# Finite time ruin problems for the Markov-modulated risk model

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**Abstract**

We consider finite time ruin problems in the Markov-modulated risk model. We start by considering the number of claims and the aggregate claim amount over a finite time interval, in each case giving both a general approach for an arbitrary number of environment states and a specific approach for the special case when the number of environment states is two. We then consider the density of the time of ruin and the joint density of the time of ruin and the deficit at ruin, before developing a recursive approach to the calculation of the moments of the time of ruin. We conclude with a study of some discrete random variables associated with a Markov-modulated risk model. Numerical illustrations are provided.

**Keywords:** Markov-modulated risk model, claim number distribution, aggregate claims distribution, finite time ruin probability, deficit at ruin, moments of time of ruin, discrete phase-type distribution

# 1 Introduction

Markov-modulated risk models have been studied by a number of authors such as Reinhard (1984), Asmussen (1989), Zhu and Yang (2008) and Li and Lu (2008). The basic idea of such models is that claim arrivals and claim amounts are influenced by an external environment process. For example, in motor vehicle insurance we might consider modelling weather conditions by such a process, with motorists being subject to a lower claim rate in good weather.

The external environment process is denoted by $\{J(t);\, t \geq 0\}$, which is a homogeneous, irreducible and recurrent Markov process, with finite state space $\mathcal{E} = \{1, 2, \ldots, m\}$. We denote the intensity matrix of $\{J(t);\, t \geq 0\}$ by $\mathbf{A} = (\alpha_{i,j})_{i,j=1}^{m}$, with $\alpha_{i,i} := -\alpha_i$ for $i \in \mathcal{E}$, and we denote the stationary distribution of $\{J(t);\, t \geq 0\}$ by $\vec{\boldsymbol{\pi}} = (\pi_1, \pi_2, \ldots, \pi_m)$.

Next, let $N(t)$ denote the number of claims that occur in the time interval $(0, t]$. Then, if $J(s) = i$ for all $s$ in a small time interval $(t, t+h]$, the number of claims occurring in that interval, i.e. $N(t+h)-N(t)$, has a Poisson distribution with parameter $\lambda_i\, h\,(> 0)$. Further, we assume that the process $\{N(t); t \geq 0\}$ has independent increments, given the process $\{J(t);\, t \geq 0\}$. Thus,

$$\Pr\left(N(t+h) = n+1 \middle| N(t) = n, J(s) = i \text{ for } \; t < s \leq t+h\right) = \lambda_i\, h + o(h).$$

We call the process $\{N(t); t \geq 0\}$ a Markov-modulated Poisson process.

We next consider claim amounts and premiums. If the environment process is in state $i$, i.e. if $J(t) = i$, the claim amounts have distribution function $F_i$, density function $f_i$ and finite mean $\mu_i$ ($i \in \mathcal{E}$). Further, we assume that premiums are received continuously at a constant rate $c$. The surplus process $\{U(t); t \geq 0\}$ is then given by

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i\,, \qquad t \geq 0\,, \tag{1.1}$$

where $u \geq 0$ is the initial surplus and $X_i$ is the amount of the $i$-th claim. We assume throughout that the positive loading condition holds, i.e.

$$\sum_{i=1}^{m} \pi_i(c - \lambda_i \mu_i) > 0.$$

Now define $T_u = \inf\{t \geq 0 : U(t) < 0\}$ to be the time of ruin, with $T_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$, and define for $\delta \geq 0$, $u \geq 0$ and $i, j \in \mathcal{E}$

$$\phi_{i,j}(u) = \mathrm{E}\left[e^{-\delta\, T_u} I(T_u < \infty, J(T_u) = j)\ |J(0) = i\right]$$

to be the Laplace transform of the time of ruin with ruin caused by a claim in environment state $j$, given initial surplus $u$ and initial environment state $i \in \mathcal{E}$. Then

$$\phi_i(u) = \sum_{j=1}^{m} \phi_{i,j}(u), \qquad u \geq 0,\, i \in \mathcal{E},$$

is the Laplace transform of the time of ruin, given initial surplus $u$ and initial environment state $i \in \mathcal{E}$. In particular, when $\delta = 0$, $\phi_{i,j}(u)$ is denoted $\psi_{i,j}(u)$ which is defined as

$$\psi_{i,j}(u) = \Pr\left(T_u < \infty, J(T_u) = j \middle| J(0) = i\right), \qquad i\,, j \in \mathcal{E}.$$

So $\psi_{i,j}(u)$ is the ultimate ruin probability with ruin caused by a claim in environment state $j$, given that the initial surplus is $u$ and the initial environment state is $i$. Hence $\psi_i(u) = \sum_{j=1}^{m} \psi_{i,j}(u)$ is the probability of ultimate ruin given that the initial surplus is $u$ and the initial environment state is $i$, and $\chi_i(u) = 1 - \psi_i(u)$ is the non-ruin probability given that the initial surplus is $u$ and the initial environment state is $i$.

Li and Lu (2008) give a detailed study of Gerber-Shiu functions for the Markov-modulated risk model. In this paper, we investigate the density of time of ruin, the joint distribution of the time of ruin and the deficit at ruin, and the moments of the time of ruin. The paper is set out as follows. In Section 2 we consider the distribution of the number of claims in the time interval $(0, t]$, paying particular attention to the special case when there are only 2 environment states. We then turn our attention to aggregate claims distributions in Section 3. As well as considering the special case from Section 2, we illustrate a numerical scheme for calculating these distributions. We then turn our attention to finite time ruin probabilities in Section 4, and to the joint distribution of the time of ruin and the deficit at ruin in Section 5. Moments of the time of ruin are considered in Section 6, and various counting distributions are considered in Section 7.

# 2 The distribution of the number of claims in $(0, t]$

In this section we consider the number of claims in the time interval $(0, t]$. We start with a general approach, then consider the special case when the number of environment states is 2, in which case we can make use of known results for occupation times in a two-state Markov chain.

Define

$$q_{i,j}^{(n)}(t) = \Pr(N(t) = n, J(t) = j | J(0) = i), \quad i, j \in \mathcal{E}, n \in N$$

to be the probability that $n$ claims occur before time $t$, with the initial state being $i$ and the state at time $t$ being $j$. Further, define

$$q_i^{(n)}(t) = \Pr(N(t) = n | J(0) = i) = \sum_{j=1}^{m} q_{i,j}^{(n)}(t), \quad i \in \mathcal{E},\, n \in N$$

to be the probability that $n$ claims occur up to time $t$ with the initial state being $i$.

Using the same arguments as Li and Lu (2008), we obtain the following differential equations for $q_{i,j}^{(n)}(t)$ by conditioning on the possible events in an infinitesimal interval $(0, h]$. For $i \in \mathcal{E}$,

$$q_{i,j}^{(0)\prime}(t) = -\lambda_i q_{i,j}^{(0)}(t) + \sum_{k=1}^{m} \alpha_{i,k} q_{k,j}^{(0)}(t), \tag{2.1}$$

$$q_{i,j}^{(n)\prime}(t) = -\lambda_i q_{i,j}^{(n)}(t) + \lambda_i q_{i,j}^{(n-1)}(t) + \sum_{k=1}^{m} \alpha_{i,k} q_{k,j}^{(n)}(t), \quad i, j \in \mathcal{E},\, n = 1, 2, \ldots. \tag{2.2}$$

Equations (2.1) and (2.2) can be written in matrix form as

$$\mathbf{q}_0'(t) = -(\mathbf{\Lambda} - \mathbf{A})\mathbf{q}_0(t), \tag{2.3}$$

$$\mathbf{q}_n'(t) = -(\mathbf{\Lambda} - \mathbf{A})\mathbf{q}_n(t) + \mathbf{\Lambda}\mathbf{q}_{n-1}(t), \quad n = 1, 2, \ldots, \tag{2.4}$$

where $\mathbf{q}_n(t) = \left(q_{i,j}^{(n)}(t)\right)_{m\times m}$. Solving Equations (2.3) and (2.4) gives

$$\mathbf{q}_0(t) = e^{-(\mathbf{\Lambda}-\mathbf{A})t}, \tag{2.5}$$

$$\mathbf{q}_n(t) = \int_0^t e^{-(\mathbf{\Lambda}-\mathbf{A})(t-x)} \mathbf{\Lambda}\mathbf{q}_{n-1}(x)dx, \quad n = 1, 2, \ldots. \tag{2.6}$$

Equation (2.6) is a recursive formula for $\mathbf{q}_n(t)$ which is not computationally tractable. However, Equation (2.4) can be solved using Laplace transforms. Let $\tilde{\mathbf{q}}_n(s) = \int_0^\infty e^{-st}\mathbf{q}_n(t)dt$ be the Laplace transform of $\mathbf{q}_n(t)$. Taking the Laplace transform of Equation (2.4) and re-arranging, we have for $n = 1, 2, \ldots,$

$$\begin{aligned} \tilde{\mathbf{q}}_n(s) &= [s\mathbf{I} + (\mathbf{\Lambda} - \mathbf{A})]^{-1}\mathbf{\Lambda}\tilde{\mathbf{q}}_{n-1}(s) \\ &= \left\{[s\mathbf{I} + (\mathbf{\Lambda} - \mathbf{A})]^{-1}\mathbf{\Lambda}\right\}^n \tilde{\mathbf{q}}_0(s) \\ &= \left\{[s\mathbf{I} + (\mathbf{\Lambda} - \mathbf{A})]^{-1}\mathbf{\Lambda}\right\}^n [s\mathbf{I} + (\mathbf{\Lambda} - \mathbf{A})]^{-1}. \end{aligned} \tag{2.7}$$

Let $\mathbf{H}(s) = s\mathbf{I} + (\mathbf{\Lambda} - \mathbf{A})$ and let $\mathbf{H}^*(s)$ be the adjoint matrix of $\mathbf{H}(s)$. Then Equation (2.7) can be rewritten as

$$\tilde{\mathbf{q}}_n(s) = \frac{[\mathbf{H}^*(s)\mathbf{\Lambda}]^n\, \mathbf{H}^*(s)}{[\mathrm{Det}\, \mathbf{H}(s)]^{n+1}}. \tag{2.8}$$

We remark that the denominator $[\text{Det}\,\mathbf{H}(s)]^{n+1}$ is a polynomial of degree $m(n+1)$, and the numerator is a matrix with each element being a polynomial of degree $(n+1)(m-1)$. Therefore $\tilde{\mathbf{q}}_n(s)$ is a matrix with each element being a rational function which can be inverted by partial fractions.

