right)^2.$$

If the Lambert function is not available, one may use the approximation (Corless *et al.* [1], Eq.(4.19))

$$\mathcal{W}(z) \;\approx\; \log z - \log\log z + \frac{\log\log z}{\log z}.$$

It is good for z as small as 10, and improves as z increases.

3 The Markov Regime Switching Volatility Model

We now derive some of the properties of the Markov chain model for stochastic volatility, in particular the Laplace transform of U_T (defined below). The latter will be used in Section 4, in order to apply the method described in Section 2. We begin with the general N-state model, and then turn to the particular cases $N = 2$ and 3, where more explicit results are available.

We assume that under the risk-neutral measure the volatility V_t takes one of the values $v_1, \dots, v_N$, and that transition from volatility v_j to volatility v_k occurs with intensity λ_{jk}. The integrated squared volatility over $[0, T]$ is

$$U_T = \int_0^T V_s^2 \, ds.$$

Define

$$\begin{aligned}
\lambda_j &= \sum_{k \neq j} \lambda_{jk} \\
L_j(r, T) &= \mathbb{E}[\exp\{-rU_T\} | V_0 = v_j^2] \\
\vec{L}(r, T) &= (L_1(r, T), \dots, L_N(r, T))' \\
\Lambda(r) &= \begin{pmatrix} -\lambda_1 - rv_1^2 & \lambda_{12} & \dots & \lambda_{1N} \\ \lambda_{21} & -\lambda_2 - rv_2^2 & \dots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \dots & -\lambda_N - rv_N^2 \end{pmatrix}
\end{aligned}$$

$$\Lambda = \Lambda(0), \quad D = \operatorname{diag}(v_1^2, \dots, v_N^2), \quad \vec{1} = (1, \dots, 1)', \quad \vec{v} = D\vec{1} = (v_1^2, \dots, v_N^2)'.$$

3.1 Laplace transform and moments of U_T

Proposition 3.1. *For $\Re(r) > 0$,*

$$\vec{L}(r, T) = e^{\Lambda(r)T} \vec{1},$$

and, for $\Re(s)$ large enough,

$$\int_0^\infty e^{-sT} \vec{L}(r, T) \, dT = (sI - \Lambda(r))^{-1} \vec{1}.$$

Proof. By the Markov property,

$$\begin{aligned}
L_j(r, T) &= e^{-rTv_j^2} e^{-\lambda_j T} + \sum_{k \neq j} \lambda_{jk} \int_0^T e^{-\lambda_j t - rtv_j^2} L_k(r, T - t) \, dt \\
&= e^{-(\lambda_j + rv_j^2)T} \Big(1 + \sum_{k \neq j} \lambda_{jk} \int_0^T e^{(\lambda_j + rv_j^2)x} L_k(r, x) \, dx\Big).
\end{aligned}$$

Then

$$\frac{\partial}{\partial T} L_j(r, T) = -(\lambda_j + rv_j^2) L_j(r, T) + \sum_{k \neq j} \lambda_{jk} L_k(r, T), \tag{3.1}$$

or, in matrix form,

$$\frac{\partial}{\partial T}\vec{L}(r,T) = \Lambda(r)\vec{L}(r,T).$$

This implies $\vec{L}(r,T) = e^{\Lambda(r)T}\vec{1}$. The Laplace transform in T of $\vec{L}(r,T)$ follows from well-known properties of matrices. For a matrix M with norm $|||M||| < s$,

$$\int_0^\infty e^{-st}e^{Mt}\,dt \;=\; \sum_{n=0}^\infty M^n \int_0^\infty e^{-st}\frac{t^n}{n!}\,dt \;=\; \sum_{n=0}^\infty s^{-n-1}M^n$$

by dominated convergence. The last expression must be $(sI - M)^{-1}$, because

$$(sI - M)\sum_{n=0}^{p} s^{-n-1}M^n \;=\; I - s^{-p-1}M^{p+1} \;\to\; I$$

as $p \to \infty$. □

Define $m_{nj}(t) = \mathbb{E}(U_t^n \,|\, V_0 = v_j)$ and $m_n(t) = (m_{n1}(t), \ldots, m_{nN}(t))'$.

