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APPROXIMATE CALCULATION OF MOMENTS OF RUIN RELATED DISTRIBUTIONS

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1. Introduction and Notation

In the classical insurance surplus process, the insurer's surplus at time t is

$$U(t) = u + ct - S(t)$$

where u is the insurer's initial surplus, c is the insurer's premium income per unit time and $S(t)$ is the aggregate claim amount up to time t. The aggregate claims process is a compound Poisson process with Poisson parameter λ and individual claim amount distribution $P(x)$. We shall assume that $P(x)$ is a continuous distribution with density function $p(x)$, and we shall denote by p_k the $k-th$ moment of this distribution. We will further assume that $P(0) = 0$. The insurer's premium income per unit time will be written as $c = (1+\theta)\lambda p_1$ where $\theta > 0$ is the insurer's premium loading factor.

Let T denote the time to ruin for this process, starting from initial surplus u. We define

$$T = \begin{cases} \inf\{t : U(t) < 0\} \\ \infty \text{ if } U(t) \geq 0 \text{ for all } t \geq 0 \end{cases}$$

and define the ultimate ruin probability as

$$\psi(u) = \Pr(T < \infty)$$

The complementary probability is known as the survival probability and is denoted $\delta(u)$ so that $\delta(u) = 1 - \psi(u)$. We next define

$$G(u, y) = \Pr(T < \infty \quad \text{and} \quad U(T) > -y)$$

so that $G(u, y)$ denotes the probability that ruin occurs from initial surplus u and that the insurer's deficit at the time of ruin, or severity of ruin, is less than y. The associated (defective) density is denoted $g(u, y)$. Finally, we define

$$F(u, x) = \Pr(T < \infty \quad \text{and} \quad U(\tilde{T}) < x)$$

to be the probability that ruin occurs from initial surplus u and that the insurer's surplus immediately prior to ruin, denoted by $U(\tilde{T})$, is less than x. The associated (defective) density is denoted $f(u, x)$.

[1]This paper contains some results presented at the A.C. Aitken Centenary Conference in Dunedin, August 1995.

Dickson *et al (1995)* show how to calculate lower and upper bounds for $G(u,y)$ and $F(u,x)$. They also show that the average of these bounds can be used to approximate these functions. In this paper we will show that their method can also be applied to find moments of distributions. Let us define

$$E(Y^k|u) = \int_0^\infty y^k g(u,y)dy$$

$$E(\tilde{Y}^k|u) = E(Y^k|u)/\psi(u)$$

$$E(X^k|u) = \int_0^\infty x^k f(u,x)dx$$

$$E(\tilde{X}^k|u) = E(X^k|u)/\psi(u)$$

where the (defective) random variables X and Y denote the insurer's surplus prior to ruin and the insurer's deficit at ruin. In the following sections we will find bounds and approximations for these quantities.

2. The Severity of Ruin

Let us first make two assumptions which will apply throughout the paper. First, we assume that the surplus process has been rescaled such that p_1 is "large". In our examples we will set $p_1 = 100$. Second, we assume that u is an integer. Rescaling the surplus process has no effect on the ultimate ruin probability, nor on the related distributions, except, of course, for a change in scale. Asssuming that u is an integer is not particularly restrictive. For example, if $p_1 = 1$ in the original process and we use a rescaling factor of 100, then calculating for $u = 0, 1, 2, \ldots$ in the rescaled process is equivalent to calculating for $u = 0, 0.01, 0.02, \ldots$ in the original process.

As in Dickson et al (1995) our starting point is

$$g(u,y) = \frac{1}{\delta(0)} \left(\frac{\lambda}{c} \int_0^u p(y+z)\psi(u-z)dz + g(0,u+y) - \psi(u)g(0,y) \right) \qquad (2.1)$$

where

$$g(0,y) = \frac{\lambda}{c}(1 - P(y)) \qquad \text{for } y > 0$$

(see, for example, Gerber et al (1987)).