We now consider an application of these results. Suppose that there are 2 environment states with intensity matrix

$$\mathbf{A} = \begin{pmatrix} -1/4 & 1/4 \\ 3/4 & -3/4 \end{pmatrix},$$

with Poisson parameters $\lambda_1 = 1$ and $\lambda_2 = 2/3$. Then we obtain

$$\tilde{\mathbf{q}}_0(s) = \begin{pmatrix} \frac{17+12s}{19+32s+12s^2} & \frac{3}{19+32s+12s^2} \\ \\ \frac{9}{19+32s+12s^2} & \frac{15+12s}{19+32s+12s^2} \end{pmatrix},$$

which is easily inverted. For example,

$$q_{1,2}^{(0)}(t) = \frac{3}{4\sqrt{7}} \left( e^{-(4/3-\sqrt{7}/6)t} - e^{-(4/3+\sqrt{7}/6)t} \right). \tag{2.9}$$

It is straightforward but slightly tedious to apply either Equation (2.7) or (2.8) with software to obtain $\tilde{\mathbf{q}}_n(s)$ and hence $\mathbf{q}_n(t)$. Whilst this approach leads to the probability function of the number of claims in $(0, t]$ given $J(0)$, it does not give probabilities associated with how many claims occur in each state in $(0, t]$, which are useful in the context of aggregate claims distributions. We now show how to find such probabilities in the special case when there are 2 environment states.

We now make use of results by Pedler (1971) who studied occupation times for two-state Markov chains. We condition here on the environment process starting in state 1, i.e. on $J(0) = 1$, and the following arguments are easily adapted to conditioning on $J(0) = 2$. Let $S_1(t)$ denote the time spent in state 1 in the interval $(0, t]$ given that $J(0) = 1$. Pedler (1971) gives the density of $S_1(t)$ as

$$h(s,t) = e^{-\alpha_1 s - \alpha_2 (t-s)} \left( \sqrt{\frac{\alpha_1\,\alpha_2\,s}{t-s}}\, I_1\left(\sqrt{4\alpha_1\,\alpha_2\;s\,(t-s)}\right) + \alpha_1\, I_0\left(\sqrt{4\alpha_1\,\alpha_2\;s\,(t-s)}\right)\right),$$

where $I_v(t) = \sum_{n=0}^{\infty} (t/2)^{2n+v}/(n!\,(n+v)!)$ is the modified Bessel function of order $v$ (e.g., Abramovitz and Stegun (1965)), with $\Pr(S_1(t) = t) = e^{-\alpha_1 t}$.

Now let $M_i(t)$ denote the number of claims occurring in state $i$ in $(0, t]$, for $i = 1, 2$. Then

$$
\begin{aligned}
\Pr(M_1(t) = j,\ M_2(t) = k \mid J(0) = 1) \\
= \quad & I(k=0)\, e^{-(\lambda_1+\alpha_1)t} \frac{(\lambda_1 t)^j}{j!} \\
& + \int_0^t h(s,t) e^{-\lambda_1 s} \frac{(\lambda_1 s)^j}{j!}\, e^{-\lambda_2 (t-s)} \frac{(\lambda_2(t-s))^k}{k!}\, ds \qquad (2.10)
\end{aligned}
$$

where $I$ is the indicator function. This follows from the fact that if the time spent in state 1 is $s$, then the distribution of the number of claims in state 1 (in $(0,t]$) is Poisson with parameter $\lambda_1 s$ and the distribution of the number of claims in state 2 is Poisson with parameter $\lambda_2(t-s)$.

Further, Pedler (1971) gives the distribution of the total number of transitions in $(0,t]$, and expresses $h(s,t)$ in terms of this probability function. Assuming that the process starts in state 1, if the number of transitions in $(0,t]$ is either 0 or an even number, then the process is in state 1 at time $t$; if the number of transitions is odd, then the process is in state 2 at time $t$. Straightforward manipulation of Pedler's final formula for $h(s,t)$ gives

$$h(s,t) = h_1(s,t) + h_2(s,t)$$

where

$$h_1(s,t) = \alpha_1\, \alpha_2 s e^{-\alpha_1 s - \alpha_2 (t-s)}\, {}_0F_1(2; \alpha_1\, \alpha_2\, s(t-s))$$

is the density associated with duration $s$ in state 1 in $(0,t]$ with the state at time $t$ being 1, and

$$h_2(s,t) = \alpha_1\, e^{-\alpha_1 s - \alpha_2 (t-s)}\, {}_0F_1(1; \alpha_1\, \alpha_2\, s(t-s))$$

is the density associated with duration $s$ in State 1, with the state at time $t$ being 2, with ${}_0F_1$ being the hypergeometric function (e.g., Abramovitz and Stegun (1965)) given by

$${}_0F_1(a;x) = \sum_{j=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+j)} \frac{x^j}{j!}\,.$$

Then for $j, k = 0, 1, 2, \ldots$, and $i = 1, 2$,

$$
\begin{aligned}
\Pr(M_1(t) = j,\ M_2(t) = k,\ J(t) = i \mid J(0) = 1) \\
= \quad & I(k=0)\, I(i=1)\, e^{-(\lambda_1+\alpha_1)t} \frac{(\lambda_1 t)^j}{j!} \\
& + \int_0^t h_i(s,t) e^{-\lambda_1 s} \frac{(\lambda_1 s)^j}{j!}\, e^{-\lambda_2 (t-s)} \frac{(\lambda_2(t-s))^k}{k!}\, ds\,. \qquad (2.11)
\end{aligned}
$$

Given the form of both $h(s,t)$ and $h_i(s,t)$, $i = 1,2$, it is possible to integrate out in both (2.10) and (2.11). However, the nature of the integrands is such that it is very efficient computationally to use numerical integration to obtain probabilities.

We extend our previous notation and write

$$q_{i,j}^{(m,n)}(t) = \Pr(M_1(t) = m,\ M_2(t) = n,\ J(t) = j \mid J(0) = i)$$

for $m, n = 0, 1, 2, \ldots$ and $i, j = 1, 2$. Table 2.1 shows some values of $q_{1,j}^{(m,n)}(5)$, $j = 1, 2$, as well as values of

$$q_{1,j}^{(n)}(5) = \sum_{k=0}^{n} q_{1,j}^{(k,n-k)}(5)\,.$$

The parameter values are the same as earlier in this section. We remark that each of the values in this table was calculated instantaneously in Mathematica, despite the use of numerical integration. The value of $q_{1,2}^{(0)}(5)$ can also be obtained from formula (2.9).

| $m$ | $n$ | $q_{1,1}^{(m,n)}(5)$ | $q_{1,2}^{(m,n)}(5)$ | $q_{1,1}^{(m)}(5)$ | $q_{1,2}^{(m)}(5)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0.0069 | 0.0032 | 0.0069 | 0.0032 |
| 0 | 1 | 0.0052 | 0.0048 | | |
| 1 | 0 | 0.0267 | 0.0090 | 0.0320 | 0.0138 |
| 0 | 2 | 0.0040 | 0.0047 | | |
| 1 | 1 | 0.0140 | 0.0099 | | |
| 2 | 0 | 0.0563 | 0.0150 | 0.0744 | 0.0296 |
| 0 | 3 | 0.0025 | 0.0035 | | |
| 1 | 2 | 0.0089 | 0.0077 | | |
| 2 | 1 | 0.0217 | 0.0132 | | |
| 3 | 0 | 0.0830 | 0.0184 | 0.1162 | 0.0428 |

Table 2.1: Values of $q_{1,j}^{(m,n)}(5)$ and $q_{1,j}^{(m)}(5)$

# 3 The distribution of aggregate claims over $(0,t]$

We now consider aggregate claims distributions. Define

$$G_{i,j}(x,t) = \Pr(S(t) \le x, J(t) = j | J(0) = i), \quad i, j \in \mathcal{E},\ x, t \ge 0,$$

to be the probability that the aggregate claim amount at time $t$ is at most $x$ and $J(t) = j$, given that $J(0) = i$. Then $G_i(x,t) = \sum_{j=1}^{m} G_{i,j}(x,t)$ is the distribution function of the aggregate claim amount at time $t$ given that $J(0) = i$. From these definitions, it is clear that

$$G_{i,j}(0,t) = q_{i,j}^{(0)}(t), \quad G_i(0,t) = q_i^{(0)}(t), \quad G_{i,j}(x,0) = I(i=j) \text{ and } G_i(x,0) = 1.$$

Next, for $x > 0$ let $g_{i,j}(x,t) = \partial G_{i,j}(x,t)/\partial x$ and $g_i(x,t) = \partial G_i(x,t)/\partial x$, so that the latter is the density of the aggregate claim amount at time $t$ given that $J(0) = i$.

## 3.1 General results

Conditioning on the possible events that can occur in an infinitesimal interval $[t, t+h]$, we have

$$\begin{aligned} G_{i,j}(x,t+h) &= [1-(\lambda_j+\alpha_j)h]G_{i,j}(x,t) + \lambda_j h \int_0^x G_{i,j}(x-y,t) f_j(y)dy \\ &\quad + \sum_{k\neq j} G_{i,k}(x,t)\alpha_{k,j}h + o(h). \end{aligned} \tag{3.1}$$

A system of integro-differential equations can be derived from (3.1) as

$$\frac{\partial \mathbf{G}(x,t)}{\partial t} = \mathbf{G}(x,t)(\mathbf{A}-\mathbf{\Lambda}) + \int_0^x \mathbf{G}(x-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy, \tag{3.2}$$

where $\mathbf{G}(x,t) = \big(G_{i,j}(x,t)\big)_{m\times m}$ and $\mathbf{f}(x) = \mathrm{diag}(f_1(x), f_2(x), \ldots, f_m(x))$. Taking the derivative of Equation (3.2) with respect to $x$ gives

$$\frac{\partial \mathbf{g}(x,t)}{\partial t} = \mathbf{g}(x,t)(\mathbf{A}-\mathbf{\Lambda}) + \int_0^x \mathbf{g}(x-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy + \mathbf{G}(0,t)\mathbf{\Lambda}\mathbf{f}(x), \tag{3.3}$$

where $\mathbf{G}(0,t) = \mathbf{q}_0(t)$ and $\mathbf{g}(x,t) = \big(g_{i,j}(x,t)\big)_{m\times m}$.

We remark that when $m = 1$, the Markov-modulated risk model simplifies to the compound Poisson risk model with $\mathbf{A} = 0$ and $\mathbf{\Lambda} = \lambda$, $\mathbf{G}(x,t)$ simplifies to $G(x,t)$ and Equation (3.2) simplifies to

$$\frac{\partial G(x,t)}{\partial t} = -\lambda G(x,t) + \lambda \int_0^x G(x-y,t)f(y)\,dy.$$

This integro-differential equation has the following solution:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\,(\lambda t)^n}{n!} F^{n*}(x),$$

where $F^{n*}$ is the $n$-fold convolution of $F$ with itself, with $F^{0*}(x)$ defined to be 1 for $x \geq 0$.