Proposition 3.2. *Let $m_{0j}(t) \equiv 1$. Then, for $n = 1, 2, \ldots$, $T \geq 0$,*

$$\begin{aligned}
\frac{\partial}{\partial T}m_n(T) &= \Lambda m_n(T) + nDm_{n-1}(T) \\
m_n(T) &= ne^{\Lambda T}\int_0^T e^{-\Lambda t}Dm_{n-1}(t)\,dt \\
m_1(T) &= \int_0^T e^{\Lambda u}\,du\,\vec{v}.
\end{aligned}$$

Proof. From (3.1),

$$\frac{\partial}{\partial T}\frac{\partial^n}{\partial r^n}L_j(r,T) = -(\lambda_j + rv_j^2)\frac{\partial^n}{\partial r^n}L_j(r,T) - nv_j^2\frac{\partial^{n-1}}{\partial r^{n-1}}L_j(r,T) + \sum_{k\neq j}\lambda_{jk}\frac{\partial^n}{\partial r^n}L_k(r,T)$$

(this is because

$$\frac{\partial^n}{\partial r^n}\left[(a+br)f(r)\right] = (a+br)f^{(n)}(r) + bnf^{(n-1)}(r)).$$

Next, multiply by $(-1)^n$ and set $r = 0$ to get:

$$\frac{\partial}{\partial T}m_{nj}(T) \;=\; -\lambda_j m_{nj}(T) + nv_j^2 m_{n-1,j}(T) + \sum_{k\neq j}\lambda_{jk}m_{nk}(T).$$

In vector form, this is the first equation in the statement of the proposition, which implies

$$e^{-\Lambda T}\left[\frac{\partial}{\partial T}m_n(T) - \Lambda m_n(T)\right] \;=\; \frac{\partial}{\partial T}\left[e^{-\Lambda T}m_n(T)\right] \;=\; ne^{-\Lambda T}Dm_{n-1}(T).$$

The other two equations then follow. □

3.2 The case $N = 2$

Use Proposition 3.1:

$$\begin{aligned}
(sI - \Lambda(r))^{-1} &= \begin{pmatrix} s + rv_1^2 + \lambda_{12} & -\lambda_{12} \\ -\lambda_{21} & s + rv_2^2 + \lambda_{21} \end{pmatrix}^{-1} \\
&= \frac{1}{(s + rv_1^2 + \lambda_{12})(s + rv_2^2 + \lambda_{21}) - \lambda_{12}\lambda_{21}} \begin{pmatrix} s + rv_2^2 + \lambda_{21} & \lambda_{12} \\ \lambda_{21} & s + rv_1^2 + \lambda_{12} \end{pmatrix}.
\end{aligned}$$

Hence,

$$\int_0^\infty e^{-sT} L_1(r,T)\, dT \;=\; \frac{s + rv_2^2 + \lambda_{12} + \lambda_{21}}{(s + rv_1^2 + \lambda_{12})(s + rv_2^2 + \lambda_{21}) - \lambda_{12}\lambda_{21}}$$

(same expression for $L_2(r,T)$, just reverse the roles of subscripts "1" and "2"). The inversion may be achieved by first observing that if one defines J_T as the amount of time the chain spends in state 1 during the period $[0,T]$, then, on the one hand,

$$U_T \;=\; v_1^2 J_T + v_2^2 (T - J_T)$$

and, on the other hand, the double Laplace transform of J_T is the one above with (v_1^2, v_2^2) replaced with (1,0). Hence, one needs to invert

$$\begin{aligned}
\frac{s + \lambda_{12} + \lambda_{21}}{(r + s + \lambda_{12})(s + \lambda_{21}) - \lambda_{12}\lambda_{21}} &= \frac{s + \lambda_{12} + \lambda_{21}}{r(s + \lambda_{21}) + s(s + \lambda_{21} + \lambda_{12})} \\
&= \frac{s + \lambda_{12} + \lambda_{21}}{s + \lambda_{21}} \, \frac{1}{r + s(1 + \frac{\lambda_{12}}{s + \lambda_{21}})}
\end{aligned}$$

in r and s to get the law of J_T. We will make use of the generalized hypergeometric function

$${}_0F_1(c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(c)_n}, \quad z \in \mathbb{C}, \quad -c \notin \mathbb{N}.$$

Here the shifted factorials $(c)_0 = 1$, $(c)_n = c(c+1)\cdots c(c+n-1)$, are used. See Lebedev [7] p.238, for more details.