We can use these formulae to derive lower and upper bounds for $E(Y^k|u)$. Systematic derivation of these bounds is somewhat tedious, but basically straightforward. We will therefore illustrate the method to find bounds for $E(Y|u)$ and will quote the formulae for the bounds for $E(Y^2|u)$.

To find bounds for $E(Y|u)$ we first multiply equation (2.1) by y, then integrate with respect to y over $(0,\infty)$. This gives

$$E(Y|u) = \frac{1}{\delta(0)}\left(\frac{\lambda}{c}\int_0^\infty y\int_0^u p(y+z)\psi(u-z)dzdy + \int_0^\infty yg(0,u+y)dy - \psi(u)\int_0^\infty yg(0,y)dy\right)$$

Now replace the integral expressions as follows. First,

$$\int_0^\infty y\int_0^u p(y+z)\psi(u-z)dzdy = \int_0^u \psi(u-z)E[\max(0,W-z)]dz$$

where W has density function $p(x)$. This identity follows by changing the order of integration. Next

$$\int_0^\infty yg(0,u+y)dy = \frac{\lambda p_2}{2c} - \psi(0)E[\min(Y_0,u)]$$

where Y_0 has density function $(1-P(x))/p_1$, and finally

$$\int_0^\infty yg(0,y)dy = \frac{\lambda p_2}{2c}$$

Hence

$$\begin{aligned} E(Y|u) &= \frac{1}{\delta(0)}\left(\frac{\lambda}{c}\int_0^u \psi(u-z)E[\max(0,W-z)]dz + \frac{\lambda p_2}{2c} - \psi(0)E[\min(Y_0,u)] - \psi(u)\frac{\lambda p_2}{2c}\right) \\ &= \frac{\lambda}{c\delta(0)}\left(\sum_{j=0}^{u-1}\int_j^{j+1}\psi(u-z)E[\max(0,W-z)]dz + \frac{1}{2}\delta(u)p_2 - p_1E[\min(Y_0,u)]\right) \qquad (2.2)\end{aligned}$$

since $\psi(0) = \lambda p_1/c$. Now let $\delta^l(u)$ and $\delta^h(u)$ be lower and upper bounds for $\delta(u)$ and let $\psi^l(u)$ and $\psi^h(u)$ be lower and upper bounds for $\psi(u)$. In our examples we will calculate these bounds by the method described by Dufresne and Gerber (1989). It then follows from (2.2) that a lower bound for $E(Y|u)$ is

$$E^l(Y|u) = \frac{\lambda}{c\delta(0)}\left(\sum_{j=0}^{u-1}\psi^l(u-j)I_1(j) + \tfrac{1}{2}\delta^l(u)p_2 - p_1E[\min(Y_0,u)]\right)$$

where

$$I_1(j) = \int_j^{j+1} E[\max(0,W-z)]dz$$

and an upper bound is

$$E^h(Y|u) = \frac{\lambda}{c\delta(0)} \left(\sum_{j=0}^{u-1} \psi^h(u-j-1)I_1(j) + \tfrac{1}{2}\delta^h(u)p_2 - p_1 E[\min(Y_0, u)] \right)$$

Using the same approach we can show that a lower bound for $E(Y^2|u)$ is

$$E^l(Y^2|u) = \frac{1}{\delta(0)} \left(\frac{\lambda}{c} \sum_{j=0}^{u-1} \psi^l(u-j)I_2(j) + \phi(u) - \psi^h(u)\frac{\lambda p_3}{3c} \right)$$

where

$$\phi(u) = \frac{\lambda}{c}\left(\frac{p_3}{3} - p_1 E[\min(Y_0, u)^2]\right) - 2u\left(\frac{p_2}{2} - p_1 E[\min(Y_0, u)]\right)$$

and an upper bound is

$$E^h(Y^2|u) = \frac{1}{\delta(0)} \left(\frac{\lambda}{c} \sum_{j=0}^{u-1} \psi^h(u-j-1)I_2(j) + \phi(u) - \psi^l(u)\frac{\lambda p_3}{3c} \right)$$

