STOCHASTIC OPTIMAL GROWTH WITH UNBOUNDED SHOCK

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Abstract. This paper considers a neoclassical optimal growth problem where the shock that perturbs the economy in each time period is potentially unbounded on the state space. Sufficient conditions for existence, uniqueness and stability of equilibria are derived in terms of the primitives of the model using new techniques from the field of perturbed dynamical systems.

1. Introduction

This paper studies equilibria in the stochastic optimal growth economy of Brock and Mirman [5] without their assumption that the shock which perturbs production is realized within a bounded interval. It provides sufficient conditions for existence, uniqueness and stability of equilibria in terms of the primitives of the one-sector accumulation problem, namely the utility function u, the per capita production function f and the distribution ψ of the disturbance term ε. The arguments are based on recent innovations in the theory of stochastically perturbed dynamical systems.

The original work of Brock and Mirman extends the deterministic optimal growth problem of Ramsey [33], Cass [6], Koopmans [21] and others to a stochastic setting. With regard to equilibria, they show that the existence, uniqueness and stability results of the deterministic case are also realized in a stochastic model under similar assumptions

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on preferences and production technology. In their analysis the productivity shock is restricted to a bounded interval of the real line.[1]

The problem of characterizing equilibria and long-run behavior in Brock-Mirman economies with bounded shock has subsequently been studied by Mirman [30], Mirman and Zilcha [31], Brock and Majumdar [4], Razin and Yahav [34], Donaldson and Mehra [7], Majumdar and Zilcha [28], Stokey et al. [37], Hopenhayn and Prescott [16] and Amir [1]. The analogous problem for the overlapping generations model with bounded shock has been studied by Laitner [23] and Wang [41]. The related question of ergodicity in moments for the Solow-Swan model with a shock that is unbounded above but cannot be arbitrarily small is investigated in Binder and Pesaran [3]. Evstigneev and Flåm [10] and Amir and Evstigneev [2] investigate the asymptotic distributions of aggregate rewards accumulated along equilibrium and optimal paths. More general studies of stochastic equilibria in economics include Futia [12] and Duffie et al. [8].

Stochastic growth with unbounded shock is treated in Mirman [29], who provides an existence result and proves that the equilibrium measure is not concentrated at zero. However, the sufficient conditions pertain to a class of consumption policies that may or may not be optimal (Mirman [29], A1–A3, p. 275). In other words, the savings rate is exogenously given, and the conditions are not stated in terms of the primitives u, f and ψ. Further, the problems of uniqueness and stability are not treated. In the present paper conditions for existence, uniqueness and stability are obtained in terms of the triple (u, f, ψ) and the restrictions imposed by optimizing behavior.

[1]Such a shock is said to have compact support. For simplicity these shocks are referred to as "bounded". Shocks where no restrictions are placed on the support are called "unbounded".

The mathematical techniques used in the paper are based on recent innovations in the theory of perturbed dynamical systems. See Lasota [24], Lasota and Mackey [26] and Horbacz [18] in particular.[2]

In addition to identifying and characterizing equilibria in Brock-Mirman economies with unbounded shock, the paper also makes the following contributions. First, a formulation of Markovian systems known as the L_1 approach is introduced to stochastic growth theory. Second, the notions of strong contractiveness and Lagrange stability are developed in the context of stochastic optimal growth. Third, new proofs are given for the two major fixed point results used in the paper. The first result states that every Lagrange stable Markov system has at least one fixed point, and the second that every strongly contractive and Lagrange stable Markov system is asymptotically stable. The proof of the former uses a Brouwer-type convexity argument, while that of the latter is based on the properties of contractive operators on a compactum.

Section 2 previews the mathematical arguments used in the paper. Section 3 formulates the stochastic optimal growth problem. Section 4 states the main result. The proof is then developed over Sections 5–7. Section 8 concludes.

[2]Previously the same methods have been applied to the study of particle energy in an ideal gas (Lasota [24]), fluctuations in the brightness of the Milky Way (Lasota and Mackey [26]), propagation of annual plants with seed-bank (Horbacz [18]), and cell growth in a proliferating cell population (Lasota and Mackey [25], Tyson and Hannsgen [40], Trycha [39], Lasota, Mackey and Trycha [27] and Lasota [24]). For an earlier application to economics see Stachurski [36].

2. Mathematical Techniques

This section gives an introduction to the mathematical techniques used in the paper, and to the L_1 method for Markov processes in particular.[3]

2.1. Outline.

2.1. Outline. For dynamic economic models, an equilibrium (or steady state) is defined to be a point in the state space that is stationary under the period-to-period transition rule. If such a point is obtained then no further change is observed in the system. As well as this invariance property, equilibria may be attractive for points in the surrounding state space, which is to say that the transition rule moves nearby points closer to the equilibrium.

In the case of stochastic models, a state cannot be stationary in the same sense as those in deterministic systems, given that shocks continue to disturb activity in each period. Instead a steady state must be viewed as a situation where the probabilistic laws that govern the state variables cease to change over time (Green and Majumdar [13]). For stochastic economies the notion of stable equilibrium can be approached as follows. Since the path of the economy is a stochastic process, the state at any time in the future can be known only up to a probability distribution. Hence the state space is reinterpreted to be the collection of all density functions on the original space. Densities can be identified with points on the unit sphere in the space of integrable functions.[4] Thus any stable stochastic equilibrium can be

[3]Operator-theoretic treatment of Markov processes begins with Krylov and Bogolioubov [22]. Important contributions include Kakutani and Yoshida [20] and Hopf [17]. For an early operator-theoretic treatment of optimal stochastic growth see Brock and Majumdar [4].

[4]The set of densities coincides with the intersection of the positive cone and the surface of the unit sphere.

viewed as a point on this infinite dimensional sphere to which nearby points are attracted as time evolves.

In this sense, deterministic and stochastic equilibria can be thought of as differing not conceptually but rather in the nature (in particular, in the dimension) of the space in which they are located. Here the above identification of stochastic equilibria with attractors on the unit sphere of the space of integrable functions is exploited to obtain sufficient conditions for the existence of stable equilibria in the stochastic neoclassical growth model.

2.2. **Discrete Dynamical Systems.** To formalize these ideas, consider an abstract system characterized at each time t by a vector of state variables x_t taking values in state space U. Evolution is governed by a first-order difference equation

$$x_{t+1} = Tx_t, \quad x \in U, \quad T \colon U \to U. \tag{1}$$

The map T encodes the structure of the economic system, which is in turn determined by the primitives of the model, such as preferences, technology and market conditions. A realization or *trajectory* for the system is a sequence $(T^n x)$ in U generated by iterating the map T on initial state x.[5] An equilibrium is a fixed point of T on U.