Returning to the case $m > 1$, we define $\tilde{\mathbf{G}}(s,t) = \int_0^\infty e^{-sx}\mathbf{G}(x,t)dx$ to be the Laplace transform of $\mathbf{G}(x,t)$ with respect to $x$ and define $\tilde{\mathbf{f}}(s) = \int_0^\infty e^{-sx}\mathbf{f}(x)dx$ to be the Laplace transform of $\mathbf{f}(x)$. Taking the Laplace transforms of Equation (3.2) yields

$$\frac{\partial}{\partial t}\tilde{\mathbf{G}}(s,t) = \tilde{\mathbf{G}}(s,t)\big[(\mathbf{A}-\mathbf{\Lambda}) + \mathbf{\Lambda}\tilde{\mathbf{f}}(s)\big], \tag{3.4}$$

with $\tilde{\mathbf{G}}(s,0) = \frac{1}{s}\mathbf{I}$. Solving Equation (3.4) gives

$$\tilde{\mathbf{G}}(s,t) = \frac{1}{s}e^{[(\mathbf{A}-\mathbf{\Lambda})+\mathbf{\Lambda}\tilde{\mathbf{f}}(s)]t}.$$

## 3.2 The case $m = 2$

In the case when $m = 2$ we can take a more conventional approach based on the number of claims occurring in each state in $(0,t]$ and write

$$G_{i,j}(x,t) = q_i^{(0)}(t) + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} q_{i,j}^{(m,n)}(t)\, F_1^{m*} * F_2^{n*}(x) + \sum_{n=1}^{\infty} q_{i,j}^{(0,n)}(t)\, F_2^{n*}(x)$$

for $x, t \geq 0$ where $F_1^{m*} * F_2^{0*} \equiv F_1^{m*}$, with

$$g_{i,j}(x,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} q_{i,j}^{(m,n)}(t)\, f_1^{m*} * f_2^{n*}(x) + \sum_{n=1}^{\infty} q_{i,j}^{(0,n)}(t)\, f_2^{n*}(x) \tag{3.5}$$

for $x > 0$ with $f_1^{m*} * f_2^{0*} \equiv f_1^{m*}$. Although it is also possible to extend this approach to the case $m > 2$, it is not clear how to implement formulae, whereas in the case $m = 2$ we can make use of results from Section 2 to calculate probabilities $q_{i,j}^{m,n}(t)$. Thus, for certain forms of individual claim amount distribution, e.g. exponential claims in each environment state, it is possible to evaluate convolutions of the form $f_1^{m*} * f_2^{n*}(x)$ to obtain $g_{i,j}(x,t)$.

## 3.3 Numerical approach

We now present a numerical approach to calculating approximate values of $\mathbf{G}(x,t)$ and $\mathbf{g}(x,t)$. From Equations (3.2) and (3.3), we know that for small $h$,

$$\mathbf{G}(x,t+h) = \Big[\mathbf{G}(x,t)(\mathbf{A}-\mathbf{\Lambda}) + \int_0^x \mathbf{G}(x-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy\Big]h + \mathbf{G}(x,t),$$

and

$$\mathbf{g}(x,t+h) = \Big[\mathbf{g}(x,t)(\mathbf{A}-\mathbf{\Lambda})+\int_0^x \mathbf{g}(x-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy+\mathbf{G}(0,t)\mathbf{\Lambda}\mathbf{f}(x)\Big]h+\mathbf{g}(x,t).$$

We then use the trapezoidal rule to calculate the integrals, assuming that the arguments of $\mathbf{G}$ are integer multiples of $h$. First, for $m = 1, 2, 3, \ldots$,

$$\mathbf{G}(0, mh) = \mathbf{G}(0,(m-1)h)(\mathbf{A}-\mathbf{\Lambda})h + \mathbf{G}(0,(m-1)h).$$

Then for $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$,

$$\begin{aligned}\mathbf{G}(nh, mh) \approx\; & \Big[\mathbf{G}(nh,(m-1)h)(\mathbf{A}-\mathbf{\Lambda}) + \frac{h}{2}\mathbf{G}(nh,(m-1)h)\mathbf{\Lambda}\mathbf{f}(0) \\ & + \sum_{k=1}^{n-1} h\mathbf{G}((n-k)h,(m-1)h)\mathbf{\Lambda}\mathbf{f}(kh) \\ & + \frac{h}{2}\mathbf{G}(0,(m-1)h)\mathbf{\Lambda}\mathbf{f}(nh)\Big]h + \mathbf{G}(nh,(m-1)h).\end{aligned}$$

The recursive calculation of $\mathbf{G}(nh, mh)$ requires $\mathbf{G}(nh, 0)$ for $n = 0, 1, 2, \ldots$, whose $(i, j)$-th element is $I(i = j)$. Similarly,

$$\mathbf{g}(0, mh) = \Big[\mathbf{g}(0,(m-1)h)(\mathbf{A}-\mathbf{\Lambda}) + \mathbf{G}(0,mh)\mathbf{\Lambda}\mathbf{f}(0)\Big]h + \mathbf{g}(0,(m-1)h)$$

and

$$\begin{aligned}\mathbf{g}(nh, mh) \approx\; & \Big[\mathbf{g}(nh,(m-1)h)(\mathbf{A}-\mathbf{\Lambda}) + \frac{h}{2}\mathbf{g}(nh,(m-1)h)\mathbf{\Lambda}\mathbf{f}(0) \\ & + \sum_{k=1}^{n-1} h\mathbf{g}((n-k)h,(m-1)h)\mathbf{\Lambda}\mathbf{f}(kh) + \frac{h}{2}\mathbf{g}(0,(m-1)h)\mathbf{\Lambda}\mathbf{f}(nh) \\ & + \mathbf{G}(0,mh)\mathbf{\Lambda}\mathbf{f}(nh)\Big]h + \mathbf{g}(nh,(m-1)h),\end{aligned}$$

with $\mathbf{g}(x, 0) = 0$ if $x > 0$ and $\mathbf{g}(0, 0) = 0$. We perform calculations by first setting $x = h$ and calculating for $t = h, 2h, \ldots$ as far as required, then set $x = 2h$ and repeat the process, and so on.

Table 3.1 shows some exact and approximate values of $g_{1,j}(x, 5)$ for $j = 1, 2$ for the same parameters $\mathbf{A}$ and $\mathbf{\Lambda}$ as in Section 2, with $f_1(x) = e^{-x}$ and $f_2(x) = 4xe^{-2x}$. In this case we can calculate the convolutions in formula (3.5) as

$$f_1^{m*} * f_2^{n*}(x) = \frac{2^{2m}\, x^{2m+n-1}\, e^{-x}}{\Gamma(2m+n)}\, {}_1F_1(2m, 2m+n, -x)$$

where ${}_1F_1$ is the confluent hypergeometric function (e.g. Abramovitz and Stegun (1965)). The exact values have been calculated by truncating the infinite sums in formula (3.5) and the approximate values have been calculated as described above with $h = 0.005$. The densities are plotted in Figure 3.1, and Figure 3.2 shows the corresponding functions when $t = 10$. In each case the higher lines show $g_{1,1}(x,t)$ and the darker lines represent the exact calculation. Both figures suggest that the approximation works well.

| | Exact | | Approximate | |
|---|---|---|---|---|
| $x$ | $g_{1,1}(x,5)$ | $g_{1,2}(x,5)$ | $g_{1,1}(x,5)$ | $g_{1,2}(x,5)$ |
| 0 | 0.0267 | 0.0009 | 0.0266 | 0.0089 |
| 5 | 0.0906 | 0.0295 | 0.0902 | 0.0294 |
| 10 | 0.0203 | 0.0055 | 0.0202 | 0.0054 |
| 15 | 0.0022 | 0.0005 | 0.0022 | 0.0005 |
| 20 | 0.0002 | 0.0000 | 0.0002 | 0.0000 |

Table 3.1: Values of $g_{1,1}(x,5)$ and $g_{1,2}(x,5)$

We note that an advantage of our approximations is that we do not need to be able to calculate convolutions in order to calculate these functions. The disadvantage of the approximations from a computational point of view is that we need to store values in arrays, particularly for our subsequent applications.

# 4 The distribution of the time of ruin

Define

$$\chi_{i,j}(u,t) = \Pr(T_u > t, J(t) = j | J(0) = i), \qquad i,j \in \mathcal{E},$$

to be the probability that ruin does not occur by time $t$ and that the environment state at time $t$ is $j$, given that the initial environment state is $i$. Then $\chi_i(u,t) = \sum_{j=1}^m \chi_{i,j}(u,t) = \Pr(T_u > t | J(0) = i)$ is the non-ruin probability by time $t$ given that the initial environment state is $i$. Let $w_{i,j}(u,t) = -\partial\chi_{i,j}(u,t)/\partial t$ denote the defective density of the time of ruin with ruin caused by a claim in environment state $j$ given that the initial environment state is $i$, and let $w_i(u,t) = \sum_{j=1}^m w_{i,j}(u,t)$ denote the defective density of time of ruin given the initial environment state is $i$.

Reinhard (1984) uses Laplace transform techniques to show that

$$\chi_{i,j}(u,t) = G_{i,j}(u+ct,t) - c\sum_{k=1}^m \int_0^t g_{i,k}(u+cs,s)\chi_{k,j}(0,t-s)ds \qquad (4.1)$$

