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ON THE DISTRIBUTION OF THE DURATION OF NEGATIVE SURPLUS

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On the Distribution of the Duration of Negative Surplus

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ABSTRACT

In the classical risk model we allow the surplus process to continue if the surplus falls below zero. We consider the distributions of the duration of a single period of negative surplus and of the total duration of negative surplus. We derive explicit results where possible and show how to approximate these distributions.

KEYWORDS

Ruin theory; negative surplus; severity of ruin; failure rate; recursive calculation

1. INTRODUCTION AND NOTATION

The duration of negative surplus for the classical continuous time risk model was studied by dos Reis (1993). In his paper he finds the distribution of the number of occasions on which the surplus falls below zero and presents results for moments of the duration of a single period of negative surplus and the total duration of negative surplus.

The objective of this paper is to calculate the distribution of the duration of negative surplus, for both single periods and total duration. We will do this in two ways. First, we establish a formula for the density function of the duration of a single period of negative surplus. This leads to the distribution function which we apply in a recursion formula for the distribution function of the total duration of negative surplus. Second, we find results for a discrete time risk model and use these results to approximate the relevant quantities in the classical continuous time model.

In the classical continuous time risk model the insurer's surplus at time t is

$$U(t) = u + ct - S(t)$$

[1]Support from FISEG is gratefully acknowledged.

where u is the insurer's initial surplus, c is the insurer's premium income per unit time and $S(t)$ denotes aggregate claims up to time t. The aggregate claims process is a compound Poisson process with Poisson parameter λ. Individual claim amounts have distribution function $P(x)$, where $P(0) = 0$, density function $p(x)$ and mean p_1. We assume throughout that $c = (1 + \theta)\lambda p_1$, where $\theta > 0$ is the premium loading factor. We denote by $F(x,t)$ and $f(x,t)$ the distribution function and density function of aggregate claims up to time t.

The time to ruin is denoted T and defined by

$$T = \begin{cases} \inf\{t : U(t) < 0\} \\ \infty \quad \text{if } U(t) \geq 0 \quad \text{for all } t > 0 \end{cases}$$

and the probability of ultimate ruin from initial surplus u is $\psi(u) = \Pr(T < \infty)$. The survival probability is denoted $\delta(u)$ and defined as $\delta(u) = 1 - \psi(u)$. The finite time ruin probability $\Pr(T \leq t)$ is denoted $\psi(u,t)$. We define

$$G(u,y) = \Pr(T < \infty \quad \text{and} \quad U(T) > -y)$$

to be the probability that ruin occurs from initial surplus u and that the deficit at the time of ruin is less than y. The associated (defective) density is denoted $g(u,y)$ (see Gerber *et al* (1987) for details). Let $Y(u)$ denote the deficit at the time of ruin, given that ruin occurs from initial surplus u. We denote by $\tilde{G}(u,y)$ and $\tilde{g}(u,y)$ the distribution function and density function of $Y(u)$. Note that $\tilde{g}(u,y) = g(u,y)/\psi(u)$ and $\tilde{G}(u,y) = G(u,y)/\psi(u)$.

Next, we define T_x to be the time of the first passage of the surplus process through the fixed positive level x starting from initial surplus 0. Define $H(t,x)$ and $h(t,x)$ to be the distribution function and density function respectively of T_x. From Dickson and Gray (1984, Section 4) we have

$$\Pr(T_x = x/c) = e^{-\lambda x/c}$$

and

$$h(t,x) = \frac{x}{t} f(ct - x, t) \quad \text{for } t > x/c$$

We will let the surplus process continue if it falls below zero. We define N to be the number of occasions on which the surplus process falls below zero, T_i to be the duration of the i-th period of negative surplus, and TT to be the total duration of negative surplus, so that

$$TT = \sum_{i=1}^{N} T_i \qquad (= 0 \quad \text{if} \quad N = 0)$$

Dos Reis (1993) shows that

$$p_n = \Pr(N = n) = \begin{cases} \delta(u) & \text{for } n = 0 \\ \psi(u)[\psi(0)]^{n-1}\delta(0) & \text{for } n = 1, 2, 3, \dots \end{cases}$$