More generally, a *semidynamical system* is a pair (U, T), where U is a metric space with distance d, and T is a continuous mapping of U into itself.[6] An *equilibrium* or *steady state* of (U, T) is a fixed point of T on U, i.e. a point $p \in U$ such that $Tp = p$. Fixed points are said to be *stationary* or *invariant* under T. Similar terminology also applies to sets. In particular, if $TA \subset A$ then A is said to be *invariant* under T. If p is a fixed point of T on U then the *stable set* $S_T(p)$ of p is that

[5] Here $T^n x \overset{\text{df}}{=} T(T^{n-1}(x))$, $T^1 x \overset{\text{df}}{=} Tx$.

[6] The system is called *dynamical* if, in addition, the mapping T is invertible with continuous inverse (i.e. is a homeomorphism).

subset of U which is convergent to p under iteration of T:

$$S_T(p) \stackrel{\text{df}}{=} \{x \in U \,:\, T^n x \to p \;\; (n \to \infty)\}. \tag{2}$$

The point p is said to be *stable*, or an *attractor*, whenever there exists a set G open in U such that $p \in G$ and $S_T(p) \supset G$. In particular:

Definition 2.1. (U, T) is said to be *asymptotically stable* if there exists a unique fixed point p and $S_T(p) = U$.

Figure 1 shows motion induced by iteration of an arbitrary map T, $U = \mathbf{R}^2$. Continued iteration generates a sequence in the plane.

2.3. **Stochastically Perturbed Dynamical Systems.** Suppose that the system (1) is perturbed at each transition from state x_t to state x_{t+1} by serially uncorrelated, U-valued shock ε_t with distribution given by density ψ:

$$x_{t+1} = T(x_t, \varepsilon_t), \quad x \in U, \quad \varepsilon_t \sim \psi. \tag{3}$$

For each fixed $x_t \in U$, x_{t+1} is a random variable with distribution uniquely determined by the value of x_t, the density ψ and the map T. Let the density of this conditional distribution be $p(x_t, \cdot)$. That is,

$$p \colon U^2 \to \mathbf{R}_+, \quad \text{Prob}(x_{t+1} \in B | x_t) = \int_B p(x_t, x_{t+1}) dx_{t+1}, \tag{4}$$

where $\text{Prob}(x_{t+1} \in B | x_t)$ is the probability that the state vector is in $B \subset U$ at time $t + 1$ given its current location at x_t. Figure 2 shows a perturbed system with additive shock in state space $\mathbf{R}^2$. The circles represent contour lines for the conditional density $p(x_t, \cdot)$. The bold arrows are sample realizations of the process.

Equation (3) and the function p in (4) contain essentially the same information. However, the latter formulation is convenient for calculation of the *unconditional* distribution of the state vector at each point in time. In particular, suppose that the unconditional (marginal) distribution of x_t is known, and is given by density φ_t. Then

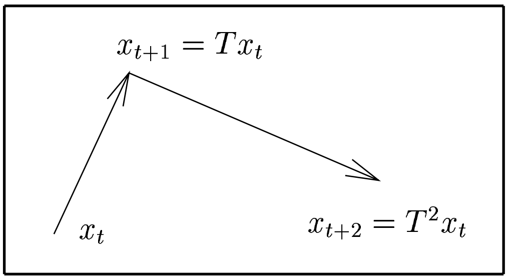

FIGURE 1. Deterministic system in the plane.

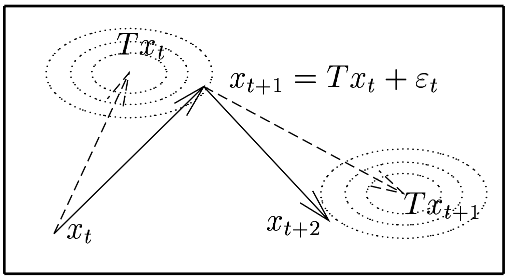

FIGURE 2. The perturbed system.

$$\varphi_{t+1}(x_{t+1}) = \int_U p(x_t, x_{t+1}) \varphi_t(x_t) dx_t \tag{5}$$

defines the unconditional density of the state at time $t+1$. The intuition is that the integral sums the probability $p(x_t, x_{t+1})$ of traveling to x_{t+1} from x_t for all $x_t \in U$, weighted at each point by the likelihood $\varphi_t(x_t)$ of x_t occurring as the current state. The importance of the recursion (5) is that it provides a way to calculate the entire sequence of densities (φ_t) that represent the marginal distributions for the stochastic process (x_t) from any initial density φ_0 ($x_0 \sim \varphi_0$).

In analyzing the behavior of this sequence, one possibility is to use standard techniques from the classical theory of Markov processes (see, for example, Shiryaev [35], Chapter 8). However, it is also possible to frame the same problem as a semidynamical system (Lasota [24]). The idea is to reinterpret the state space to be the collection of all densities on U. Call this set $D(L_1(U))$. The other half of the pair is the operator (call it P) that associates current-period with next-period densities through the integration defined in (5).

In this notation, (5) can be rewritten as

$$\varphi_{t+1} = P\varphi_t, \quad \varphi \in D(L_1(U)), \quad P\colon D(L_1(U)) \to D(L_1(U)). \tag{6}$$

The recursion (6) is now in exactly the same formula as the deterministic system (1), which means that similar techniques can be applied to its analysis. Translation of the perturbed system (3) into a deterministic map on the space of all density functions is called the L_1 approach to Markov processes.[7] Evolution of the economy is characterized by a sequence of densities generated by iterating P on some initial density φ_0. An equilibrium is a fixed point of the semidynamical system $(D(L_1(U)), P)$. The economy has a unique, globally stable equilibrium whenever $(D(L_1(U)), P)$ is asymptotically stable in the sense of Definition 2.1.

These definitions are conceptually equivalent to those used in previous studies.[8] However, the space of possible states $D(L_1(U))$ and hence equilibria has been constructed to include only those distributions that can be represented by density functions.[9] Thus probability mass cannot be concentrated at a point. In particular, this means that the model does not include the deterministic system as a special case; the distribution of the disturbance term ε must be non-degenerate.

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[7]Much of the early L_1 theory is due to Hopf [17]. The monograph of Foguel [11] contains an extensive survey of asymptotic results. Lasota and Mackey [26] use L_1 techniques to study chaotic systems.

[8]The operator P corresponds to T^* in Brock and Majumdar [4], (4.3), Futia [12], p. 380, and Stokey et al. [37], (2), p. 213, and to T in Hopenhayn and Prescott [16], p. 1392.

[9]The distributions which are absolutely continuous with respect to Lebesgue measure. For an earlier density-based treatment see Mirman [30].