First, invert with respect to r; the last expression is equal to:

$$\begin{aligned}
&\int_0^\infty e^{-rx} \frac{s + \lambda_{12} + \lambda_{21}}{s + \lambda_{21}} \exp\Big\{ - \frac{s(s + \lambda_{12} + \lambda_{21})}{s + \lambda_{21}} x \Big\}\, dx \\
&\qquad = \int_0^\infty e^{-rx} \Big(1 + \frac{\lambda_{12}}{s + \lambda_{21}}\Big) \exp\Big\{ -(s + \lambda_{12})x + \frac{\lambda_{12}\lambda_{21}}{s + \lambda_{21}} x \Big\}\, dx \\
&\qquad = \int_0^\infty e^{-rx} \Big(1 + \frac{\lambda_{12}}{s + \lambda_{21}}\Big) e^{-(s+\lambda_{12})x} \sum_{n=0}^{\infty} \frac{(\lambda_{12} x)^n}{n!} \Big(\frac{\lambda_{21}}{s + \lambda_{21}}\Big)^n dx.
\end{aligned}$$

Next, split expression $1 + \frac{\lambda_{12}}{s+\lambda_{21}}$ into two parts, so that the integrand is now $A+B$, where

$$\begin{aligned} A &= e^{-(s+\lambda_{12})x} \sum_{n=0}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \Big(\frac{\lambda_{21}}{s+\lambda_{21}}\Big)^n \\ &= \int_0^{\infty} e^{-sT} e^{-\lambda_{12}x} \Big(\delta_x(T) + \sum_{n=1}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \frac{\lambda_{21}^n (T-x)^{n-1} e^{-\lambda_{21}(T-x)}}{(n-1)!} \mathbb{1}_{\{T>x\}}\Big)\, dT \\ &= \int_0^{\infty} e^{-sT} e^{-\lambda_{12}x} \Big(\delta_x(T) + \lambda_{12}\lambda_{21} x e^{-\lambda_{21}(T-x)} {}_0F_1(2; \lambda_{12}\lambda_{21}x(T-x)) \mathbb{1}_{\{T>x\}}\Big)\, dT, \end{aligned}$$

and

$$\begin{aligned} B &= e^{-(s+\lambda_{12})x} \sum_{n=0}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \Big(\frac{\lambda_{21}}{s+\lambda_{21}}\Big)^n \frac{\lambda_{12}}{s+\lambda_{21}} \\ &= \int_0^{\infty} e^{-sT} e^{-\lambda_{12}x} \sum_{n=0}^{\infty} \frac{(\lambda_{21}x)^n}{n!} \frac{\lambda_{12}^{n+1} (T-x)^n e^{-\lambda_{21}(T-x)}}{n!} \mathbb{1}_{\{T>x\}}\, dT \\ &= \int_0^{\infty} e^{-sT} \lambda_{12} e^{-\lambda_{12}x - \lambda_{21}(T-x)} {}_0F_1(1; \lambda_{12}\lambda_{21}x(T-x))_{\{T>x\}}\, dT. \end{aligned}$$

Finally, we have found that the distribution of J_T is:

$$\mathbb{P}(J_T \in dx) \;=\; e^{-\lambda_{12}T}\mathbf{1}_{\{x=T\}} + \lambda_{12} e^{-\lambda_{12}x-\lambda_{21}(T-x)}[\lambda_{21}x\, {}_0F_1(2;\lambda_{12}\lambda_{21}x(T-x)) + {}_0F_1(1;\lambda_{12}\lambda_{21}x(T-x))]\mathbf{1}_{\{0<x<T\}} dx.$$

This may also be written in terms of Bessel functions (see Di Masi *et al.* [2]). The distribution of U_T may be found from that of J_T.

3.3 The case $N = 3$

When there are three states the algebra is more involved. However, our approach only requires the Laplace transform of the distribution of U_T, not the distribution itself. Apply Propostion 2.1 once again:

$$(sI - \Lambda(r))^{-1} = \begin{pmatrix} s+\lambda_1+rv_1^2 & -\lambda_{12} & -\lambda_{13} \\ -\lambda_{21} & s+\lambda_2+rv_2^2 & -\lambda_{23} \\ -\lambda_{31} & -\lambda_{32} & s+\lambda_3+rv_3^2 \end{pmatrix}^{-1} = \frac{1}{D(r,s)} M,$$

where

$$\begin{aligned} D(r,s) = {} & \Big((s+rv_1^2+\lambda_1)(s+rv_2^2+\lambda_2) - \lambda_{12}\lambda_{21}\Big)(s+rv_3^2+\lambda_3) - \Big(\lambda_{13}(s+rv_2^2+\lambda_2) + \lambda_{12}\lambda_{23}\Big)\lambda_{31} \\ & - \Big(\lambda_{13}\lambda_{21} + (s+rv_1^2+\lambda_1)\lambda_{23}\Big)\lambda_{32} \end{aligned}$$