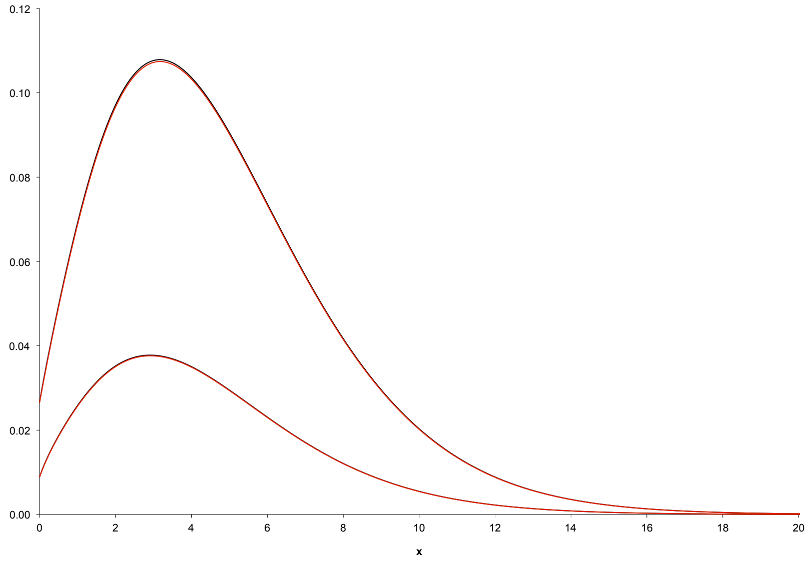


Figure 3.1: $g_{1,1}(x, 5)$ and $g_{1,2}(x, 5)$

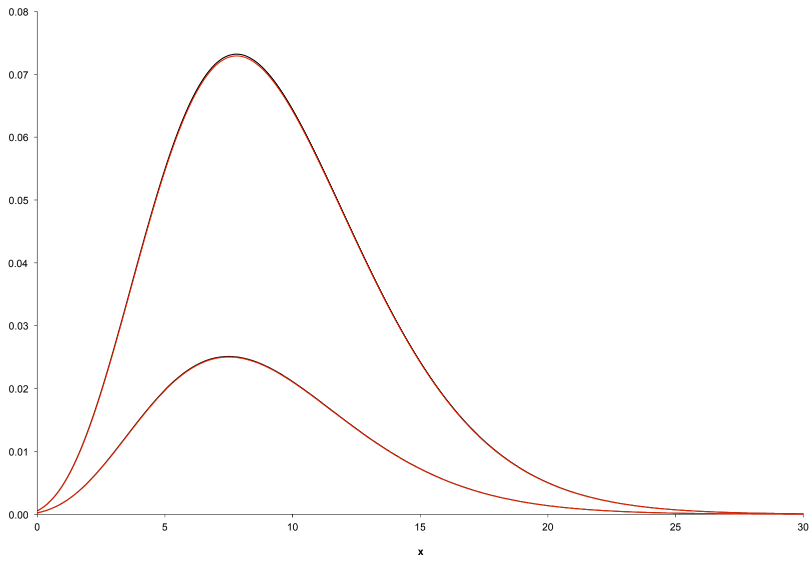


Figure 3.2: $g_{1,1}(x, 10)$ and $g_{1,2}(x, 10)$

from which we obtain

$$\begin{aligned}\chi_i(u,t) &= \Pr[S(t) \leq u+ct | J(0)=i] - c\sum_{j=1}^{m}\int_0^t g_{i,j}(u+cs,s)\chi_j(0,t-s)ds \\ &= G_i(u+ct,t) - c\sum_{j=1}^{m}\int_0^t g_{i,j}(u+cs,s)\chi_j(0,t-s)ds. \qquad (4.2)\end{aligned}$$

Reinhard does not discuss the interpretation of these equations. As we explain later, the interpretation of these formulae is virtually the same as the interpretation of Prabhu's (1961) formula for finite time non-ruin probabilities in the classical risk model.

By differentiating formula (4.1) with respect to $t$, we obtain

$$\begin{aligned}w_{i,j}(u,t) &= -\frac{\partial}{\partial t}\chi_{i,j}(u,t) \\ &= -\frac{\partial}{\partial t}G_{i,j}(u+ct,t) + c\sum_{k=1}^{m} g_{i,k}(u+ct,t)\chi_{k,j}(0,0) \\ &\quad -c\sum_{k=1}^{m}\int_0^t g_{i,k}(u+cs,s)w_{k,j}(0,t-s)ds. \qquad (4.3)\end{aligned}$$

Note that $\chi_{k,j}(0,0) = I(k=j)$ and so the second term in Equation (4.3) above simplifies to $cg_{i,j}(u+ct,t)$. Then Equation (4.3) can be written in matrix form as

$$\mathbf{w}(u,t) = -\frac{\partial}{\partial t}\mathbf{G}(u+ct,t) + c\mathbf{g}(u+ct,t) - c\int_0^t \mathbf{g}(u+cs,s)\mathbf{w}(0,t-s)ds. \qquad (4.4)$$

It follows from Equation (3.2) that

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{G}(u+ct,t) &= c\mathbf{g}(u+ct,t) + \mathbf{G}(u+ct,t)(\mathbf{A}-\mathbf{\Lambda}) \\ &\quad + \int_0^{u+ct}\mathbf{G}(u+ct-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy\,,\end{aligned}$$

and therefore Equation (4.4) simplifies to

$$\begin{aligned}\mathbf{w}(u,t) &= \mathbf{G}(u+ct,t)(\mathbf{\Lambda}-\mathbf{A}) - \int_0^{u+ct}\mathbf{G}(u+ct-y,t)\mathbf{\Lambda}\mathbf{f}(y)dy \\ &\quad -c\int_0^t \mathbf{g}(u+cs,s)\mathbf{w}(0,t-s)ds\end{aligned}$$

$$\begin{aligned} &= -\mathbf{G}(u+ct,t)\mathbf{A} + \mathbf{G}(0,t)\mathbf{\Lambda}[\mathbf{I} - \mathbf{F}(u+ct)] \\ &\quad + \int_0^{u+ct} \mathbf{g}(x,t)\mathbf{\Lambda}[\mathbf{I} - \mathbf{F}(u+ct-x)]dx \\ &\quad -c\int_0^t \mathbf{g}(u+cs,s)\mathbf{w}(0,t-s)ds\,, \end{aligned} \tag{4.5}$$

where $\mathbf{F}(x) = \text{diag}\big(F_1(x), F_2(x), \ldots, F_m(x)\big)$.

Let $\vec{\mathbf{w}}(u,t) = (w_1(u,t), w_2(u,t), \ldots, w_m(u,t))^\top$ and $\vec{\mathbf{1}} = (1,1,\ldots,1)^\top$. Then $\vec{\mathbf{w}}(u,t) = \mathbf{w}(u,t)\vec{\mathbf{1}}$, and it follows from Equation (4.5) and the fact that $\mathbf{A}\vec{\mathbf{1}} = \vec{\mathbf{0}} = (0,0,\ldots,0)^\top$ that

$$\begin{aligned} \vec{\mathbf{w}}(u,t) &= \mathbf{G}(0,t)\mathbf{\Lambda}[\vec{\mathbf{1}} - \vec{\mathbf{F}}(u+ct)] + \int_0^{u+ct} \mathbf{g}(x,t)\mathbf{\Lambda}[\vec{\mathbf{1}} - \vec{\mathbf{F}}(u+ct-x)]dx \\ &\quad -c\int_0^t \mathbf{g}(u+cs,s)\vec{\mathbf{w}}(0,t-s)ds\,, \end{aligned} \tag{4.6}$$

where $\vec{\mathbf{F}}(x) = \big(F_1(x), F_2(x), \ldots, F_m(x)\big)^\top$ and $\mathbf{G}(0,t) = \mathbf{q}_0(t)$.

We remark that when $m = 1$, we have $\mathbf{\Lambda} = \lambda$, $\vec{\mathbf{F}}(x) = F(x)$, $\mathbf{G}(0,t) = G(0,t) = e^{-\lambda t}$, $\vec{\mathbf{w}}(u,t)$ simplifies to $w(u,t)$, the density of the time of ruin in the classical risk model, and so we have

$$\begin{aligned} w(u,t) &= \lambda e^{-\lambda t}[1 - F(u+ct)] + \lambda\int_0^{u+ct} g(x,t)[1 - F(u+ct-x)]dx \\ &\quad -c\int_0^t g(u+cs,s)w(0,t-s)ds, \end{aligned} \tag{4.7}$$

an equation that can be found in Dickson (2007).

Further, it follows from formula (4.6) that

$$\begin{aligned} w_i(u,t) &= \sum_{j=1}^m q_{i,j}^{(0)}(t)\lambda_j[1 - F_j(u+ct)] \\ &\quad + \sum_{j=1}^m \int_0^{u+ct} g_{i,j}(x,t)\lambda_j[1 - F_j(u+ct-x)]dx \\ &\quad -c\sum_{j=1}^m \int_0^t g_{i,j}(u+cs,s)w_j(0,t-s)ds. \end{aligned} \tag{4.8}$$

We now give an interpretation of Equation (4.8) which allows an extension to the joint distribution of the time of ruin and the deficit at ruin in Section 5. We interpret $w_i(u,t)dt$ as the probability of ruin occurring in the interval

$(t, t+dt)$ given that the initial environment state is $i$. For ruin to occur in $(t, t+dt)$, we require that the surplus is positive but not greater that $u+ct$ at time $t$, and a claim must occur in this interval. The first term on the right hand side represents the probability that there are no claims by time $t$ with $J(t) = j \in \mathcal{E}$ so that the surplus at time $t$ is $u + ct$, and there is a claim in environment state $j$ occurring within $(t, t+dt)$, with probability $\lambda_j dt$, of size greater than $u+ct$. The term $g_{i,j}(x,t)dx$ in the second term of Equation (4.8) represents the probability that the aggregate claim amount at time $t$ is between $(x, x+dx)$ and $J(t) = j$, so that the surplus at time $t$ is $u+ct-x$, the term $\lambda_j dt[1 - F_j(u + ct - x)]$ represents the probability that there is a claim in environment state $j$ within $(t, t+dt)$ with probability $\lambda_j dt$ and the claim size is greater than $u+ct-x$ so that ruin occurs within $(t, t+dt)$. The final term adjusts by allowing for the surplus having been below zero before time $t$, where $s < t$ is the last time when the surplus upcrosses through zero in environment state $j \in \mathcal{E}$, and then ruin occurs from current state $j$ and current level 0 at time $s$ within the interval $(t, t+dt)$, with density $w_j(0, t-s)$. Similar interpretations apply to formulae (4.1) and (4.2).