3. Formulation of the Problem

This section contains a formulation of the stochastic optimal growth problem studied by Brock and Mirman [5]. $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{R}_{++}$ denote the reals, the nonnegative reals and the positive reals respectively. If X is any metric space then $\mathcal{B}(X)$ is the Borel sets of X. m is Lebesgue measure.

The accumulation problem evolves as follows. At the start of period t the (representative) agent receives income x_t. In response a level of consumption $c_t \leq x_t$ is chosen, yielding current utility $u(c_t)$. The remainder is invested in production, returning in the following period output $x_{t+1} = f(x_t - c_t)\varepsilon_t$. Here f is the production function and ε is a nonnegative random variable.[10] The process then repeats.

3.1. **Assumptions.**

All of the assumptions on (u, f, ψ) are standard, apart from the lack of a boundedness assumption on the shock. The production function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ is zero at zero, strictly increasing, strictly concave, differentiable and satisfies the Inada conditions

$$\lim_{x \downarrow 0} f'(x) = \infty, \quad \lim_{x \uparrow \infty} f'(x) = 0. \tag{7}$$

The utility function $u\colon \mathbf{R}_+ \to \mathbf{R}$ is strictly increasing, strictly concave, differentiable and satisfies the interiority condition

$$\lim_{x \downarrow 0} u'(x) = \infty. \tag{8}$$

Future utility is discounted geometrically at rate $\beta \in (0, 1)$.

[10] Following Stokey et al. [37] and Hopenhayn and Prescott [16], it is assumed that the disturbance term ε is multiplicative. Brock and Mirman use the more general formulation $x_{t+1} = f(x_t - c_t, \varepsilon_t)$. See Amir [1] for an even more general technology.

The shocks to production are uncorrelated and identically distributed. The distribution of ε is represented by density ψ.[11] It is assumed that ε has finite mean $E(\varepsilon)$. Without any loss of generality, let $E(\varepsilon) = 1$.[12]

3.2. **Technology.** The conditional density for next-period output given income x and consumption c is, by a change of variable argument,

$$y \mapsto \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}. \tag{9}$$

Given that $f(0) = 0$, (9) is not defined when consumption is equal to income. In this case (when $c = x$), next-period income is zero with probability one. Such a probability cannot be represented by a density. Consequently, the fully specified technology associating savings $x - c$ to next-period income will be defined by probability $\mathcal{B}(\mathbf{R}_+) \ni B \mapsto \mathbf{Q}(x, c; B)$, where

$$\mathbf{Q}(x, c; B) = \int_B \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}dy. \tag{10}$$

when $c < x$, and by the probability concentrated at zero when $c = x$. Thus $\mathbf{Q}(x, c; B)$ is the probability that next-period output is in B given that current income is x and consumption is $c \in [0, x]$.

3.3. **The optimal policy.** The agent selects a sequence (c_t) to solve

$$\max E\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right] \tag{11}$$

subject to the feasibility constraint $0 \leq c_{t+1} \leq f(x_t - c_t)\varepsilon_t$.

The meaning of the expectations operator in (11) is not immediately clear. A more formal statement of the problem is that the agent seeks

[11]The function ψ is required to be measurable. All integration is with respect to Lebesgue measure unless otherwise stated.

[12]Suppose $E(\varepsilon) = \mu < \infty$. Define $f^* = \mu f$ and $\varepsilon^* = \varepsilon/\mu$, with ψ^* the density of ε^*. Then $E(\varepsilon^*) = 1$ and (u, f^*, ψ^*) satisfies all of the preceding assumptions whenever (u, f, ψ) does.

a control policy $g\colon \mathbf{R}_+ \ni x_t \mapsto c_t \in \mathbf{R}_+$ that is feasible $(0 \le g(x) \le x)$ and maximizes $v(x, g)$, where

$$v(x,g) \stackrel{\text{df}}{=} E_x^g \left[\sum_{t=0}^{\infty} \beta^t u(g(x_t)) \right]. \tag{12}$$

Here E_x^g signals integration with respect to the (well-defined and unique) Markovian distribution over infinite-dimensional sequence space $\mathbf{R}_+^\infty$ generated by Markov transition kernel $\mathbf{Q}(x, g(x); dy)$.[13]

The value function V for the problem is defined at x as the supremum of $v(x, g)$ over the set of all feasible policies. A feasible policy g^* is called *optimal* if $v(x, g^*) = V(x)$ for all x.

The following results are well-known.

Theorem 3.1. *Let u, f and ψ be as in Section 3.1. Then*

(i) The value function V is finite and satisfies the Bellman equation

$$V(x) = \max_{0 \le c \le x} \left\{ u(c) + \beta \int V(y) \mathbf{Q}(x, c; dy) \right\}. \tag{13}$$

(ii) There exists a unique optimal policy g and

$$V(x) = u(g(x)) + \beta \int V(y) \mathbf{Q}(x, g(x); dy). \tag{14}$$

(iii) V is nondecreasing, concave and differentiable with

$$V'(x) = u(g(x)); \text{ and} \tag{15}$$

(iv) If g is optimal then $0 < g(x) < x$, $\forall x > 0$, and both $x \mapsto g(x)$ and $x \mapsto x - g(x)$ are nondecreasing (savings and consumption both increase with income).

Proof. See, for example, Mirman and Zilcha [31], p. 331–2. (For a formal discussion of Markov control programs with unbounded reward see Hernández-Lerma and Lasserre [15], Chapter 8.) Here (i)–(iii) $\Longrightarrow$ (iv). □

[13] See, for example, Hernández-Lerma and Lasserre [14].

Substitution of the optimal control into the production relation yields the closed-loop law of motion

$$x_{t+1} = f(x_t - g(x_t))\varepsilon_t. \tag{16}$$

In this study (16) corresponds to (3) in Section 2.

4. Statement of Results

It is now possible to state the main result of the paper, which gives sufficient conditions for existence, uniqueness and stability of equilibria in the stochastic growth model of the previous section. The conditions provided have the advantage that they are very easy to check in applications.

Theorem 4.1. *Let u, f and ψ satisfy the assumptions of Section 3.1.*

- *(i) If $E(\varepsilon^{-1}) < 1$ then (u, f, ψ) has at least one (nonzero) equilibrium.*
- *(ii) If in addition ψ is everywhere positive then the equilibrium is unique and globally stable.*

The proof is developed in stages through the remaining sections. The approach is to represent the economy (u, f, ψ) as a semidynamical system and then apply two concepts used in the theory of such systems, namely Lagrange stability and strong contractiveness.[14] In Section 5 Lagrange stability and strong contractiveness are defined. Further, it is shown (a) that every semidynamical system that is Lagrange stable and has certain linearity properties has at least one fixed point, and (b) that every semidynamical system which is both Lagrange stable and strongly contractive is asymptotically stable. New proofs are offered for both results. In Section 6 it is shown that (u, f, ψ) can be represented as a semidynamical system. If it can be established under

[14]The utility of these concepts for identifying equilibria in stochastically perturbed dynamical systems has recently been emphasized by Lasota [24].

the hypotheses of Theorem 4.1, part (i), that this semidynamical system generated by (u, f, ψ) is Lagrange stable then (a) can be used to establish the existence of at least one equilibrium. If, in addition, it can be shown that positivity of ψ in part (ii) of the theorem implies strong contractiveness then by (b) the system is also asymptotically stable, which is to say that there exists a unique and globally stable equilibrium. These two results are established in Section 7, completing the proof of the theorem.