and M is

$$\begin{pmatrix} (s+rv_2^2+\lambda_2)(s+rv_3^2+\lambda_3) - \lambda_{23}\lambda_{32} & \lambda_{12}(s+rv_3^2+\lambda_3) + \lambda_{13}\lambda_{32} & \lambda_{13}(s+rv_2^2+\lambda_2) + \lambda_{12}\lambda_{23} \\ \lambda_{21}(s+rv_3^2+\lambda_3) + \lambda_{23}\lambda_{31} & (s+rv_1^2+\lambda_1)(s+rv_3^2+\lambda_3) - \lambda_{13}\lambda_{31} & \lambda_{23}(s+rv_1^2+\lambda_1) + \lambda_{13}\lambda_{21} \\ \lambda_{31}(s+rv_2^2+\lambda_2) + \lambda_{21}\lambda_{32} & \lambda_{32}(s+rv_1^2+\lambda_1) + \lambda_{12}\lambda_{31} & (s+rv_1^2+\lambda_1)(s+rv_2^2+\lambda_2) - \lambda_{12}\lambda_{21} \end{pmatrix}.$$

Multiplying $(sI-\Lambda(r))^{-1}$ by $\vec{1}=(1,1,1)'$, we get

$$\int_0^\infty e^{-sT}L_1(r,T)\,dT = \frac{s^2+\Big(r(v_2^2+v_3^2)+\lambda_{12}+\lambda_{13}+\lambda_2+\lambda_3\Big)s+r^2(v_2^2v_3^2)+r\Big(v_2^2(\lambda_{13}+\lambda_3)+v_3^2(\lambda_{12}+\lambda_2)\Big)+c_1}{D(r,s)}$$

with $c_1=\lambda_{12}(\lambda_{23}+\lambda_3)+\lambda_{13}(\lambda_{32}+\lambda_2)+\lambda_2\lambda_3-\lambda_{23}\lambda_{32}$. The last expression may be inverted with respect to s, yielding a combination of exponentials times polynomials, that can be inserted into the Parseval integral of Section 2. It is obvious that $\mathbb{E}e^{-rU_T}$ is finite for all r.

4 Numerical results

We show European put prices computed using the Parseval formula, when the stochastic volatility is a Markov chain with $N=2$ or 3 states. When $N=2$, these may be compared with the exact price, found using the density of U_T, which is explicit, as shown in Section 3. When $N=3$, comparison is made with Monte Carlo simulation results. Because the Brownian motion and the volatility are independent, it is not required to simulate the Brownian motion, since

$$e^{-rT}\mathbb{E}[(K-S_T)_+]=\mathbb{E}[e^{-rT}\mathbb{E}(K-S_0e^{rT-\frac{U_T}{2}+\sqrt{U_T}Z})_+\,|\,V].$$

The conditional expectation on the right is simply the Black-Scholes formula for a put with maturity T, strike K and volatility $\sigma=\sqrt{(U_T/T)}$. Hence it is sufficient to simulate the Black-Scholes formula for such a put and take the average.

In the case $N=2$, the generator of the Markov chain is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

and there are two different volatility vectors, $(v_1,v_2)=(0.1,0.3)$ and $(0.1,0.9)$ (Tables 1 and 2). There are two different generators in the case $N=3$ (Tables 3 and 4):

$$Q_1=\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},\quad Q_2=\begin{pmatrix} -4 & 3 & 1 \\ 2 & -4 & 2 \\ 1 & 3 & -4 \end{pmatrix}$$

and one volatility vector $(v_1,v_2,v_3)=(0.1,0.2,0.3)$. European put prices are computed for various maturities T and three strikes, one out of money, one at the money and one in the money. In the case of simulation numbers, the 95% confidence intervals are shown in brackets; MC 1 and MC 2 refer to the Monte Carlo simulation using one million runs when the Markov chain starts in state 1 and 2, respectively.