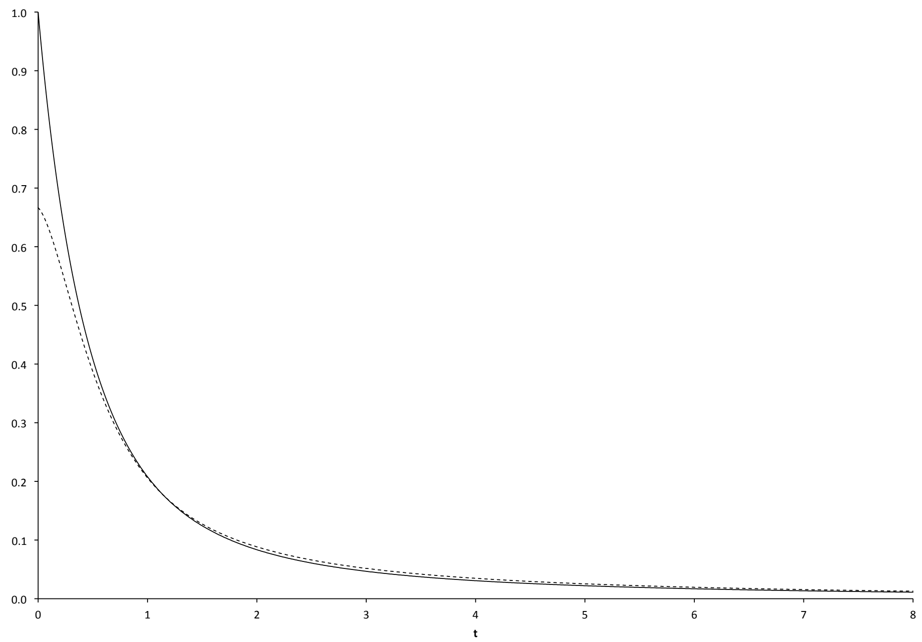


Figure 4.1: $w_1(0,t)$ (solid line) and $w_2(0,t)$

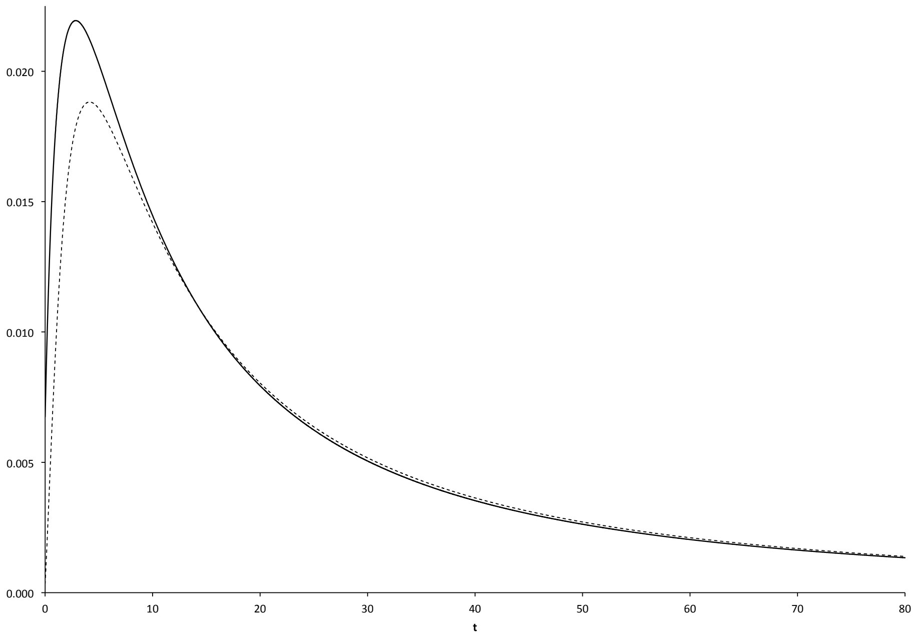


Figure 4.2: $w_1(5, t)$ (solid line) and $w_2(5, t)$

Figures 4.1 and 4.2 show the densities $w_i(u, t)$ for $u = 0$ and $u = 5$ respectively, and $i = 1, 2$, using the same parameters and distributions as in Section 3 with $c = 1$, so that the implied loading factor is 1/12. Figure 4.1 was produced using exact values of the density functions $g_{i,j}$ from formula (3.5). However, use of this formula was not practical to produce Figure 4.2 due to very lengthy computation times, so the numerical approach of Section 3.3 was adopted. These plots are consistent with plots of the density of the time of ruin in the classical risk model (e.g. Dickson (2008)), with the densities being positively skewed.

# 5 The joint distribution of the time of ruin and the deficit at ruin

We now let $w_i(u, t, y)$ denote the joint defective density of the time of ruin, $T_u$, and the deficit at ruin, $|U(T_u)|$, given that the initial environment state is $i \in \mathcal{E}$. The interpretation of Equation (4.8) leads to the following formula for the joint density of the time of ruin and the deficit at ruin:

$$\begin{aligned} w_i(u,t,y) &= \sum_{j=1}^{m} q_{i,j}^{(0)}(t)\lambda_j f_j(u+ct+y) \\ &\quad + \sum_{j=1}^{m} \int_0^{u+ct} g_{i,j}(x,t)\lambda_j f_j(u+ct-x+y)dx \\ &\quad - c\sum_{j=1}^{m} \int_0^{t} g_{i,j}(u+cs,s) w_j(0,t-s,y)ds. \end{aligned} \tag{5.1}$$

Equation (5.1) can be rewritten in matrix form as

$$\begin{aligned} \vec{\mathbf{w}}(u,t,y) &= \mathbf{G}(0,t)\boldsymbol{\Lambda}\vec{\mathbf{f}}(u+ct+y) + \int_0^{u+ct} \mathbf{g}(x,t)\boldsymbol{\Lambda}\vec{\mathbf{f}}(u+ct-x+y)dx \\ &\quad - c\int_0^{t} \mathbf{g}(u+cs,s)\vec{\mathbf{w}}(0,t-s,y)ds\,, \end{aligned} \tag{5.2}$$

where $w_i(u,t,y)$ is the $i$-th entry of the column vector $\vec{\mathbf{w}}(u,t,y)$ and $f_i(x)$ is the $i$-th entry of the column vector $\vec{\mathbf{f}}(x)$.

Now consider the case where $f_i(x)$ can be decomposed as

$$f_i(x+y) = \sum_{k=1}^{r} \eta_{i,k}(x)\tau_k(y), \tag{5.3}$$

where $\{\eta_{i,k}(x)\}_{k=1}^{r}$ are known functions and $\{\tau_k(y)\}_{k=1}^{r}$ are density functions. This factorization was introduced by Willmot (2007) who shows that many well-known density functions have this form. When this factorization applies we find by inverting the Gerber-Shiu function in Li and Lu (2008) (with penalty function $w(x,y) = e^{-sy}$) that

$$w_i(0,t,y) = \sum_{k=1}^{r} h_{i,k}(0,t)\tau_k(y), \tag{5.4}$$

and Equations (5.3) and (5.4) can be expressed in matrix form as

$$\vec{\mathbf{f}}(x+y) = \boldsymbol{\eta}(x)\vec{\boldsymbol{\tau}}(y), \tag{5.5}$$

$$\vec{\mathbf{w}}(0,t,y) = \mathbf{h}(0,t)\vec{\boldsymbol{\tau}}(y), \tag{5.6}$$

where $\boldsymbol{\eta}(x) = \big(\eta_{i,k}(x)\big)_{m\times r}$, $\vec{\boldsymbol{\tau}}(y) = \big(\tau_1(y),\tau_2(y),\ldots,\tau_r(y)\big)^{\top}$, and $\mathbf{h}(0,t) = \big(h_{i,k}(0,t)\big)_{m\times r}$. Substituting Equations (5.5) and (5.6) into Equation (5.2) we obtain

$$\vec{\mathbf{w}}(u,t,y) = \mathbf{h}(u,t)\vec{\boldsymbol{\tau}}(y), \tag{5.7}$$

where the $m \times r$ matrix $\mathbf{h}(u,t)$ satisfies

$$\begin{aligned}\mathbf{h}(u,t) &= \mathbf{G}(0,t)\boldsymbol{\Lambda}\boldsymbol{\eta}(u+ct) + \int_0^{u+ct} \mathbf{g}(x,t)\boldsymbol{\Lambda}\boldsymbol{\eta}(u+ct-x)dx \\ &\quad -c\int_0^t \mathbf{g}(u+cs,s)\mathbf{h}(0,t-s)ds\,. \qquad (5.8)\end{aligned}$$

For example, if $f_i(x) = \beta_i e^{-\beta_i x}$, $i = 1,2,\ldots,m$, then $\vec{\mathbf{f}}(x+y) = \boldsymbol{\eta}(x)\vec{\boldsymbol{\tau}}(y)$, where

$$\begin{aligned}\vec{\boldsymbol{\tau}}(y) &= \big(\beta_1 e^{-\beta_1 y}, \beta_2 e^{-\beta_2 y}, \ldots, \beta_m e^{-\beta_m y}\big)^\top, \\ \boldsymbol{\eta}(x) &= \mathrm{diag}\big(e^{-\beta_1 x}, e^{-\beta_2 x}, \ldots, e^{-\beta_m x}\big).\end{aligned}$$

# 6 The moments of the time of ruin

We now consider moments of the time of ruin. Define

$$\psi_{i,j}^{(n)}(u) = \mathrm{E}\big[T_u^n I(T_u < \infty, J(T_u) = j)\big| J(0) = i\big], \quad n \in N,$$

to be the $n$-th moment of the time to ruin with ruin caused by a claim in environment state $j$, given that the initial surplus is $u$ and the initial environment state is $i$. Further, $\psi_i^{(n)}(u) = \sum_{j\in\mathcal{E}} \psi_{i,j}^{(n)}(u)$ is the $n$-th moment of the time of ruin, given that the initial surplus is $u$ and the initial environment state is $i$. We note that

$$\psi_{i,j}^{(0)}(u) = \psi_{i,j}(u) = 1 - \lim_{t\to\infty} \chi_{i,j}(u,t).$$

Further, let $\boldsymbol{\phi}(u) = \big(\phi_{i,j}(u)\big)_{m\times m}$ and $\boldsymbol{\psi}_n(u) = \big(\psi_{i,j}^{(n)}(u)\big)_{m\times m}$ with $\boldsymbol{\psi}_0(u) = \boldsymbol{\psi}(u) = \big(\psi_{i,j}(u)\big)_{m\times m}$, and let $\tilde{\boldsymbol{\phi}}(s) = \big(\tilde{\phi}_{i,j}(s)\big)_{m\times m}$ and $\tilde{\boldsymbol{\psi}}_n(s) = \big(\tilde{\psi}_{i,j}^{(n)}(s)\big)_{m\times m}$ be their Laplace transforms with respect to $s$.