The proof of Lagrange stability (Proposition 7.1) constitutes the core contribution of the paper in a mathematical sense. As expected, the Inada conditions and the concavity of the program are crucial to the proof.

In contrast, strong contractiveness is a straightforward consequence of assumed positivity of the distribution ψ. This important result is relatively well-known in recent literature on stochastic systems, but does not appear to have been used in economics to date.

4.1. **Remarks.** The restriction

$$E(\varepsilon^{-1}) \stackrel{\text{df}}{=} \int_0^\infty \frac{1}{x}\psi(x)dx < 1 \tag{17}$$

in part (i) of the theorem has a simple interpretation. Previous work has assumed that ε is realized in a compact interval $[a, b]$, $0 < a \leq b < \infty$. Here, in contrast, the shock may be arbitrarily large or arbitrarily close to zero. (17) implies that ε is "unlikely" to be very close to zero, or, in other words, that the left-hand tail of the density ψ is relatively small. To understand the implication of (17), define for nonnegative summable functions V and h on $\mathbf{R}_{++}$ the (possibly infinite) number

$$E(V|h) \stackrel{\text{df}}{=} \int_{\mathbf{R}_{++}} V(x)h(x)dx, \tag{18}$$

as well as the set $G_a \stackrel{\text{df}}{=} \{x \in \mathbf{R}_{++} : V(x) < a\}$. Evidently

$$E(V|h) \geq \int_{\mathbf{R}_{++}\setminus G_a} V(x)h(x)dx, \tag{19}$$

implying

$$\int_{\mathbf{R}_{++}\setminus G_a} h(x)dx \leq \frac{E(V|h)}{a}. \tag{20}$$

(This is a version of Chebychev's inequality.) Substituting $I^{-1}\colon x \mapsto x^{-1}$ for V, ψ for h and $1/r$ for a gives

$$\int_0^r \psi(x)dx \leq rE(\varepsilon^{-1}). \tag{21}$$

Thus (17) is a restriction on the left-hand tail of ψ.

There is no explicit assumption on the right-hand tail in Theorem 4.1. However, it has already been assumed that $E(\varepsilon) = 1$. This is a restriction on the right-hand tail. To see this, substitute $I\colon x \mapsto x$ for V and ψ for h to obtain

$$\int_a^\infty \psi(x)dx \leq \frac{E(\varepsilon)}{a}. \tag{22}$$

These restrictions on the tails of ψ can be thought of as a generalization of the assumption that ψ is zero below a and above b made in previous studies. (As a caveat to the claim that the restrictions on ψ are a generalization of boundedness, recall that in this paper—in contrast to the majority of previous work—the shock must be non-degenerate and representable by a density function.)

An example of a density ψ on the income space that is everywhere positive and satisfies $E(\varepsilon^{-1}) < 1$ is the lognormal distribution $\log \varepsilon \sim N(\mu, \sigma^2)$ with $\sigma^2 < 2\mu$.

Finally, the assumption $E(\varepsilon^{-1}) < 1$ is far from necessary. In fact the proof requires only

$$E(\varepsilon^{-1}) < \frac{f(x - g(x))}{x} \tag{23}$$

on $(0, \delta)$ for some $\delta > 0$. If consumption is a constant fraction of x, for example, then (23) is satisfied whenever $E(\varepsilon^{-1})$ is finite, as holds for *any* lognormal shock.

The assumption on positivity of ψ in part (ii) of the theorem is akin to the "communication" assumptions used in traditional Markov chain theory (Shiryaev [35], Chapter 8).

5. Lagrange Stability and Contractive Systems

In this section the notions of Lagrange stability and strong contractiveness are developed and two fixed point results are established.

5.1. **Lagrange Stability.**

The notion of Lagrange stability has been used extensively in the study of nonlinear differential equations and iterated function systems. Lagrange's original stability work was on the N-body problem of planetary motion. He showed that a first-order approximation of the system does not grow without bounds. The concept of Lagrange stability retains this meaning.

Recall that a set $A \subset U$ is *precompact* if every sequence in A has a convergent subsequence. (A is compact if, in addition, the limit of the sequence is always in A.) A sequence (x_n) in U is defined to be precompact whenever $\{x_n : n \in \mathbf{N}\}$ is a precompact subset of U.

Definition 5.1. Semidynamical system (U, T) is called *Lagrange stable* if the trajectory of x is precompact for every $x \in U$.[15]

A fixed point result for Lagrange stable systems is now stated. An alternative proof based on spectral decomposition can be found in Lasota and Mackey [26], Proposition 5.4.1 (the notation and formulation

[15] In finite dimensional space precompactness is equivalent to boundedness by the Bolzano-Weierstrass theorem. Thus for such a space Lagrange stability corresponds to the idea that none of the possible trajectories for the state variables grow without bounds.

is somewhat different). Here a new proof is offered based on an infinite dimensional Brouwer fixed point theorem.

Theorem 5.1. *Let X be a normed linear space and let U be a nonempty convex closed subset of X. Let T be linear and continuous on X with $TU \subset U$. Suppose further that (U, T) is Lagrange stable. Then T has a fixed point in U.*

Proof. Take any $x \in U$. Define $\gamma(x)$ to be the set $\{T^n x : n \in \mathbf{N}\}$, let $\hat{\gamma}(x)$ be its convex hull and let $\mathrm{cl}(\hat{\gamma}(x))$ be the closure of the latter. Since the convex hull of a precompact set is again precompact it follows that $\hat{\gamma}(x)$ is precompact. Since the closure of a precompact set is compact, $\mathrm{cl}(\hat{\gamma}(x))$ must be compact. Using the linearity of T, if $a \in \hat{\gamma}(x)$ then evidently Ta is again in $\hat{\gamma}(x)$, or $T\hat{\gamma}(x) \subset \hat{\gamma}(x)$. But then $T\mathrm{cl}(\hat{\gamma}(x)) \subset \mathrm{cl}(\hat{\gamma}(x))$.[16] Thus T is invariant on nonempty convex compact set $\mathrm{cl}(\hat{\gamma}(x))$. It follows that T has a fixed point in $\mathrm{cl}(\hat{\gamma}(x))$.[17] Finally, since $\mathrm{cl}(\hat{\gamma}(x)) \subset U$ by the assumption that U is closed and convex, the fixed point must also be in U. □

5.2. **Strongly Contractive Systems.** Next strong contractiveness and its relationship to Lagrange stability are discussed.