The Fourier integral performs very well compared with Monte Carlo simulation. Execution times are shown at the bottom of each column. (These are the times it took to compute all 18 numbers in each column, on a MacBookPro 2.66Ghz with 4GB RAM; Parseval integrals were computed using Mathematica,

T	Exact 1	Parseval 1	MC 1	Exact 2	Parseval 2	MC 2
K=80						
0.25	0.02059	0.02059	0.02055 (±0.00012)	0.28111	0.28111	0.28115 (±0.00021)
0.50	0.14842	0.14842	0.14822 (±0.00055)	0.85408	0.85408	0.85431 (±0.00076)
0.75	0.36137	0.36137	0.36030 (±0.00104)	1.33859	1.33859	1.33972 (±0.00129)
1.00	0.61403	0.61403	0.61264 (±0.00147)	1.72833	1.72833	1.72976 (±0.00172)
2.00	1.60916	1.60916	1.60917 (±0.00254)	2.75008	2.75008	2.74993 (±0.00267)
3.00	2.34793	2.34793	2.34847 (±0.00294)	3.34532	3.34532	3.34481 (±0.00300)
K=100						
0.25	1.91183	1.91180	1.91136 (±0.00206)	4.99372	4.99134	4.99435 (±0.00161)
0.50	2.90070	2.90092	2.89899 (±0.00341)	6.33464	6.33409	6.33599 (±0.00271)
0.75	3.72364	3.72215	3.72786 (±0.00430)	7.10134	7.10119	7.09788 (±0.00347)
1.00	4.41744	4.41765	4.41763 (±0.00487)	7.60750	7.60747	7.60724 (±0.00399)
2.00	6.25551	6.25551	6.25523 (±0.00567)	8.61588	8.61588	8.61627 (±0.00491)
3.00	7.17574	7.17574	7.17008 (±0.00563)	8.99007	8.99007	8.99512 (±0.00507)
K=120						
0.25	18.59363	18.59363	18.59392 (±0.00043)	19.41942	19.41942	19.41900 (±0.00060)
0.50	17.58961	17.58959	17.58825 (±0.00167)	19.49572	19.49571	19.49697 (±0.00180)
0.75	16.96791	16.96790	16.96665 (±0.00298)	19.48010	19.48010	19.48141 (±0.00287)
1.00	16.60070	16.60068	16.59939 (±0.00409)	19.38516	19.38517	19.38635 (±0.00371)
2.00	16.08430	16.08429	16.08301 (±0.00639)	18.73199	18.73199	18.73264 (±0.00548)
3.00	15.79802	15.79800	15.79693 (±0.00689)	17.99058	17.99059	17.99078 (±0.00603)
Time (sec)	5.502	0.440	13.923	4.893	0.551	13.940

Table 1: Put option prices for $N = 2$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.3$, $\lambda_{12} = 1$, $\lambda_{21} = 1$, $r = 0.05$.

simulations were coded in C). It is of course much faster to compute a single integral numerically than to perform one million simulation runs, and the coding is easier as well. Using the explicit density (case $N = 2$) is not as fast as one would have thought, no doubt because the density is a special function. On the downside it was observed that the parameter α does affect the numerical value of the integral, although in theory it should not (this sort of behaviour often happens with Fourier integrals). By trial and error we found that good results are obtained for $0 < \alpha < 10$. For a long maturity T it seems that a smaller α, say 0.5, does better. For short maturities the integrand is more oscillatory, which leads to less accurate results, as the tables show (a few of the Parseval values are outside the 95% confidence interval). These integrals were computed using the "NIntegrate" command in Mathematica. We did not specify any options in NIntegrate; an expert may be able to improve the accuracy of the numerical integration for this particular type of integrands, but the accuracy of the results in the tables is already quite good.

T	Exact 1	Parseval 1	MC 1	Exact 2	Parseval 2	MC 2
K=80						
0.25	0.78430	0.78430	0.78341 (±0.00359)	6.71019	6.71019	6.71111 (±0.00368)
0.50	2.53096	2.53096	2.52852 (±0.00774)	10.65711	10.65712	10.65951 (±0.00703)
0.75	4.53410	4.53410	4.52420 (±0.01087)	13.22986	13.22986	13.23922 (±0.00938)
1.00	6.53248	6.53248	6.52164 (±0.01312)	15.13610	15.13610	15.14754 (±0.01102)
2.00	13.20313	13.20313	13.20442 (±0.01683)	20.02428	20.02427	20.02546 (±0.01390)
3.00	17.73534	17.73535	17.73808 (±0.01704)	23.02361	23.02361	23.02093 (±0.01432)
K=100						
0.25	3.56511	3.56511	3.56367 (±0.00871)	15.86796	15.86794	15.87049 (±0.00574)
0.50	6.92071	6.92090	6.91402 (±0.01439)	20.57866	20.57873	20.58338 (±0.00960)
0.75	10.13820	10.13671	10.15602 (±0.01806)	23.50175	23.50172	23.48957 (±0.01225)
1.00	13.07611	13.07633	13.07644 (±0.02036)	25.61279	25.61279	25.61180 (±0.01402)
2.00	21.89143	21.89144	21.89083 (±0.02290)	30.83590	30.83590	30.83712 (±0.01694)
3.00	27.28129	27.28129	27.26103 (±0.02182)	33.88203	33.86987	33.88203 (±0.01716)
K=120						
0.25	19.83049	19.83047	19.83444 (±0.00581)	28.91152	28.91152	28.90798 (±0.00523)
0.50	21.26640	21.26640	21.25964 (±0.01219)	33.47179	33.47179	33.47997 (±0.00964)
0.75	23.30443	23.30441	23.29877 (±0.01703)	36.30230	36.30230	36.30913 (±0.01275)
1.00	25.51878	25.51876	25.51311 (±0.02042)	38.32272	38.32272	38.32707 (±0.01492)
2.00	33.33241	33.33243	33.32724 (±0.02531)	43.19682	43.19681	43.19839 (±0.01863)
3.00	38.51275	38.51273	38.50698 (±0.02467)	45.91883	45.91881	45.91940 (±0.01906)
Time (sec)	9.348	0.778	13.885	7.614	2.073	13.835