We introduce the following notation:

$$\begin{aligned} K(t) &= \Pr(TT \le t) \\ A(t) &= \Pr(T_1 \le t) \\ D(t) &= \Pr(T_i \le t) \quad \text{for } i = 2,3,4,\ldots \end{aligned}$$

and denote by $k(t)$, $a(t)$ and $d(t)$ the corresponding densities. $K(t)$ is of course a mixed distribution with $K(0) = \Pr(N = 0) = \delta(u)$. We denote by $\tilde{K}(t)$ the conditional distribution of TT given that ruin occurs. Finally, we define $a * d^{(n-1)*}(t)$ to be the density of $\sum_{i=1}^{n} T_i$ for $n = 1, 2, 3, \ldots$.

In calculating $K(t)$ there appear to be three cases to consider:

(i) when $u = 0$, in which case N has a geometric distribution and TT has a compound geometric distribution;

(ii) when $u > 0$ and the distribution of T_1 is the same as that of T_i, where $i > 1$; and

(iii) when $u > 0$ and the distribution of T_1 is different to that of T_i, where $i > 1$.

However, case (ii) can be treated easily by noting that the conditional distribution of TT given that ruin occurs is independent of the initial surplus. Thus, if we can calculate the distribution of TT when $u = 0$, we can also calculate it for $u > 0$ provided we can calculate $\delta(u)$. In the next two sections we consider the calculation of $K(t)$ for cases (i) and (iii). There is in fact just one situation for case (ii), and that is when the individual claim amount distribution is exponential. For this individual claim amount distribution the distribution of the severity of ruin, given that ruin occurs, is independent of the initial surplus: see, for example, Bowers *et al* (1986).

2. THE CASE WHEN $u = 0$

When $u = 0$ we can find lower and upper bounds for $K(t)$ if we can calculate $A(t)$, which in this case is the same as $D(t)$. Dufresne and Gerber (1989) describe a method for calculating lower and upper bounds for tail probabilities for a compound geometric distribution and we can easily apply this method to calculate bounds for $K(t)$.

It is straightforward to write down an expression for $A(t)$ using the fact that T_1 and $T \,|\, T < \infty$ have the same distribution, as noted by dos Reis (1993). Thus $A(t) = D(t) = \psi(0,t)/\psi(0)$.

EXAMPLE 1. Let the individual claim amount distribution be exponential with mean 1, and let $\lambda = 1$. Seal (1969, formula (4.10)) gives a formula for $\psi(0,t)$ from which we can easily find values for $A(t)$. Table 1 shows values of $A(t)$ and lower and upper bounds for $K(t)$ (for $t > 0$) when $\theta = 0.3$. In calculating these bounds we have bounded $A(t)$ with discrete distributions with span 0.01, using rescaled versions of Dufresne and Gerber's (1989) formulae (8) and (9).

Table 1

t	$A(t)$	Lower bound for $K(t)$	Upper bound for $K(t)$
0	0	0.2308	0.2308
10	0.9236	0.7124	0.7131
20	0.9654	0.8308	0.8313
30	0.9806	0.8908	0.8911
40	0.9880	0.9262	0.9263
50	0.9922	0.9485	0.9487

Figure 1 shows the density $a(t)$ for three values of θ, namely 0.1, 0.3 and 0.5. Although this figure illustrates the shape of the densities it hides the fact that each pair of densities has exactly one point of intersection.

3. THE CASE WHEN $u > 0$

When $u > 0$, we can still establish a recursion formula as follows.