In many fields of economics, Banach's contraction principle is used to locate equilibria and solve dynamic programs.[18] Let U be a metric space and T an operator on U. T is said to be a contraction mapping in the sense of Banach if there exists a $\lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \forall x, y \in U. \tag{24}$$

[16]If A is any set with $TA \subset A$ and A' is the closure of A then $a' \in A'$ implies the existence of a sequence $(a_n) \subset A$, $a_n \to a'$. But then $Ta' = T \lim a_n = \lim Ta_n$, which, as the limit of a sequence in A, must again be in A'. Hence $TA' \subset A'$.

[17]This is the Brouwer argument. See Joshi and Bose [19], Theorem 4.3.10.

[18]See, for example, Stokey et al. [37], Lemma 11.11 and Section 17.2.

Banach's contraction principle is essentially equivalent to the statement that every semidynamical system (U, T) where U is complete and T satisfies (24) is asymptotically stable (Joshi and Bose [19], Theorem 4.1.1).

Unfortunately, for the semidynamical systems generated by stochastic growth models with unbounded shock, (24) either does not hold or is difficult to verify. In contrast, the following slightly weaker condition, defined here as strong contractiveness, will be demonstrated to be an immediate consequence of positivity of the distribution ψ. (Recall Theorem 4.1, part (ii).)

Definition 5.2. Semidynamical system (U, T) is called *contractive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in U. \tag{25}$$

The system is called *strongly contractive* if, in addition,

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in U, \; x \neq y. \tag{26}$$

Evidently (24) $\implies$ (26) $\implies$ (25). Like contractiveness in the sense of Banach, strong contractiveness implies uniqueness of equilibrium. (Suppose otherwise. In particular, let distinct points x and y be stationary under T. Then both $d(x, y) = d(Tx, Ty)$ and $d(Tx, Ty) < d(x, y)$. Contradiction.) However, strong contractiveness does not guarantee existence. (For example, consider $U = \mathbf{R}_+$, $T \colon x \mapsto x + e^{-x}$.) Nevertheless, existence and stability can be obtained if strong contractiveness is supplemented by compactness of U:

Lemma 5.1. *Let (U, T) be strongly contractive, and let U be a compactum. Then (U, T) is asymptotically stable.*

Proof. See Joshi and Bose [19], Theorem 4.1.6, Corollary 1. □

In the arguments that follow, the underlying space U corresponds to the space of density functions on $\mathbf{R}_{++}$, which is defined below as the intersection of the positive cone and the surface of the unit sphere in the space of summable functions $L_1(\mathbf{R}_{++})$. This set is not compact, and hence Lemma 5.1 is not immediately applicable. However, it is closed, and in this case compactness can be replaced by Lagrange stability. For discrete dynamical systems this fact was recently proved in the context of Hausdorff space using Liapunov methods (Lasota [24], Theorem 2.1). Here a simple new proof is given.

Theorem 5.2. *Let X be a metric space, let U be a nonempty closed subset of X and let $T\colon X \to X$ be a continuous function invariant on U. If (U, T) is both Lagrange stable and strongly contractive then it is asymptotically stable.*

Proof. Fix $x \in U$. Define $\Gamma(x)$ to be the closure of $\{T^n x : n \in \mathbf{N}\}$. Since (U, T) is Lagrange stable, $\Gamma(x)$ is a compact subset of X. Moreover, $T\Gamma(x) \subset \Gamma(x)$.[19] Therefore $(\Gamma(x), T)$ is itself a strongly contractive semidynamical system on a compactum, and, by Lemma 5.1, has a unique fixed point p with $T^n x \to p$. The point p is in U because U is closed and hence $\Gamma(x) \subset U$. Moreover, (U, T) has at most one fixed point by strong contractiveness. The result follows. $\square$

6. Markov Chains as Semidynamical Systems

In this section it is shown that (u, f, ψ) can be interpreted as a semidynamical system. Mathematically, the exposition in based on Hopf [17], Foguel [11], Lasota [24] and Lasota and Mackey [26]. Although the formal structure is somewhat different, our approach to Markovian economic models benefits from previous operator-theoretic treatments, such as Brock and Majumdar [4], Futia [12], Stokey et al. [37] and Hopenhayn and Prescott [16].

Consider the Markov system (3). Let Σ be a σ-algebra of subsets of U, and let μ be a σ-finite measure on (U, Σ). As usual, $L_1(U)$

[19] Take $u \in \Gamma(x)$. There exists a subsequence $(T^{n_k} x)$ such that $\lim_k T^{n_k} x = u$. But then $Tu = T \lim_k T^{n_k} x = \lim_k T^{n_k + 1} x \in \Gamma(x)$ by continuity of T.

denotes the normed linear space of μ-integrable functions on U with norm $\|f\| = \int |f| d\mu$. $L_1(U)$ is endowed with a distance metric in the usual way, that is $d(f,g) = \|f - g\|$. Functions in $L_1(U)$ are defined only up to the complement of a μ-null set, and "almost everywhere" notation is suppressed throughout.

A formal representation of p in (4) is now possible:

Definition 6.1. A *stochastic kernel* for measure space (U, Σ, μ) is a nonnegative, real-valued and Σ^2-measurable function p on U^2 such that

$$\int p(x,y)\mu(dy) = 1, \quad \forall x \in U. \tag{27}$$

Thus $y \mapsto p(x,y)$ is a density for each $x \in U$. More precisely, $p(x,\cdot)$ is the conditional density of the state in the next period, given that the current location is x.

6.1. **A Definition of Hopf.** Let X be a normed linear space with norm $\|\cdot\|$. A partial order $\leq$ on X is defined in terms of a cone $X_+ \subset X$, which is declared to be *positive*. Elements in the cone are called positive. The statement $x \leq y$ is interpreted to mean $y - x \in X_+$. Clearly x is positive iff $0 \leq x$. An operator $P\colon X \to X$ is called positive if $PX_+ \subset X_+$. P is called *isometric* on $A \subset X$ if $\|Pa\| = \|a\|$ whenever $a \in A$. For $L_1(U)$ the positive cone is the nonnegative functions.

The following definition is due to Hopf [17].

Definition 6.2. Let X be a normed linear space with partial order. A *Markov operator* on X is an operator $P\colon X \to X$ such that P is (i) linear, (ii) positive and (iii) isometric on X_+.[20]

Let p be a stochastic kernel on (U, Σ, μ), and define an operator $P\colon L_1(U) \to L_1(U)$ by

$$(Pf)(y) = \int_U p(x,y) f(x) \mu(dx). \tag{28}$$

[20] A Markov process is a pair (X, P), where X is a normed linear space with partial order and P is a Markov operator on X.