Table 2: Put option prices for $N = 2$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.9$, $\lambda_{12} = 1$, $\lambda_{21} = 1$, $r = 0.05$.

T	Parseval 1	MC 1	Parseval 2	MC 2	Parseval 3	MC 3
K=80						
0.25	0.01775	0.02032 (±0.00011)	0.04996	0.04693 (±0.00012)	0.24883	0.25006 (±0.00023)
0.50	0.15100	0.15154 (±0.00049)	0.28280	0.28285 (±0.00049)	0.73069	0.73040 (±0.00076)
0.75	0.37060	0.37014 (±0.00090)	0.58655	0.58668 (±0.00087)	1.13127	1.13092 (±0.00119)
1.00	0.62574	0.62582 (±0.00123)	0.88973	0.88978 (±0.00118)	1.45892	1.45812 (±0.00150)
2.00	1.58298	1.58471 (±0.00194)	1.87080	1.87131 (±0.00186)	2.37901	2.37691 (±0.00206)
3.00	2.26221	2.26279 (±0.00216)	2.51625	2.51701 (±0.00209)	2.94683	2.94640 (±0.00221)
K=100						
0.25	2.08875	2.08939 (±0.00206)	3.41536	3.41461 (±0.00129)	4.83678	4.83618 (±0.00167)
0.50	3.22523	3.22592 (±0.00312)	4.53187	4.53239 (±0.00215)	6.02554	6.02513 (±0.00258)
0.75	4.11529	4.11312 (±0.00367)	5.28096	5.27976 (±0.00271)	6.69266	6.69416 (±0.00310)
1.00	4.81814	4.81971 (±0.00395)	5.83955	5.83928 (±0.00309)	7.13966	7.13771 (±0.00340)
2.00	6.48258	6.48377 (±0.00413)	7.13525	7.13585 (±0.00365)	8.07702	8.07624 (±0.00376)
3.00	7.21571	7.21609 (±0.00397)	7.69815	7.70055 (±0.00368)	8.43366	8.43443 (±0.00373)
K=120						
0.25	18.60032	18.59057 (±0.01356)	18.75741	18.75122 (±0.01895)	19.34014	19.33543 (±0.02401)
0.50	17.66806	17.65049 (±0.02018)	18.20442	18.18064 (±0.02441)	19.24750	19.21611 (±0.02877)
0.75	17.14425	17.16052 (±0.02446)	17.90352	17.92049 (±0.02771)	19.09712	19.11786 (±0.03147)
1.00	16.85663	16.83267 (±0.02739)	17.69456	17.67183 (±0.02995)	18.91365	18.89484 (±0.03323)
2.00	16.35320	16.34778 (±0.03336)	17.08704	17.08034 (±0.03476)	18.13733	18.13401 (±0.03688)
3.00	15.90152	15.90113 (±0.03560)	16.48694	16.48735 (±0.03661)	17.36083	17.32379 (±0.03818)
Time (sec)	7.907	18.299	8.759	18.378	11.654	18.301

Table 3: Put option prices with $N = 3$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.2$, $v_3 = 0.3$, $\lambda_{12} = \lambda_{13} = \lambda_{21} = \lambda_{23} = \lambda_{31} = \lambda_{32} = 1$, $r = 0.05$.