It follows from Li and Lu (2008) that

$$\begin{aligned}c\,\frac{d}{du}\,\phi_{i,j}(u) &= (\lambda_i+\delta)\,\phi_{i,j}(u) - \lambda_i \int_0^u \phi_{i,j}(u-x)\,f_i(x)dx \\ &\quad -\lambda_i I(i=j)\bar{F}_i(u) - \sum_{k=1}^m \alpha_{i,k}\,\phi_{k,j}(u)\,. \qquad (6.1)\end{aligned}$$

Taking the Laplace transform of Equation (6.1) gives

$$\left[s - \frac{\lambda_i+\delta}{c} + \frac{\lambda_i}{c}\,\tilde{f}_i(s)\right]\tilde{\phi}_{i,j}(s) + \frac{1}{c}\sum_{k=1}^m \alpha_{i,k}\,\tilde{\phi}_{k,j}(s) = \phi_{i,j}(0) - \frac{\lambda_i}{c}\tilde{\bar{F}}_i(s)I(i=j)\,. \qquad (6.2)$$

Further, for simplicity, define $S_i(s) = s - \delta/c - (\lambda_i/c)(1 - \hat{f}_i(s))$, for $i \in \mathcal{E}$. Then Equation (6.2) can be rewritten in matrix form as

$$\mathbf{B}_\delta(s)\tilde{\boldsymbol{\phi}}(s) \;=\; \boldsymbol{\phi}(0) - (\boldsymbol{\Lambda}/c)\tilde{\bar{\boldsymbol{F}}}(s)\,, \tag{6.3}$$

with

$$\mathbf{B}_0(s)\tilde{\boldsymbol{\psi}}(s) \;=\; \boldsymbol{\psi}(0) - (\boldsymbol{\Lambda}/c)\tilde{\bar{\boldsymbol{F}}}(s), \tag{6.4}$$

where

$$\begin{aligned}
\mathbf{B}_\delta(s) &= \mathrm{diag}(S_1(s), S_2(s), \ldots, S_m(s)) + \mathbf{A}/c, \\
\tilde{\bar{\boldsymbol{F}}}(s) &= \mathrm{diag}\big(\tilde{\bar{F}}_1(s), \tilde{\bar{F}}_2(s), \ldots, \tilde{\bar{F}}_i(s)\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\boldsymbol{\psi}_n(u) &= (-1)^n \frac{\partial^n \boldsymbol{\phi}(u)}{\partial \delta^n}\Big|_{\delta=0}, \\
\tilde{\boldsymbol{\psi}}_n(s) &= (-1)^n \frac{\partial^n \tilde{\boldsymbol{\phi}}(s)}{\partial \delta^n}\Big|_{\delta=0}.
\end{aligned}$$

Differentiating Equation (6.2) $n$ times with respect to $\delta$, then setting $\delta = 0$, yields

$$\left[s - \frac{\lambda_i}{c}[1 - \tilde{f}_i(s)]\right] \tilde{\psi}^{(n)}_{i,j}(s) + \frac{n}{c}\,\tilde{\psi}^{(n-1)}_{i,j}(s) + \frac{1}{c}\sum_{j=1}^{m} \alpha_{i,j}\tilde{\psi}^{(n)}_{i,j}(s) = \phi^{(n)}_{i,j}(0)\,. \tag{6.5}$$

Equation (6.5) can be rewritten in matrix form as

$$\tilde{\boldsymbol{\psi}}_n(s) = [\mathbf{B}_0(s)]^{-1}\left[\boldsymbol{\psi}_n(0) - \frac{n}{c}\tilde{\boldsymbol{\psi}}_{n-1}(s)\right], \quad s > 0. \tag{6.6}$$

Now let $\tilde{\mathbf{v}}(s) = \int_0^\infty e^{-su}\mathbf{v}(u)du = [\mathbf{A}_0(s)]^{-1}$. Li and Lu (2008) show that

$$\mathbf{v}(u) = \boldsymbol{\Phi}(u) - \int_0^u \boldsymbol{\Phi}(x)\boldsymbol{\Theta}e^{-\boldsymbol{\Theta}(u-x)}dx, \tag{6.7}$$

with $\lim_{u\to\infty}\mathbf{v}(u) = [\mathbf{I} - \boldsymbol{\psi}(0)]^{-1}$, where $\boldsymbol{\Phi}(u) = [\mathbf{I} - \boldsymbol{\psi}(u)][\mathbf{I} - \boldsymbol{\psi}(0)]^{-1}$ and $\boldsymbol{\Theta} = \mathbf{A}[\mathbf{I} - \boldsymbol{\psi}(0)]^{-1}/c$. The fact that $\sum_{j=1}^m \psi_{i,j}(u) = \psi_i(u) < 1$ shows that $\mathbf{I} - \boldsymbol{\psi}(u)$ is diagonally dominant, and therefore $\mathbf{I} - \boldsymbol{\psi}(u)$ is invertible for $u \geq 0$.

Multiplying both sides of Equation (6.6) and applying the Final Value Theorem, we have

$$\mathbf{0} \;=\; \lim_{u\to\infty}\boldsymbol{\psi}_n(u) = \lim_{s\to 0} s\tilde{\boldsymbol{\psi}}_n(s)$$

$$
\begin{aligned}
&= \lim_{s\to 0} s\tilde{\mathbf{v}}(s) \lim_{s\to 0}\left[\boldsymbol{\psi}_n(0) - \frac{n}{c}\tilde{\boldsymbol{\psi}}_{n-1}(s)\right] \\
&= \lim_{u\to\infty} \mathbf{v}(u)\left[\boldsymbol{\psi}_n(0) - \frac{n}{c}\int_0^\infty \boldsymbol{\psi}_{n-1}(x)dx\right].
\end{aligned}
$$

This gives

$$
\boldsymbol{\psi}_n(0) = \frac{n}{c}\int_0^\infty \boldsymbol{\psi}_{n-1}(x)dx\,, n = 1, 2, \ldots,
$$

and Equation (6.6) simplifies to

$$
\tilde{\boldsymbol{\psi}}_n(s) = \frac{n}{c}\tilde{\boldsymbol{v}}(s)\left[\int_0^\infty \boldsymbol{\psi}_{n-1}(x)dx - \tilde{\boldsymbol{\psi}}_{n-1}(s)\right]. \tag{6.8}
$$

Inverting Equation (6.8) yields, for $n = 1, 2, \ldots,$

$$
\begin{aligned}
\boldsymbol{\psi}_n(u) &= \frac{n}{c}\left[\boldsymbol{\Phi}(u)\int_0^\infty \boldsymbol{\psi}_{n-1}(x)dx - \int_0^u \boldsymbol{\Phi}(u-x)\boldsymbol{\psi}_{n-1}(x)dx\right] \\
&\quad + \frac{n}{c}\int_0^u \boldsymbol{\Phi}(x)\boldsymbol{\Theta}e^{-\boldsymbol{\Theta}(u-x)}\int_{u-x}^\infty \boldsymbol{\psi}_{n-1}(y)dydx\,, \qquad (6.9)
\end{aligned}
$$

with $\boldsymbol{\psi}_0(u) = \boldsymbol{\psi}(u)$ and $\boldsymbol{\Phi}(u) = [\mathbf{I} - \boldsymbol{\psi}(u)][\mathbf{I} - \boldsymbol{\psi}(0)]^{-1}$.

In particular, if $m = 1$, the Markov-modulated risk model reduces to the classical risk model with $\mathbf{A} = 0, \boldsymbol{\Theta} = 0, \boldsymbol{\Phi}(u) = [1 - \psi(u)]/[1 - \psi(0)] = c[1 - \psi(u)]/(c - \lambda\mu)$, $\boldsymbol{\psi}_n(u) = \psi_n(u)$, and Equation (6.9) simplifies to

$$
\begin{aligned}
\psi_n(u) &= \frac{n}{c - \lambda\mu}\Big[\int_0^u \psi(u-x)\psi_{n-1}(x)dx + \int_u^\infty \psi_{n-1}(x)dx \\
&\quad -\psi(u)\int_0^\infty \psi_{n-1}(x)dx\Big], \qquad (6.10)
\end{aligned}
$$

for $n = 1, 2, \ldots,$ with $\psi_0(u) = \psi(u)$.

We now illustrate the above results for the two-state model using the same parameters that were used for Figures 4.1 and 4.2 in Section 4. We do not give full details as the formulae are lengthy, and are most easily implemented using software like Mathematica, but will sketch the steps. We start with $n = 1$ in Equation (6.6), giving

$$
\begin{aligned}
\tilde{\boldsymbol{\psi}}_1(s) &= [\mathbf{B}_0(s)]^{-1}\left[\boldsymbol{\psi}_1(0) - \frac{1}{c}\tilde{\boldsymbol{\psi}}_0(s)\right] \\
&= [\mathbf{B}_0(s)]^{-1}\left[-\frac{d}{d\delta}\boldsymbol{\phi}(0)\Big|_{\delta=0} - \frac{1}{c}\tilde{\boldsymbol{\psi}}_0(s)\right].
\end{aligned}
$$

Li and Lu (2008, page 58) give a formula for $\boldsymbol{\phi}(0)$ in terms of $\rho_i \equiv \rho_i(\delta)$, $i = 1, 2$, which are the positive solutions of $\mathrm{Det}[\mathbf{B}_\delta(s)] = 0$. This allows us to find $(d/d\delta)\,\boldsymbol{\phi}(0)$. We can find $\tilde{\boldsymbol{\psi}}_0(s)$ from formula (2.4) of Li and Lu (2008) for $\boldsymbol{\phi}(s)$ by setting $\delta = 0$. Matrix multiplication then gives $\tilde{\boldsymbol{\psi}}_1(s)$, which can be inverted to yield $\boldsymbol{\psi}_1(u)$.

Once we have $\tilde{\boldsymbol{\psi}}_1(s)$ we can again use Equation (6.6), this time with $n = 2$ so that

$$\tilde{\boldsymbol{\psi}}_2(s) = [\mathbf{B}_0(s)]^{-1}\left[\boldsymbol{\psi}_2(0) - \frac{2}{c}\tilde{\boldsymbol{\psi}}_1(s)\right]$$

and we can find $\boldsymbol{\psi}_2(0)$ by differentiating $\boldsymbol{\phi}(0)$ twice with respect to $\delta$, and then set $\delta = 0$.

It does not seem possible to obtain general explicit solutions from these formulae. However, we can solve explicitly for particular parameter values and individual claim amount distributions. Using the same parameters and distributions as in Section 3 we obtain the following results:

$$\begin{aligned}
\psi_1(u) &= 0.9210\,e^{-0.0851u} - 0.0004\,e^{-0.9390u} - 0.0005\,e^{-2.7992u}\,,\\
\psi_1^{(1)}(u) &= (10.208 + 10.109u)\,e^{-0.0851u} + (0.05653 - 0.00011u)\,e^{-0.9390u}\\
&\quad + (0.06639 + 0.00004u)\,e^{-2.7992u}\,,\\
\psi_1^{(2)}(u) &= 110.97(0.96801 + u)(26.74583 + u)\,e^{-0.0851u}\\
&\quad - 3.07 \times 10^{-5}(-1369.9 + u)(380.99 + u)\,e^{-0.9390u}\\
&\quad - 3.9 \times 10^{-6}(-1140.7 + u)(4123.8 + u)\,e^{-2.7992u}\,.
\end{aligned}$$

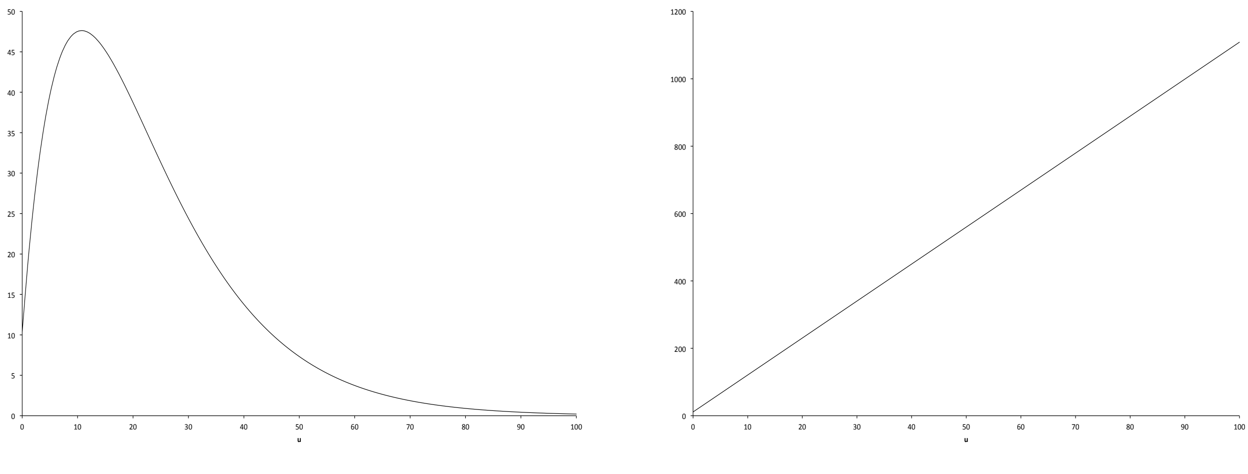


Figure 6.1: Unconditional (left) and conditional first moments of $T_u$.