Then P is a Markov operator on $L_1(U)$.[21] Here (28) is the operation in (5), and P corresponds to the operator in (6).

The next lemma begins to draw connections between Markov processes and contractive systems.

Lemma 6.1 (Lasota and Mackey). *Let P be a Markov operator on $L_1(U)$. Then P is contractive on the same.*

Proof. Fix $f \in L_1(U)$. Define $f^{\pm} \stackrel{\text{df}}{=} \max(\pm f, 0)$. By linearity and positivity,

$$|Pf| = |Pf^+ - Pf^-| \leq Pf^+ + Pf^- = P|f|. \tag{29}$$

Integration obtains

$$\|Pf\| \stackrel{\text{df}}{=} \int |Pf| d\mu \leq \int P|f| d\mu = \|f\|. \tag{30}$$

An application of linearity yields (25). □

Let P be a Markov operator on $L_1(U)$, and let $D(L_1(U))$ be the collection of all densities on the same. That is,

$$D(L_1(U)) \stackrel{\text{df}}{=} \{f \in L_1(U) \,:\, f \geq 0 \text{ and } \|f\| = 1\} \tag{31}$$

$D(L_1(U))$ is a metric space with distance $d(f,g) = \|f - g\|$. Lemma 6.1 shows that in this topology the operator P is Lipschitz and hence continuous. Moreover, it is clear from Definition 6.2 that $PD(L_1(U)) \subset D(L_1(U))$.[22] Hence $(D(L_1(U)), P)$ is a semidynamical system.

6.2. **The Brock-Mirman Process.** The stochastic kernel, Markov operator and semidynamical system associated with the Brock-Mirman economy (u, f, ψ) are derived from the law of motion (16).

By a change of variable argument, the stochastic kernel is

$$\mathbf{R}^2_{++} \ni (x,y) \mapsto \psi\left(\frac{y}{f(x - g(x))}\right) \frac{1}{f(x - g(x))}. \tag{32}$$

[21] Evidently P is linear and positive. That $\|Pf\| = \|f\|$ when $f \geq 0$ follows from an application of Fubini's theorem.

[22] This justifies (6).

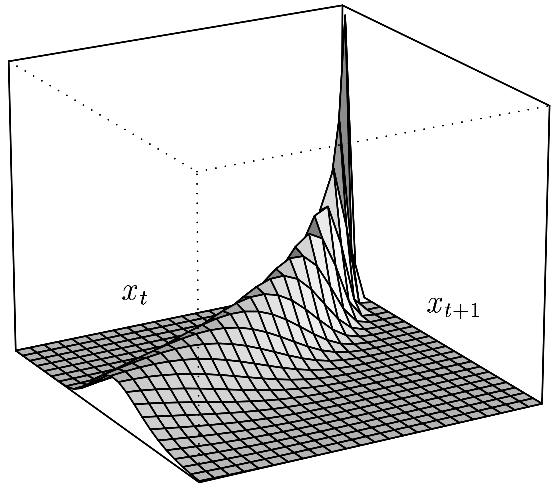

FIGURE 3. Stochastic kernel (32), lognormal shock.

Define Q to be the Markov operator associated with (32) by (28). The semidynamical system for the process is then $(D(L_1(\mathbf{R}_{++})), Q)$.[23] Thus if initial income x_0 is distributed according to φ_0 then time t income is distributed according to $Q^t\varphi_0$.

A plot of (32) is shown in Figure 3 for the parameterization $f\colon x \mapsto x^{1/2}$, $u\colon x \mapsto \log x$, ε lognormal. The origin is the corner of the graph furthest from the viewer. For each x_t a density function runs parallel to the x_{t+1} axis. The density governs the likelihood that income per head takes values along that axis, given that the current state is x_t.[24]

7. Proof of the Theorem

The proof of Theorem 4.1 proceeds as follows. In Section 7.1 Lagrange stability of (u, f, ψ) is established. In Section 7.2 strong contractiveness of the economy is established using the additional hypothesis of positivity of ψ. The proof is completed in Section 7.3.

[23] $L_1(\mathbf{R}_{++})$ is understood to be the summable functions on measure space $(\mathbf{R}_{++}, \mathcal{B}(\mathbf{R}_{++}), m)$.

[24] For a kernel estimated nonparametrically from actual growth data see Quah [32], Figures 5 and 6.

7.1. **Proof of Lagrange Stability.** The first result provides a way to identify weakly precompact sets in $L_1(\mathbf{R}_{++})$.[25]

Lemma 7.1. *A bounded set $\mathcal{M}$ of nonnegative functions in $L_1(\mathbf{R}_{++})$ is weakly precompact whenever*

(i) $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that if* $A \in \mathcal{B}(\mathbf{R}_{++})$ *and* $m(A) < \delta$ *then*

$$\int_A h(x)dx < \varepsilon, \quad \forall h \in \mathcal{M}; \quad \text{and} \tag{33}$$

(ii) there exists a constant M *such that* $\forall r > 0$,

$$\int_r^\infty h(x)dx \leq \frac{M}{r}, \quad \forall h \in \mathcal{M}. \tag{34}$$

Proof. See Dunford and Schwartz [9], IV.13.54. □

The next lemma is required for the proof of Lagrange stability of (u, f, ψ). Related arguments are found in Stokey et al. [37], Section 6.1.

Lemma 7.2. *Let (u, f, ψ) be as in Section 3.1, and let g be the optimal policy. Then there exists an $x_0 > 0$ such that*

$$f(x - g(x)) \geq x \text{ whenever } 0 \leq x \leq x_0. \tag{35}$$

Proof. The first order condition of (13) is

$$u'(g(x)) = \beta \int_0^\infty V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz. \tag{36}$$

Using the envelope relation (15) obtains

$$\begin{aligned} V'(x) &= \beta \int_0^\infty V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz \\ &\geq \beta \int_0^1 V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz \\ &\geq \beta \int_0^1 V'(f(x - g(x))) f'(x - g(x)) z\psi(z) dz, \end{aligned}$$

[25] $f^n \to f$ weakly in $L_1(\mathbf{R}_{++})$ if $\int f^n(x)g(x)dx \to \int f(x)g(x)dx$ for all $g \in L_\infty(\mathbf{R}_{++})$. See, for example, Lasota and Mackey, [26], Section 2.3.

where the first inequality follows from the fact that V is nondecreasing and the second from the fact that V is concave. Thus

$$(37)\qquad V'(x) \geq V'(f(x-g(x)))f'(x-g(x))M, \quad M \stackrel{\text{df}}{=} \beta \int_0^1 z\psi(z)dz.$$

Evidently $M > 0$, because $M = 0$ implies $\int_1^\infty \psi(z)dz = 1$. But then $\int_1^\infty \psi(z)dz < \int_0^\infty z\psi(z)dz$, contradicting $E(\varepsilon) = 1$. Given positivity of M and the assumptions on f, there exists an $x_0 > 0$ such that $f'(x-g(x))M \geq 1$ whenever $x \in (0, x_0]$. Therefore

$$(38)\qquad V'(x) \geq V'(f(x-g(x))) \text{ on } (0, x_0].$$

The result now follows from the concavity of V. □

The proof of the following proposition, which is the central technical result in the paper, draws heavily on Horbacz [18], Theorem 1.