T	Parseval 1	MC 1	Parseval 2	MC 2	Parseval 3	MC 3
K=80						
0.25	0.02090	0.02221 (±0.00011)	0.04110	0.03877 (±0.00011)	0.20392	0.20452 (±0.00024)
0.50	0.17284	0.17241 (±0.00044)	0.22590	0.22592 (±0.00044)	0.58535	0.58580 (±0.00069)
0.75	0.41634	0.41694 (±0.00077)	0.47072	0.47125 (±0.00075)	0.92360	0.92242 (±0.00099)
1.00	0.68808	0.68893 (±0.00102)	0.72182	0.72224 (±0.00098)	1.22324	1.22200 (±0.00120)
2.00	1.64943	1.64982 (±0.00149)	1.56147	1.56266 (±0.00143)	2.13770	2.13768 (±0.00156)
3.00	2.30506	2.30372 (±0.00162)	2.11971	2.11813 (±0.00155)	2.72216	2.72260 (±0.00164)
K=100						
0.25	2.37380	2.37279 (±0.00201)	3.21553	3.21532 (±0.00146)	4.59332	4.59418 (±0.00169)
0.50	3.66572	3.66587 (±0.00272)	4.22325	4.22417 (±0.00217)	5.63661	5.63668 (±0.00234)
0.75	4.58457	4.58381 (±0.00298)	4.91641	4.91623 (±0.00254)	6.25834	6.25852 (±0.00263)
1.00	5.26093	5.26099 (±0.00307)	5.43337	5.43287 (±0.00273)	6.70472	6.70432 (±0.00277)
2.00	6.75391	6.75295 (±0.00304)	6.58856	6.59031 (±0.00290)	7.70916	7.71117 (±0.00287)
3.00	7.37681	7.37772 (±0.00290)	7.04584	7.04656 (±0.00283)	8.10787	8.10712 (±0.00280)
K=120						
0.25	18.62048	18.62049 (±0.00039)	18.71949	18.71997 (±0.00039)	19.22279	19.22302 (±0.00067)
0.50	17.80485	17.80504 (±0.00137)	18.03066	18.03099 (±0.00125)	18.94495	18.94408 (±0.00160)
0.75	17.40262	17.40136 (±0.00216)	17.61266	17.61112 (±0.00195)	18.69958	18.70078 (±0.00220)
1.00	17.18238	17.18160 (±0.00268)	17.31686	17.31811 (±0.00244)	18.48229	18.48321 (±0.00259)
2.00	16.66320	16.66574 (±0.00340)	16.47887	16.47864 (±0.00325)	17.73044	17.72743 (±0.00321)
3.00	16.11567	16.11589 (±0.00347)	15.72008	15.72269 (±0.00340)	16.98706	16.98761 (±0.00331)
Time (sec)	6.758	23.861	7.173	23.719	9.040	23.711

Table 4: Put option prices with $N = 3$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.2$, $v_3 = 0.3$, $\lambda_{12} = \lambda_{32} = 3$, $\lambda_{13} = \lambda_{31} = 1$, $\lambda_{21} = \lambda_{23} = 2$, $r = 0.05$.

4.1 Implied volatility curves

Finally, we show some implied volatility curves obtained with Markov chain volatilities. The implied volatility is the constant volatility that makes an option price equal to what it should be in the ordinary Black-Scholes model. The *raison d'être* of stochastic volatility models is the well-known fact that implied volatilities from observed real-world option prices, when plotted against the strike price, are not constant, as they would be if the Black-Scholes model were correct. In Figure 1, we take the Markov chain volatility put prices as observed prices, and back out the implied volatility from the Black-Scholes formula. Monte Carlo prices are used for this exercise.

In all cases the volatility curves are concave. The steepest concavity is obtained for the shortest maturity (dashed line) of .25 year. Other volatility vectors and intensities would produce more or less different curves, but, in all cases we looked at, the Markov volatility model does indeed produce "volatility smiles" with shapes that are not too different from those obtained from real-world option prices.

References

[1] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., and D.E. Knuth (1996). On the Lambert W function. *Adv. Comput. Math. 5*, 329–359.

[2] Di Masi, G.B., Kabanov, Y.M., and W.J. Runggaldier (1994). Mean-variance hedging of options on stocks with Markov volatilities. *Theory of Probability and its Applications 39*, 172–182.

[3] Dufresne, D., Garrido, J., and M. Morales (2009). Fourier inversion formulas in option pricing and insurance. *Methodology and Computing in Applied Probability 11*, 359-383.

[4] Fuh, C.D., Wang, R.H., and J.C. Cheng (2002). Option pricing in a Black-Scholes model with Markov switching. Working paper, Institute of Statistical Science, Taipei.

[5] Harrison, J.M., and S.R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications 11*, 215-260.

