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MARKETS THAT DON'T REPLICATE ANY OPTION

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Markets That Don't Replicate any Option*

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Abstract

It is well known from the work of S. Ross that a securities market is complete if and only if each call option can be replicated using available securities. The present short note announces the following surprising complementary result to Ross' important contribution.

- *If the number of securities is less than half the number of states of the world, then not a single option can be replicated by traded securities.*

This provides further strong motivation for relaxing the assumption of a perfect market in the theory of option pricing and portfolio insurance.

Keywords: Backward Induction, subgame perfect equilibrium, Nash equilibrium

JEL classification: C7

1. Introduction

Investors often wish to hedge a liability by replicating the payoff of an option by holding a portfolio of traded securities. For instance, an investor who wants to guarantee a minimum payoff on the downside while capturing the upside can do so by holding a riskless asset and a portfolio that replicates the payoff of a fiduciary call option. Alternatively, the investor can hold the underlying asset and another portfolio that replicates the payoff of a protective put option. For more on who would want portfolio insurance see H. Leland [7].

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However, it is well known from the work of S. A. Ross [10] that whenever the payoff of every call or put option can be replicated, the securities market must be complete and we are in a perfect Arrow–Debreu world; see also the works of R. C. Green and R. A. Jarrow [5] and D. J. Brown and S. A. Ross [3]. Ross's conclusion is a negative result since it basically asserts that in an incomplete market one cannot expect to replicate the payoff of each option even if the underlying asset is traded.

This note presents a surprising complementary result to Ross' insight concerning the replicability of options. It establishes that if in a strongly resolving two-period securities market the number of non-redundant securities is less than half the number of states, then not a single (non-trivial) option can be replicated. Here non-trivial options are options that are not exclusively in-the-money or at-the-money or out-of-the-money.

Our result provides further strong motivation for relaxing the assumption of a perfect market in the theory of option pricing and portfolio insurance. It shows that the incompleteness of the securities market presents tremendous leverage constraints. In the absence of perfect replication of any option, hedging strategies such as super-replication of contingent claims become important. That is, although an option cannot be replicated, super-replication hedging chooses the cheapest portfolio whose payoff strictly exceeds the contingent claim; see for example the works of M. Boadie, J. Cvitanic and H. M. Romer [2], V. Naik and R. Uppal [9], and C. Edirisinghe, V. Naik, and R. Uppal [4]. The analysis in this note also provides further motivation for the literature on the pricing of options under constraints; see for instance I. Karatzas and S. G. Kou [6], where option prices are strictly higher than those predicted by the F. Black and M. Scholes [1] option pricing model.

2. Portfolios and options

In the sequel, we first briefly describe the two period securities model. For further details see [10] and [8]. After stating Ross' characterization of complete markets we shall elucidate our main result.

2.1. Securities markets

In the two-period securities model there are S states of the world. Agents trade $r_1, r_2, \ldots, r_J$ non-redundant securities in period-zero whose period-one payoffs are state contingent claims. As usual, the asset returns matrix R is the $S \times J$ matrix whose columns are the non-redundant security vectors:

$$R = \begin{bmatrix} r_1(1) & r_2(1) & \ldots & r_J(1) \\ r_1(2) & r_2(2) & \ldots & r_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ r_1(S) & r_2(S) & \ldots & r_J(S) \end{bmatrix}$$

Portfolios are linear combinations of the non-redundant securities. A portfolio is therefore simply a vector in $\mathbb{R}^J$. Portfolios are considered as column vectors and the *payoff* of a portfolio θ is the S-dimensional vector $R\theta$.

A state contingent claim, which is a vector in $\mathbb{R}^S$, is said to be a *marketed payoff* if it lies in the asset span of the securities $r_1, r_2, \ldots, r_J$; in which case, there is a portfolio of the non-redundant securities whose payoff is the state contingent claim. In fact, if r is a marketed payoff, then there exists exactly one portfolio θ (called the *replicating portfolio* of r) such that $R\theta = r$. We assume that the riskless bond $\mathbf{1} = (1, 1, \ldots, 1)$ is marketed. As usual, the marketed security $(k, k, \ldots, k)$ will be denoted by $\mathbf{k}$.

If the asset span coincides with the space $\mathbb{R}^S$ of all contingent claims, i.e., if $J = S$, then markets are said to be *complete*. When $J < S$, the markets are *incomplete* in which case some state contingent claim cannot be replicated by a portfolio.

Recall that a two-period securities market is said to be *resolving* if the collection of securities $r_1, r_2, \ldots, r_J$ is resolving; in the sense that for any two distinct states s_1 and s_2 there is some security r_i such that $r_i(s_1) \neq r_i(s_2)$. We shall also say that a two-period securities market is *strongly resolving* if for any choice of J states and any contingent claim r there is a unique portfolio whose payoff coincides with r on the J selected states. This is, of course, equivalent to saying that any $J \times J$ square submatrix of the asset returns matrix R is non-singular. It is not difficult so see that:

- *If a two-period securities market is strongly resolving, then it is also resolving.*

It is also easy to see that the set of securities markets that are not strongly resolving is small. More precisely, we have the following result.