Proposition 7.1. *Let (u, f, ψ) satisfy the hypotheses of Theorem 4.1, part (i), and let $(D(L_1(\mathbf{R}_{++})), Q)$ be the associated semidynamical system (Section 6.2). Then $(D(L_1(\mathbf{R}_{++})), Q)$ is Lagrange stable.*

Proof. It follows from Lasota [24], Proposition 3.4 and Theorem 4.1 that to establish precompactness of $(Q^n\varphi)$ for any density φ it is sufficient to find a set $\mathcal{M} \subset D(L_1(\mathbf{R}_{++}))$ such that $\mathcal{M}$ is dense in the density functions and $(Q^n h)$ is weakly precompact for every h in $\mathcal{M}$.[26]

Let $\mathcal{M}$ be the collection of densities h that satisfy

$$(39)\qquad \int_0^\infty xh(x)dx < \infty \text{ and } \int_0^\infty \frac{1}{x}h(x)dx < \infty.$$

We claim that $\mathcal{M}$ has the desired properties. To see that $\mathcal{M}$ is dense in the densities, fix $\varphi \in D(L_1(\mathbf{R}_{++}))$ and define $h_k^0 = \mathbf{1}_{(1/k,k)}\varphi$.[27] Since $\|h_k^0\| \uparrow 1$ by the monotone convergence theorem, it follows that for

[26] This holds for any integral Markov operator, that is any Markov operator derived from a stochastic kernel via (28). See Lasota [24].

[27] As usual, $\mathbf{1}_E(x)$ is 1 when $x \in E$ and zero otherwise.

some $K \in \mathbf{N}$, $\|h_k^0\| > 0$ whenever $k \geq K$. For all such k define

$$h_k \stackrel{\text{df}}{=} \|h_k^0\|^{-1} h_k^0. \tag{40}$$

It can be established that h_k satisfies (39) for each k. In addition, h_k is a density by construction, and $h_k \to \varphi$ pointwise. But then $h_k \to \varphi$ in the L_1 metric.[28] Thus $\mathcal{M}$ is dense in $D(L_1(\mathbf{R}_{++}))$.

It remains to show that that if $h \in \mathcal{M}$ then $(Q^n h)_{n \geq 1}$ is weakly precompact. Fix arbitrary $h \in \mathcal{M}$. It is sufficient to establish precompactness of $(Q^n h)_{n \geq N}$ for some fixed $N \in \mathbf{N}$, because appending a finite number of elements to a (weakly) precompact set does not alter the property of (weak) precompactness. We now show that $(Q^n h)_{n \geq N}$ is weakly precompact by verifying the conditions of Lemma 7.1.

Boundedness of the collection is satisfied because $\|Q^n h\| = \|h\| = 1$ for all n by the positive isometry property of Markov operators (Definition 6.2). We now show that conditions (i) and (ii) also hold.

For notational simplicity define $q(x) \stackrel{\text{df}}{=} f(x - g(x))$. Use will be made of the fact that

$$\frac{1}{q(x)} E(\varepsilon^{-1}) \leq \gamma \frac{1}{x} + C \tag{41}$$

for all positive x, where γ and C are nonnegative constants, $\gamma < 1$.

To verify (41), recall that $\exists x_0 > 0$ such that $q(x) \geq x$ when $x \leq x_0$ by Lemma 7.2. Choose any γ such that $E(\varepsilon^{-1}) < \gamma < 1$. Then

$$\frac{1}{q(x)} E(\varepsilon^{-1}) \leq \gamma \frac{1}{x}, \quad \forall x \leq x_0. \tag{42}$$

Moreover, on $[x_0, \infty)$, monotonicity of f and $x \mapsto x - g(x)$ implies that $q(x) \geq q(x_0)$, or

$$\frac{1}{q(x)} E(\varepsilon^{-1}) \leq \frac{1}{q(x_0)} E(\varepsilon^{-1}) \stackrel{\text{df}}{=} C. \tag{43}$$

Together, (42) and (43) imply (41).

---

[28] This is Scheffé's Lemma. See, for example, Taylor [38], Proposition 4.5.14.

Let I^{-1} be the map $x \mapsto x^{-1}$. Applying in succession (18), (28), Fubini's theorem, a change of variable argument and (41),

$$\begin{aligned} E(I^{-1}|Q^n h) &= \int_0^\infty \frac{1}{y} Q^n h(y) dy \\ &= \int_0^\infty \frac{1}{y} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} Q^{n-1} h(x) dx \right] dy \\ &= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} \frac{1}{y} dy \right] Q^{n-1} h(x) dx \\ &= \int_0^\infty \frac{1}{q(x)} E(\varepsilon^{-1}) Q^{n-1} h(x) dx \\ &\leq \int_0^\infty [\gamma \frac{1}{x} + C] Q^{n-1} h(x) dx \\ &\leq \gamma E(I^{-1}|Q^{n-1} h) + C. \end{aligned}$$

Repeating the argument,

$$E(I^{-1}|Q^n h) \leq \gamma^n E(I^{-1}|h) + \frac{C}{1-\gamma}, \tag{44}$$

or, using finiteness of $E(I^{-1}|h)$,

$$E(I^{-1}|Q^n h) \leq 1 + \frac{C}{1-\gamma} \tag{45}$$

when $n \geq K$, K suitably large.

An application of the Chebychev argument (20) gives

$$\int_0^r Q^n h(x) dx \leq r E(I^{-1}|Q^n h) \tag{46}$$

for any positive r. Therefore

$$\int_0^r Q^n h(x) dx \leq r \left(1 + \frac{C}{1-\gamma}\right), \quad n \geq K. \tag{47}$$

Now fix any $\varepsilon > 0$. According to Lemma 7.1 part (i), we require a $\delta > 0$ and a $K \in \mathbf{N}$ such that $n \geq K$ implies

$$\int_A Q^n h(x) dx < \varepsilon \tag{48}$$

whenever $m(A) \leq \delta$. For this purpose, consider the decomposition

$$\int_A Q^n h(x) dx = \int_{A \cap (0,r)} Q^n h(x) dx + \int_{A \cap (r,\infty)} Q^n h(x) dx. \tag{49}$$

Using (47) gives

$$\int_{A\cap(0,r)} Q^n h(x)dx \leq \int_0^r Q^n h(x)dx < \frac{\varepsilon}{2}. \tag{50}$$

when $r > 0$ is chosen to be sufficiently small and $n \geq K$.