[6] Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies 6*(2), 327–343.

[7] Lebedev, N.N. (1965). *Special Functions and their Applications.* Prentice Hall.

[8] Lewis, A.L. (2001). A simple option formula for general jump–diffusion and other exponential Lévy processes. Unpublished. *OptionCity.net publications:* http://optioncity.net/pubs/ExpLevy.pdf.

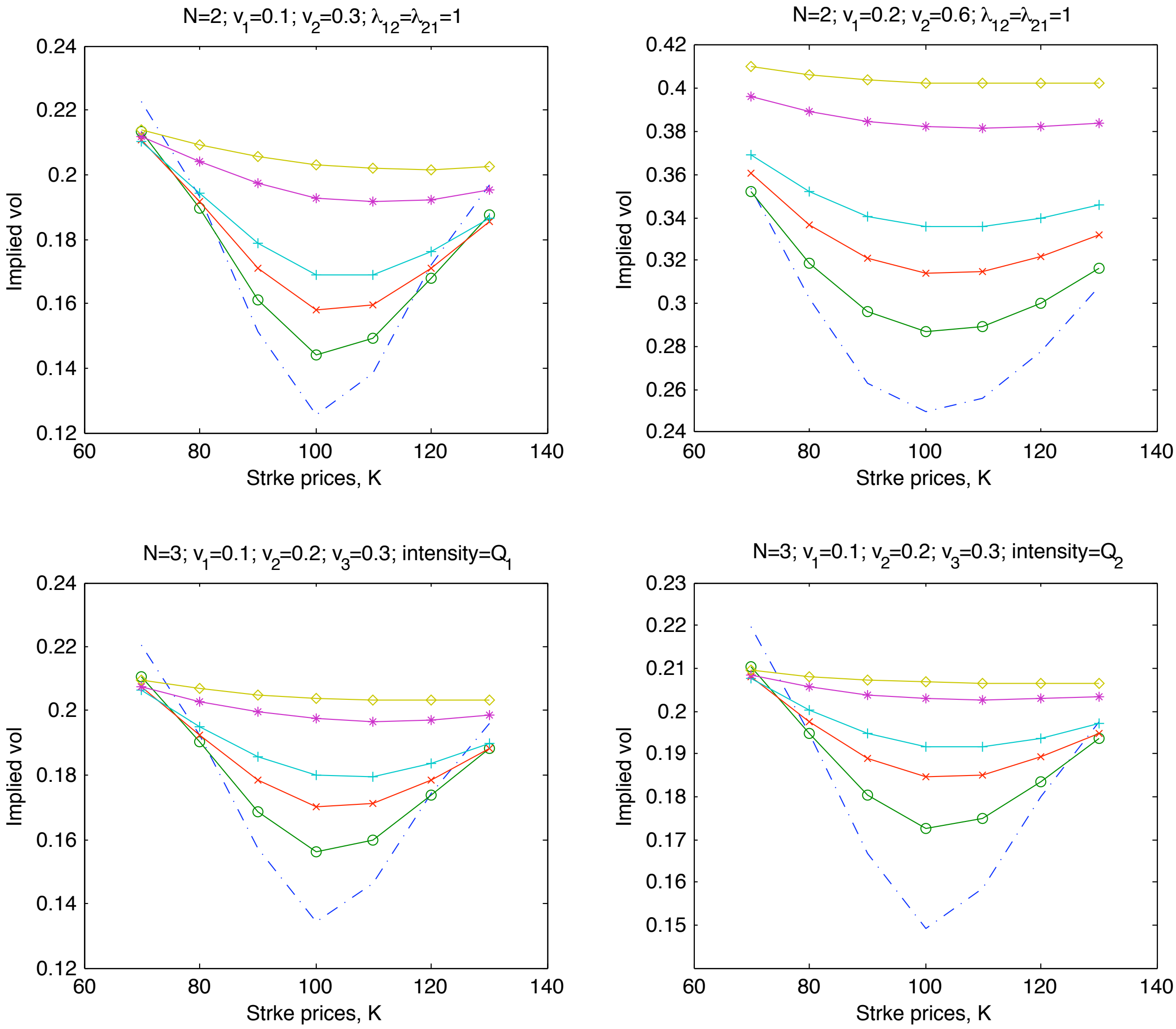

Figure 1: Implied volatility curves for maturities 0.25, 0.50, 0.75, 1, 2 and 3 years. The sharpest smile is for $T = 0.25$ and the shallowest is for $T = 3$. Spot is 100 and initial volatility is v_1.