Proposition 2.1 *If* $\mathbf{A}$ *is the set of all* $S \times J$ *asset returns matrices for which the bond* $\mathbf{1}$ *is marketed, then the collection of all strongly resolving matrices of* $\mathbf{A}$ *is open and dense.*

The notion of a strongly resolving market is *generic* in the following sense. First, arbitrarily small perturbations to the payoffs of an asset returns matrix can make it strongly resolving. Secondly, small perturbations of strongly resolving asset matrices leave them strongly resolving. Of course, one can alternatively think of strongly-resolving markets in terms of a probability-theoretic measure of size whereby the probability of randomly choosing a strongly-resolving market is one.

2.2. Options

Let us begin with an arbitrary portfolio θ and a strike price k which can be positive, or negative or zero. A *call option* on the portfolio θ at the strike price k is the contingent claim $\max\{R\theta - \mathbf{k}, 0\}$. The call option captures the upside of holding the underlying portfolio θ and short-selling k units of the riskless bond. On the other hand, the *put option* of the portfolio θ at the strike price k captures the downside of short-selling the portfolio θ and holding k units of the bond; and it is the contingent claim $\max\{\mathbf{k} - R\theta, 0\}$.

We are only interested in non-trivial options, which are in-the-money in some states and out-of-the-money in some other states. That is, a call or put option for an underlying portfolio θ at a strike price k is *non-trivial* if there exist two states s and s' such that $R\theta(s) > k$ and $k > R\theta(s')$. Clearly, trivial options, which are exclusively in-the-money or at-the-money or out-of-the-money, can be disregarded.

Since we deal with incomplete markets, some call and put options need not be marketed. This means that there exist call or put options for which we cannot find portfolios of existing securities whose payoffs replicate these call or put options. Remarkably, from the following insightful result of S. A. Ross [10] we know that whenever every call option can be replicated the market is complete.

Theorem 2.2 (Ross) *A resolving two-period securities market is complete if and only if for every portfolio the call option at each strike price can be replicated.*

Ross' theorem poses a natural question: *Can we ever replicate an option if markets are not complete?* The main result of this parer provides a surprising answer to this question. (Its proof can be found in the Appendix.)

Theorem 2.3 *If in a strongly resolving securities market we have $J \leq \frac{1}{2}(S+1)$, then no non-trivial options can be replicated.*

In view of the fact that strongly resolving markets are generic, one interpretation of this theorem is the following: *If the number of securities is less than half of the number of states, then the likelihood of replicating at least one option is zero.* In other words, the set of markets in which one option can be replicated is negligible. Alternatively, an arbitrarily small perturbation to the payoff of an asset returns matrix results in a market in which no option can be replicated. Moreover, perturbations to the payoffs of the resulting market will not alleviate the problem.

3. Conclusion

It is well known from the work of A. S. Ross[10] that whenever the payoff of every call or put option can be replicated, the securities market must be complete and we must be in a perfect Arrow–Debreu world. Ross' theorem poses a natural question: *Can we ever replicate an option if markets are not complete?*

This paper shows that (in strongly resolving markets) if the number of securities is less than half the snumber of states then not a single (non-trivial) option can be replicated.

4. Appendix: The proof of Theorem 2.3

Assume that in a strongly resolving two-period securities market we have $J \leq \frac{1}{2}(S+1)$ or $S \geq 2J-1$. If the desired conclusion is not true, then there exists a portfolio θ such that the call option at some strike price k is non-trivial and can be replicated. Therefore, there exists a portfolio η whose payoff in each state s is $R\eta(s) = \max\{R\theta(s) - k, 0\}$ and the two sets of states

$$\mathcal{S}_1 = \{s\colon\ R\theta(s) > k\} \quad \text{and} \quad \mathcal{S}_2 = \{s\colon\ R\theta(s) \leq k\}$$

are both non-empty. It follows that either $\mathcal{S}_1$ or $\mathcal{S}_2$ contains at least J states. We shall show that this leads to a contradiction.

Assume first that $\mathcal{S}_2$ contains at least J states. Fix any J states $s_1, \ldots, s_J$ in $\mathcal{S}_2$ and let R' be the $J \times J$ submatrix of R with rows $s_1, \ldots, s_J$. Since the market is strongly resolving, R' is invertible. Now from $R'\eta(s_i) = \max\{R\theta(s_i) - k, 0\} = 0$ for all $i = 1, \ldots, J$ and the invertibility of R', we see that $\eta = 0$. This implies $\mathcal{S}_1 = \emptyset$, which is impossible.

Now suppose that $\mathcal{S}_1$ contains at least J states. Since the bond $\mathbf{1}$ is marketed, there exists a portfolio θ_0 such that $R\theta_0 = \mathbf{k}$. Now consider the new portfolio

$$\mu = \eta - \theta + \theta_0\,.$$

The portfolio μ is the put option for the portfolio θ at the strike price k. That is, its payoff at each state s is $R\mu(s) = \max\{0, k - R\theta(s)\}$. This implies that $R\mu(s) = 0$ for each state $s \in \mathcal{S}_1$. Since $\mathcal{S}_1$ has at least J states, it follows (as above) that $\mu = 0$. Therefore, $\eta = \theta - \theta_0$ and so $(R\theta - \mathbf{k})^+ = R\eta = R\theta - \mathbf{k}$. The latter shows that $R\theta(s) \geq k$ holds for each state s, contrary to the non-triviality of the call option. This completes the proof of the theorem.

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