Take r as given and consider the second term in (49).

$$\begin{aligned}
\int_{A\cap(r,\infty)} Q^n h(x)dx &= \int_{A\cap(r,\infty)} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}Q^{n-1}h(x)dx\right]dy \\
&= \int_0^\infty \left[\int_{A\cap(r,\infty)} \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}dy\right]Q^{n-1}h(x)dx. \\
&= \int_0^\infty \left[\int_{\frac{A\cap(r,\infty)}{q(x)}} \psi(z)dz\right]Q^{n-1}h(x)dx.
\end{aligned}$$

The term in brackets can be written as

$$G(x) \stackrel{\text{df}}{=} \int_{\frac{r}{q(x)}}^\infty \mathbf{1}_{\frac{A}{q(x)}}(z)\psi(z)dz. \tag{51}$$

By (22) it is possible to choose $\alpha > 0$ so small that

$$\int_{\frac{r}{q(\alpha)}}^\infty \psi(z)dz < \frac{\varepsilon}{2}. \tag{52}$$

Evidently

$$G(x) < \frac{\varepsilon}{2} \text{ whenever } x \leq \alpha. \tag{53}$$

Now consider the case where $x > \alpha$. Select $\delta' > 0$ such that

$$m(B) < \delta' \implies \int_B \psi(z)dz < \frac{\varepsilon}{2}. \tag{54}$$

(Existence of such a δ' follows from absolute continuity of the integral measure with respect to m. See for example Taylor [38], 2.8.15.) Define $\delta = q(\alpha)\delta'$. Then $x > \alpha$ and $m(A) < \delta$ implies

$$G(x) \leq \int_{\frac{A}{q(x)}} \psi(z)dz < \frac{\varepsilon}{2} \tag{55}$$

because $m(A/q(x)) = m(A)/q(x) \leq m(A)/q(\alpha) < \delta/q(\alpha) = \delta'$. Thus $m(A) < \delta$ implies $G(x) < \varepsilon/2$ for all x, and hence

$$\int_{A\cap(r,\infty)} Q^n h(x)dx < \frac{\varepsilon}{2}. \tag{56}$$

Combining (49), (50) and (56) gives

$$\int_A Q^n h(x)dx < \varepsilon \tag{57}$$

when $m(A) < \delta$ and $n \geq K$. Thus condition (i) of Lemma 7.1 holds for the collection $(Q^n h)_{n\geq K}$.

Next, condition (ii) of the lemma needs to be checked for the same h. Let I be the identity map on $\mathbf{R}_{++}$. We have

$$\begin{aligned}
E(I|Q^n h) &= \int_0^\infty yQ^n h(y)dy \\
&= \int_0^\infty y\left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}Q^{n-1}h(x)dx\right]dy \\
&= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}ydy\right]Q^{n-1}h(x)dx \\
&= \int_0^\infty q(x)Q^{n-1}h(x)dx \\
&\leq \int_0^\infty f(x)Q^{n-1}h(x)dx \\
&\leq f\left(\int_0^\infty xQ^{n-1}h(x)dx\right).
\end{aligned}$$

The last inequality is that of Jensen. Thus $E(I|Q^n h) \leq f(E(I|Q^{n-1}h))$. But then $E(I|Q^n h) \leq f^n(E(I|h))$, or, using the finiteness of $E(I|h)$,

$$E(I|Q^n h) \leq D + 1, \quad n \geq M, \tag{58}$$

where $D \overset{\text{df}}{=} \lim f^n(x)$ and M is chosen to be suitably large.[29]

[29] In other words, D is defined as the limit of the sequence generated by picking any positive x and applying f iteratively. That such a sequence will converge to a point D independent of the value of the initial point x follows from the properties of f.

By (20),

$$\int_r^\infty Q^n h(x)dx \leq \frac{E(I|Q^n h)}{r} \tag{59}$$

for any n and any positive r. Hence

$$\int_r^\infty Q^n h(x)dx \leq \frac{1}{r}(D+1), \quad n \geq M, \tag{60}$$

and condition (ii) of Lemma 7.1 holds for $(Q^n h)_{n\geq M}$.

Finally, define $N \stackrel{\text{df}}{=} \max(K, M)$. Then $(Q^n h)_{n\geq N}$ satisfies all of the conditions of the lemma, completing the proof of Lagrange stability.

□

7.2. **Proof of Strong Contractiveness.** A well-known sufficient condition for strong contractiveness of Markov operators on $L_1(U)$ is as follows.

Theorem 7.1. *For given measure space (U, Σ, μ), let p be a stochastic kernel and let P be the associated Markov operator. If $p > 0$ on U^2 then $(D(L_1(U)), P)$ is strongly contractive.*

Proof. See Lasota [24], Proposition 3.1. □

Proposition 7.2. *If ψ is everywhere positive then $(D(L_1(\mathbf{R}_{++})), Q)$ is strongly contractive.*

Proof. Immediate from (32) and Theorem 7.1. □

7.3. **Proof of the Theorem.** It is now possible to prove Theorem 4.1. Proposition 7.1 shows that if (u, f, ψ) satisfies the assumptions of part (i) of the theorem then the associated semidynamical system $(D(L_1(\mathbf{R}_{++})), Q)$ is Lagrange stable. Evidently $D(L_1(\mathbf{R}_{++}))$ is a closed convex subset of $L_1(\mathbf{R}_{++})$. Moreover Q is both linear and continuous. Hence all the conditions of Theorem 5.1 are satisfied, implying the existence of an equilibrium density. Since the equilibrium is a density, probability is not concentrated at zero (i.e. it is a nonzero equilibrium). Regarding part (ii) of the theorem, if, in addition, ψ is assumed to be everywhere positive, then $(D(L_1(\mathbf{R}_{++})), Q)$ is also strongly contractive by Proposition 7.2. Existence, uniqueness and stability of equilibrium now follow from Theorem 5.2.

8. Conclusion

Sufficient conditions have been given for existence, uniqueness and stability of equilibria in a stochastic optimal growth model with unbounded shock. The conditions are notable for their simplicity. In addition, a number of new proofs have been given for pertinent results in the field of discrete dynamical systems. The techniques developed in the paper are likely to prove useful for investigating equilibrium problems in a range of stochastic economic models.

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