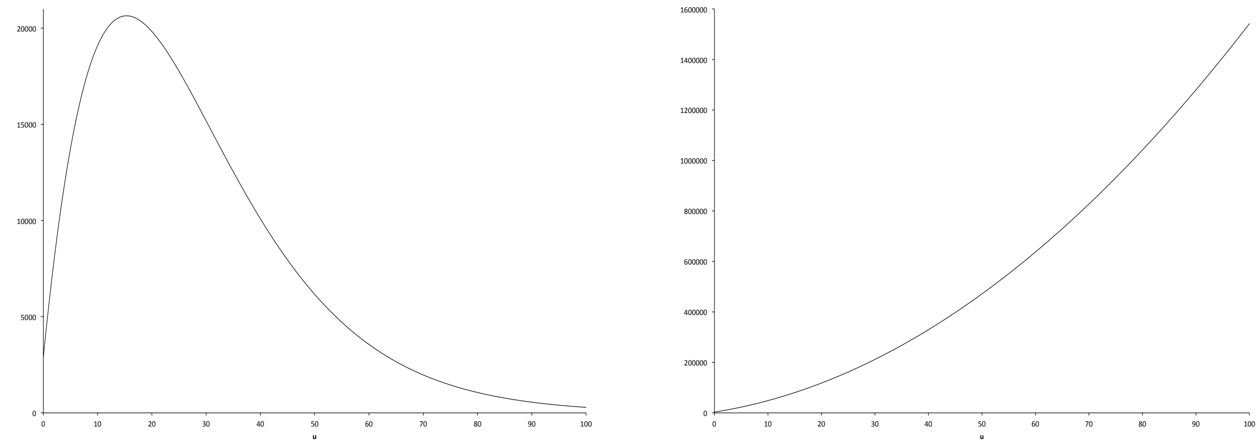


Figure 6.2: Unconditional (left) and conditional second moments of $T_u$.

Figures 6.1 and 6.2 show the first two moments of $T_u\, I(T_u < \infty)$ and $T_u \,|\, T_u < \infty$ given that $J(0) = 1$. These patterns are not particularly surprising. In the case of unconditional moments, we expect these to initially increase with $u$, then decrease and tend to 0 as $u$ increases since $\psi_1(u)$ decreases to 0 as $u$ increases. For the conditional moments the patterns are similar to patterns observed for other models (e.g. Willmot and Lin (2000) for the classical risk model and Dickson and Hipp (2001) for the Erlang(2) risk model). We found that similar patterns apply when $J(0) = 2$.

# 7 Some discrete phase-type distributions arising from the Markov-modulated risk model

In this section, we investigate some distributions relating to the number of events arising in the Markov-modulated risk model, and show that these distributions are discrete phase-type (DPH) with certain representations. First, we give a brief review of discrete phase-type distributions.

## 7.1 Review of discrete phase-type distributions

Consider a Markov chain with transient states $1, 2, \ldots, m$, one absorbing state (denoted as state 0) and an associated transition probability matrix

$$\begin{bmatrix} \mathbf{T} & \vec{\mathbf{t}} \\ \vec{\mathbf{0}}^\top & 1 \end{bmatrix},$$

where $\vec{\mathbf{0}} = (0, 0, \ldots, 0)^\top_{1\times m}$, $\vec{\mathbf{t}} = (t_{10}, t_{20}, \ldots, t_{m0})^\top$, and $\mathbf{T} = (t_{ij})_{m\times m}$. The matrix $\mathbf{T}$ is a sub-stochastic matrix, and contains the transition probabilities among the $m$ transient states, and $\vec{\mathbf{t}} = (\mathbf{I} - \mathbf{T})\vec{\mathbf{1}}$ contains the absorption probabilities into state 0 from the transient states. Given this Markov chain starts with an initial distribution $(\eta_0, \eta_1, \ldots, \eta_m)$ with $\sum_{i=0}^m \eta_i = 1$, we denote by the random variable $N$ the minimum time (number of steps) for the Markov chain to be absorbed in state 0. Then $N$ follows a DPH distribution with representation DPH$(\vec{\boldsymbol{\eta}}^\top, \mathbf{T})$, where $\vec{\boldsymbol{\eta}}^\top = (\eta_1, \ldots, \eta_m)$. The probability function of $N$ is given by

$$\begin{aligned} \Pr(N = 0) &= \eta_0, \\ \Pr(N = n) &= \vec{\boldsymbol{\eta}}^\top \mathbf{T}^{n-1}\vec{\mathbf{t}}, \quad n = 1, 2, \ldots. \end{aligned}$$

DPH distributions are one of the most general classes of distributions permitting a Markovian interpretation. This class of distributions includes probability functions with finite support on the non-negative integers, mixtures and linear combinations of geometric and negative binomial distributions, as well as their zero-truncated and zero-modified versions; see Neuts (1981, 1989), Latouche and Ramaswami (1999), and references therein.

## 7.2 The number of visits to level $b > u$ before ruin

Define

$$T_u^b = \inf\{t \geq 0 : U(t) = b\}, \quad u \leq b,$$

to be the first hitting time of level $b$ from initial surplus $u$. Then

$$\xi_{i,j}(u; b) = \Pr\left(T_u^b < T_u, J(T_u^b) = j \middle| J(0) = i\right), \quad 0 \leq u \leq b,\, i, j \in \mathcal{E},$$

with $\xi_{i,j}(b; b) = I(i = j)$, is the probability that the surplus process hits level $b$ in state $j$ before ruin occurs, given the initial state is $i$. Let $\boldsymbol{\xi}(u; b) = \big(\xi_{i,j}(u; b)\big)_{m\times m}$. Li and Lu (2007, 2008) show that $\boldsymbol{\xi}(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}$, where $\mathbf{v}(u)$ is given in (6.7).

Now define

$$H_{i,j}(u; y) = \Pr\big(T_u < \infty, J(T_u) = j, |U(T_u)| \leq y \big| J(0) = i\big), \quad u, y \geq 0,\, i, j \in \mathcal{E},$$

to be the probability that ruin occurs from a claim in state $j$ and that the deficit at ruin is at most $y$, given that the initial state is $i$. Let $h_{i,j}(u; y) = \partial H_{i,j}(u; y)/\partial y$. Then

$$k_{j,l}(b) = \sum_{k=1}^m \int_0^b h_{j,k}(0; y)\xi_{k,l}(b - y; b)dy, \quad j, l \in \mathcal{E},$$

is the probability that the surplus will visit (i.e. upcross) $b$ for the second time before ruin, and the state will be $l$ at the time of this visit if the surplus visits level $b$ in state $j$ for the first time. In matrix form, we have

$$\mathbf{k}(b) = \int_0^b \mathbf{h}(0; y)\boldsymbol{\xi}(b-y; b)dy,$$

where $\mathbf{k}(b) = \big(k_{j,l}(b)\big)_{m\times m}$ and $\mathbf{h}(0;y) = \big(h_{j,k}(0;y)\big)_{m\times m}$. We remark that $1 - \sum_{l=1}^{m} k_{j,l}(b)$ is the the probability that there are no other visits to $b$ for the surplus process before ruin after the surplus visits level $b$ at state $j$.

It follows from Li and Lu (2008) that

$$\begin{aligned}
\mathbf{k}(b) = & \left[\boldsymbol{\Phi}(b) - \boldsymbol{\Phi}(0) - \int_0^b \left[\boldsymbol{\Phi}(b-x) - \boldsymbol{\Phi}(0)\right]\mathbf{A}e^{-\mathbf{A}x}dx\right] \\
& \times \left[\boldsymbol{\Phi}(b) - \int_0^b \boldsymbol{\Phi}(b-x)\mathbf{A}e^{-\mathbf{A}x}dx\right]^{-1} \\
= & \ \mathbf{I} - e^{-\mathbf{A}b}\left[\boldsymbol{\Phi}(b) - \int_0^b \boldsymbol{\Phi}(b-x)\mathbf{A}e^{-\mathbf{A}x}dx\right]^{-1} \\
= & \ \mathbf{I} - \left[\boldsymbol{\Phi}(b)e^{\mathbf{A}b} - \int_0^b \boldsymbol{\Phi}(x)\mathbf{A}e^{-\mathbf{A}x}dx\right]^{-1}, \qquad (7.1)
\end{aligned}$$

where $\boldsymbol{\Phi}(u) = [\mathbf{I} - \boldsymbol{\psi}(u)][\mathbf{I} - \boldsymbol{\psi}(0)]^{-1}$.

Let $N_u^b$ be the number of visits of level $b$ from the initial level $u < b$ before the time of ruin. Then we have

$$\begin{aligned}
p_i(0; u, b) &= \Pr(N_u^b = 0 | J(0) = i) = 1 - \vec{\mathbf{e}}_i^\top \boldsymbol{\xi}(u; b)\vec{\mathbf{1}}, \\
p_i(n; u, b) &= \Pr(N_u^b = n | J(0) = i) = \vec{\mathbf{e}}_i^\top \boldsymbol{\xi}(u; b)[\mathbf{k}(b)]^{n-1}[\mathbf{I} - \mathbf{k}(b)]\vec{\mathbf{1}}, \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\vec{\mathbf{e}}_i^\top$ is a $1 \times m$ row vector with the $i$-th element being 1 and all other elements being 0.