Example 1: Let the individual claim amount distribution be exponential with mean 1 and let the premium loading factor be 10%. In this case we know that

$$E(Y|u) = \psi(u) = \exp\{-u/11\}/1.1$$

and

$$E(Y^2|u) = 2\psi(u)$$

since the distribution of the deficit, given that ruin occurs, is exponential with mean 1 (see, for example, Bowers et al (1987)). To apply the above methods to find bounds, let us rescale the surplus process by a factor of 100. Tables 1 and 2 below show computed values relating to $E(Y^k|u)$ for $k = 1, 2$. The legend for these tables is as follows:

(1) gives the value of u (before rescaling),

(2) gives the lower bound for $E(Y^k|u)$,

(3) gives an approximation to $E(Y^k|u)$, calculated as the average of the lower and upper bounds,

(4) gives the exact value of $E(Y^k|u)$,

(5) gives the upper bound for $E(Y^k|u)$,

(6) gives the exact value of $E(\tilde{Y}^k|u)$,

(7) gives an approximation to $E(\tilde{Y}^k|u)$, where $\psi(u)$ is approximated by the average of $\psi^l(u)$ and $\psi^h(u)$.

Table 1

(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	0.5520	0.5770	0.5770	0.6021	1.0000	1.0000
10	0.3345	0.3663	0.3663	0.3980	1.0000	1.0000
15	0.2022	0.2325	0.2325	0.2627	1.0000	1.0000
20	0.1220	0.1476	0.1476	0.1732	1.0000	1.0000
25	0.0733	0.0937	0.0937	0.1140	1.0000	1.0000

Table 2

(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	1.1040	1.1541	1.1541	1.2041	2.0000	2.0000
10	0.6690	0.7325	0.7325	0.7961	2.0000	2.0000
15	0.4044	0.4650	0.4650	0.5255	2.0000	2.0000
20	0.2439	0.2951	0.2951	0.3464	2.0000	2.0000
25	0.1467	0.1873	0.1873	0.2280	2.0000	2.0000

We can see from Tables 1 and 2 that although the computed bounds are not particularly tight, the average of the bounds provides an excellent approximation, both for conditional and unconditional moments.

Example 2: Let the individual claim amount distribution be Pareto with parameters 4 and 3 and let the premium loading factor be 10%. In this case there are no explicit results for moments when $u > 0$. Table 3 shows approximations to $E(\tilde{Y}|u)$ and $E(\tilde{Y}^2|u)$ calculated as the average of lower and upper bounds, again using a rescaling factor of 100. For comparison, the table also shows approximations to these quantities calculated by a recursive procedure described in Dickson et al (1995). The legend for Table 3 is as follows:

(1) gives the value of u (before rescaling),

(2) gives an approximation to $E(\tilde{Y}|u)$ calculated by averaging bounds for $E(Y|u)$ and $\psi(u)$,

(3) gives the approximation to $E(\tilde{Y}|u)$ given in Table 5 of Dickson et al (1995),

(4) gives an approximation to $E(\tilde{Y}^2|u)$ calculated by averaging bounds for $E(Y^2|u)$ and $\psi(u)$,

(5) gives the approximation to $E(\tilde{Y}^2|u)$ given in Table 5 of Dickson et al (1995).

Table 3

(1)	(2)	(3)	(4)	(5)
40	3.7589	3.7585	111.83	111.85
80	5.8100	5.8098	432.08	432.11
120	11.670	11.670	1,867.9	1,868.1
160	27.065	27.067	7,098.1	7,098.8
200	52.982	52.985	18,891	18,892

We can see that the results from the two methods are very similar. The method described by Dickson et al (1995) also employs a rescaling of the original process, and the numbers in the above table are also based on a rescaling factor of 100 giving us a meaningful comparison of the methods.

3. The Surplus Prior to Ruin

Dickson (1992) shows that

$$f(u,x) = \begin{cases} f(0,x)(\delta(u)-\delta(u-x))/\delta(0) & \text{for } 0<x<u \\ f(0,x)\delta(u)/\delta(0) & \text{for } x>u \end{cases}$$

from which it follows that

$$\begin{aligned} E(X^k|u) &= \int_0^\infty x^k f(0,x)\frac{\delta(u)}{\delta(0)}dx - \int_0^u x^k f(0,x)\frac{\delta(u-x)}{\delta(0)}dx \\ &= \frac{1}{\delta(0)}\left(\delta(u)\frac{\lambda}{c}\frac{p_{k+1}}{k+1} - \sum_{j=0}^{u-1}\int_j^{j+1} x^k f(0,x)\delta(u-x)dx\right) \end{aligned}$$

assuming, as in the previous section, that u is an integer, and using the fact that $f(0,x)=g(0,x)$ (see Gerber and Dufresne (1990) or Dickson (1992)). Hence a lower bound for $E(X^k|u)$ is

$$E^l(X^k|u) = \frac{1}{\delta(0)}\left(\delta^l(u)\frac{\lambda}{c}\frac{p_{k+1}}{k+1} - \sum_{j=0}^{u-1} H_k(j)\delta^h(u-j)dx\right)$$

where

$$H_k(j) = \int_j^{j+1} x^k f(0,x)dx$$

and an upper bound is

$$E^h(X^k|u) = \frac{1}{\delta(0)}\left(\delta^h(u)\frac{\lambda}{c}\frac{p_{k+1}}{k+1} - \sum_{j=0}^{u-1} H_k(j)\delta^l(u-j-1)dx\right)$$

Numerical evaluation of $H_k(j)$ poses no problem and so we can evaluate bounds for $E(X^k|u)$.

Example 3: Let the individual claim amount distribution be exponential with mean 1 and let the premium loading factor be 10%. In this case we can solve explicitly for $E(X^k|u)$. In particular,

$$E(X|u) = \psi(u)\left(2.1 - 1.1e^{-u/1.1}\right)$$

and

$$E(X^2|u) = \psi(u)\left(6.62 - (4.62 + 2.2u)e^{-u/1.1}\right)$$

with $\psi(u) = \exp\{-u/11\}/1.1$. Table 4 shows values for $E(X|u)$ and $E(\tilde{X}|u)$ and Table 5 shows the corresponding values for $E(X^2|u)$ and $E(\tilde{X}^2|u)$. As in previous examples, the scaling factor is 100. The legend for each table is as follows:

(1) gives the value of u (before rescaling),

(2) gives the lower bound for $E(X^k|u)$,

(3) gives an approximation to $E(X^k|u)$, calculated as the average of the lower and upper bounds,

(4) gives the exact value of $E(X^k|u)$,

(5) gives the upper bound for $E(X^k|u)$,

(6) gives the exact value of $E(\tilde{X}^k|u)$,

(7) gives an approximation to $E(\tilde{X}^k|u)$, where $\psi(u)$ is approximated by the average of $\psi^l(u)$ and $\psi^h(u)$.

Table 4

(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	1.1818	1.2050	1.2050	1.2282	2.0883	2.0883
10	0.7377	0.7691	0.7691	0.8005	2.0999	2.0999
15	0.4576	0.4882	0.4882	0.5188	2.1000	2.1000
20	0.2837	0.3099	0.3099	0.3360	2.1000	2.1000
25	0.1758	0.1967	0.1967	0.2176	2.1000	2.1000

Table 5

(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	3.6822	3.7243	3.7243	3.7664	6.4542	6.4542
10	2.3618	2.4235	2.4236	2.4852	6.6170	6.6170
15	1.4774	1.5390	1.5390	1.6006	6.6200	6.6199
20	0.9235	0.9769	0.9769	1.0301	6.6200	6.6199
25	0.5772	0.6201	0.6201	0.6629	6.6200	6.6198

This example has the same features as Example 1, namely that the bounds are not particularly tight, but the approximations are very good.

4. Conclusions

In this paper we have shown that the technique employed by Dickson *et al* (1995) to approximate the distributions of the surplus prior to ruin and the severity of ruin can also be applied to approximate moments of these distributions. The numerical accuracy of these methods is excellent and the methods can easily be extended to calculate higher moments than those illustrated in the paper.

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