**Remarks**

- This shows that the number of visits to level $b$ from $u(< b)$ before the time of ruin, given that the initial state is $i$, follows a DPH distribution with representation $\big(\vec{\mathbf{e}}_i^\top \boldsymbol{\xi}(u; b), \mathbf{k}(b)\big)$.
- If $m = 1$ then $\mathbf{A} = 0, \boldsymbol{\xi}(u; b) = \xi(u; b) = [1 - \psi(u)]/[1 - \psi(b)]$, and $\mathbf{k}(b) = k(b) = [\psi(0) - \psi(b)]/[1 - \psi(b)]$, giving

$$\Pr(N_u^b = 0) = 1 - \xi(u; b) = \frac{\psi(u) - \psi(b)}{1 - \psi(b)},$$

$$\begin{aligned}\Pr(N_u^b = n) &= \xi(u;b)[k(b)]^{n-1}[1-k(b)] \\ &= \frac{1-\psi(u)}{1-\psi(b)}\left[\frac{\psi(0)-\psi(b)}{1-\psi(b)}\right]^{n-1}\frac{1-\psi(0)}{1-\psi(b)}, \quad n = 1,2,\ldots.\end{aligned}$$

That is, the number of times the surplus visits level $b > u$ before the time of ruin in the classical risk model follows a zero-modified geometric distribution.

## 7.3 The number of times the record low is broken

For the surplus process $U(t) = u + ct - S(t)$, we let $L(t) = S(t) - ct$ and define

$$L_i = \sup_{t\geq 0}\{L(t)\big| J(0) = i\}$$

to be the maximum aggregate loss given that the initial state is $i$, with $L_i \geq 0$. Then we have

$$\begin{aligned}\Pr(L_i \leq u) &= \Pr\big(L(t) \leq u \,\text{for all}\, t \geq 0\big| J(0) = i\big) \\ &= \Pr(U(t) \geq 0 \,\text{for all}\ t \geq 0\,\big| J(0) = i\big) = 1 - \psi_i(u).\end{aligned}$$

It follows that $L_i$ can be decomposed as

$$L_i = \sum_{k=1}^{N_i} L_{i,k},$$

where $N_i$ represents the number of times the record low for the surplus process has been broken with $J(0) = i$, and $L_{i,k}$ represents the amount by which the record low is decreased for the $k$-th time.

Let $\{Z_n\}_{n=0}^{\infty}$ represent the state of the Markov process $\{J(t); t \geq 0\}$ when the record low for $\{U(t); t \geq 0\}$ is broken for the $n$-th time with $Z_0 = J(0)$. Then $\{Z_n\}_{n=0}^{\infty}$ is a discrete Markov chain with $m$ transition states (states $1, 2, \ldots, m$) and one absorbing state (state 0): $Z_n = 0$ if there is no record low after the $(n-1)$-th record low. The one-step transition probabilities for $\{Z_n\}_{n=0}^{\infty}$ are:

$$\begin{aligned}\Pr(Z_{n+1} = j|Z_n = i) &= \psi_{i,j}(0), \qquad i, j \in \mathcal{E}, \\ \Pr(Z_{n+1} = 0|Z_n = i) &= 1 - \psi_i(0) = 1 - \sum_{j=1}^{m}\psi_{i,j}(0).\end{aligned}$$

The transition probability matrix for $\{Z_n\}_{n=0}^{\infty}$ is

$$\begin{bmatrix} \boldsymbol{\psi}(0) & [\mathbf{I} - \boldsymbol{\psi}(0)]\vec{\mathbf{1}} \\ \vec{\mathbf{0}}^{\top} & 1 \end{bmatrix}.$$

Given $\Pr(Z_0 = i) = \Pr(J(0) = i) = 1$, i.e., the initial probability vector for $Z_0$ is $\vec{\mathbf{e}}_i^\top = (0, \ldots, 1, \ldots, 0)$.

Define $M_i = \min\{n \geq 1 : Z_n = 0 | Z_0 = i\}$ to be the first time that $Z_n$ moves to the absorbing state 0 from the initial state $i \in \mathcal{E}$. Then $M_i$ follows a DPH distribution with representation $(\vec{\mathbf{e}}_i^\top, \boldsymbol{\psi}(0))$, i.e.,

$$\Pr(M_i = n) = \vec{\mathbf{e}}_i^\top [\boldsymbol{\psi}(0)]^{n-1} [\mathbf{I} - \boldsymbol{\psi}(0)] \vec{\mathbf{1}}, \quad n = 1, 2, \ldots.$$

Then $N_i = M_i - 1$ is the number of times the record low for the surplus process has been broken with $J(0) = i$, with

$$\Pr(N_i = n) = \vec{\mathbf{e}}_i^\top [\boldsymbol{\psi}(0)]^{n} [\mathbf{I} - \boldsymbol{\psi}(0)] \vec{\mathbf{1}}^\top, \quad n = 0, 1, 2, \ldots.$$

That is, the distribution of $N_i$ is DPH with representation $\left(\vec{\mathbf{e}}_i^\top \boldsymbol{\psi}(0), \boldsymbol{\psi}(0)\right)$.

**Remarks**

- The distribution of $L_{i,k}$ depends on the state of $\{J(t); t \geq 0\}$ when the record low is broken for the $k$-th time and the state of $\{J(t); t \geq 0\}$ when the record low is broken for the $(k-1)$-th time. If the states when the record low is broken for the $(k-1)$-th and $k$-th times are $j$ and $l$, respectively, then the density of $L_{i,k}$ is $h_{j,l}(0; y)/\psi_{j,l}(0)$.

- For the classical risk model, i.e. when $m = 1$, the number of record lows for the surplus process follows a geometric distribution with parameter $1 - \psi(0)$.

## 7.4 The number of periods of negative surplus

We now allow the surplus process to continue if ruin occurs, and it is certain that the surplus will return to level 0 after ruin under the positive loading condition. We now find the distribution of the number of periods of negative surplus.

For $u \leq b$, we define $a_{i,j}(u; b)$ to be the probability that the surplus process hits level $b$ in state $j$ from initial surplus $u$ and initial state $i$. Clearly $\sum_{j=1}^{m} a_{i,j}(u; b) = 1, \forall i \in \mathcal{E}$. Let $\boldsymbol{\Gamma}(u) = \left(\gamma_{i,j}(u)\right)_{m \times m}$ be a matrix with $\gamma_{i,j}(u)$ being the probability that ruin occurs from initial surplus $u$ and initial state $i$, and the state when the surplus upcrosses through 0 is state $j$. Then we have

$$\gamma_{i,j}(u) = \sum_{k=1}^{m} \int_0^\infty h_{i,k}(u; y) a_{k,j}(0; y) dy, \qquad i, j \in \mathcal{E},$$

with $\sum_{j=1}^{m}\gamma_{i,j}(u)=\sum_{k=1}^{m}\int_0^\infty h_{i,k}(u;y)=\psi_i(u)$. In matrix form,

$$\mathbf{\Gamma}(u)=\int_0^\infty \mathbf{h}(u;y)\mathbf{a}(0;y)dy,$$

with $\mathbf{\Gamma}(u)\vec{\mathbf{1}}=\boldsymbol{\psi}(u)\vec{\mathbf{1}}=\big(\psi_1(u),\psi_2(u),\ldots,\psi_m(u)\big)^\top$.

Let $\{Y_n\}_{n=1}^\infty$ represent the state of the Markov process $\{J(t);t\geq 0\}$ when the surplus process upcrosses through zero for the $n$-th time. Then $\{Y_n\}_{n=1}^\infty$ is a discrete Markov chain with $m$ transition states (states $1,2,\ldots,m$) and one absorbing state (state 0): $Y_n=0$ if the surplus remains above zero after the $(n-1)$-th upcrossing of zero. The Markov chain $\{Y_n\}_{n=1}^\infty$ has the following properties.

1. The distribution for $Y_1$ is

$$\begin{aligned}\Pr(Y_1=j|J(0)=i) &= \gamma_{i,j}(u)=\vec{\mathbf{e}}_i^\top\mathbf{\Gamma}(u)\vec{\mathbf{e}}_j, \quad j\in\mathcal{E},\\ \Pr(Y_1=0|J(0)=i) &= 1-\sum_{j=1}^{m}\gamma_{i,j}(u)=1-\vec{\mathbf{e}}_i^\top\mathbf{\Gamma}(u)\vec{\mathbf{1}}.\end{aligned}$$

   That is, the initial probability distribution is $\big(1-\vec{\boldsymbol{\alpha}}_u^\top\vec{\mathbf{1}},\vec{\boldsymbol{\alpha}}_u^\top\big)$, where $\vec{\boldsymbol{\alpha}}_u^\top=\vec{\mathbf{e}}_i^\top\mathbf{\Gamma}(u)$.

2. The transition probability matrix for $\{Y_n\}_{n=1}^\infty$ is

$$\begin{bmatrix}\mathbf{\Gamma}(0) & [\mathbf{I}-\mathbf{\Gamma}(0)]\vec{\mathbf{1}}\\ \vec{\mathbf{0}}^\top & 1\end{bmatrix}.$$

3. The number of periods of negative surplus, $\tilde{N}_u$, is the same as the number of steps for the Markov chain $\{Y_n\}_{n=1}^\infty$ to reach the absorbing state (state 0) and the distribution of $\tilde{N}_u$, given that the initial state is $i$, is DPH with representation $\big(\vec{\boldsymbol{\alpha}}_u^\top,\mathbf{\Gamma}(0)\big)$, i.e.,

$$\begin{aligned}\Pr(\tilde{N}_u=0|J(0)=i) &= 1-\vec{\boldsymbol{\alpha}}_u^\top\vec{\mathbf{1}},\\ \Pr(\tilde{N}_u=n|J(0)=i) &= \vec{\boldsymbol{\alpha}}_u^\top[\mathbf{\Gamma}(0)]^{n-1}[\mathbf{I}-\mathbf{\Gamma}(0)]\vec{\mathbf{1}}, \quad n=1,2,\ldots.\end{aligned}$$

We remark that by using similar reasoning, we may obtain the distribution of the number of visits to level $b$ from initial surplus $u$, and we let $M_u^b$ denote this random variable. The probability function of $M_u^b$ is

$$\Pr(M_u^b=n|J(0)=i) = \vec{\mathbf{e}}_i^\top\mathbf{a}(u;b)[\mathbf{\Gamma}(0)]^{n-1}[\mathbf{I}-\mathbf{\Gamma}(0)]\vec{\mathbf{1}}, \quad n=1,2,\ldots.$$

That is, $M_u^b-1$ follows a DPH distribution with representation $\big(\mathbf{e}_i^\top\mathbf{a}(u;b),\mathbf{\Gamma}(0)\big)$.

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