THEOREM 1. $K(0) = \delta(u)$,

$$k(t) = \psi(u)\delta(0)a(t) + \psi(0)\int_0^t d(s)k(t-s)ds \qquad \text{for } t > 0 \qquad (3.1)$$

and

$$K(t) = \delta(u) + \psi(u)\delta(0)A(t) + \psi(0)\int_0^t d(s)K(t-s)ds \qquad \text{for } t > 0 \qquad (3.2)$$

Proof. As $TT = 0$ if and only if $N = 0$, $K(0) = \delta(u)$. Since $p_n = \psi(0)p_{n-1}$ for $n = 2, 3, 4, \ldots$ we can follow the proof of Theorem 2 of Sundt and Jewell (1981) to find $k(t)$. Thus:

$$\begin{aligned}
k(t) &= p_1 a(t) + \sum_{n=2}^{\infty} p_n\, a * d^{(n-1)*}(t) \\
&= p_1 a(t) + \sum_{n=1}^{\infty} p_{n+1}\, a * d^{n*}(t) \\
&= p_1 a(t) + \sum_{n=1}^{\infty} \psi(0) p_n \int_0^t d(s)\, a * d^{(n-1)*}(t-s)ds \\
&= p_1 a(t) + \psi(0)\int_0^t d(s) \sum_{n=1}^{\infty} p_n\, a * d^{(n-1)*}(t-s)ds \\
&= \psi(u)\delta(0)a(t) + \psi(0)\int_0^t d(s)k(t-s)ds
\end{aligned}$$

The result for $K(t)$ easily follows.

The special case when $a(t) = d(t)$ is covered by Theorem 2 of Sundt and Jewell (1981).

It is easy to show that we can still apply Dufresne and Gerber's (1989) method to find lower and upper bounds for $K(t)$. (We must apply a discrete version of (3.2) and this is given in Section 6.) In order to do so we require expressions for $a(t)$ and $A(t)$. We can find these by considering the deficit at the time of ruin and by using the density of the first passage time of the surplus process through a fixed level greater than its initial value. Thus we have

$$A(t) = \int_0^{ct} \tilde{g}(u,y)H(t,y)dy$$

and so

$$\begin{aligned} a(t) &= c\tilde{g}(u,ct)H(t,ct) + \int_0^{ct} \tilde{g}(u,y)h(t,y)dy \\ &= c\tilde{g}(u,ct)e^{-\lambda t} + \int_0^{ct} \tilde{g}(u,y)\ \frac{y}{t}\ f(ct-y,t)dy \end{aligned}$$

EXAMPLE 2. Let the individual claim amount distribution be exponential with mean $1/\beta$. Then $\tilde{g}(u,y) = \beta e^{-\beta y}$ and so

$$\begin{aligned} a(t) &= c\beta e^{-(\lambda+c\beta)t} + \int_0^{ct} \beta e^{-\beta y}\frac{y}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{(ct-y)^{n-1}\beta^n e^{-\beta(ct-y)}}{\Gamma(n)}dy \\ &= c\beta e^{-(\lambda+c\beta)t} + \frac{1}{t}\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\beta^{n+1}}{\Gamma(n)}e^{-\beta ct}\int_0^{ct} y(ct-y)^{n-1}dy \\ &= \sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\frac{(\beta c)^{n+1}}{(n+1)!}t^n e^{-(\lambda+\beta c)t} \end{aligned}$$

It follows that

$$A(t) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{(\beta c)^{n+1}}{(n+1)!}\frac{(2n)!}{(\lambda+c\beta)^{2n+1}}\Gamma((\lambda+c\beta)t, 2n+1)$$

where

$$\Gamma(x,\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-y}dy$$

and hence

$$\psi(0,t) = \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{(n+1)!}\frac{(\beta c)^n}{n!}\frac{(2n)!}{(\lambda+c\beta)^{2n+1}}\Gamma((\lambda+c\beta)t, 2n+1)$$

which is an alternative expression for $\psi(0,t)$ to that given by Seal (1969).

EXAMPLE 3. Let the individual claim amount distribution be Gamma(2,β), so that the distribution has mean $2/\beta$. Dos Reis (1993) shows that

$$\tilde{g}(u,y) = (1-\gamma(u))\beta e^{-\beta y} + \gamma(u)\beta^2 y e^{-\beta y}$$

where

$$\gamma(u) = \sum_{k=1}^{2} \frac{(\beta+R_k)^2}{3\beta+R_k} e^{R_k u} \Big/ \sum_{k=1}^{2} \frac{\beta+R_k}{3\beta+R_k}(3\beta+2R_k)e^{R_k u}$$

and

$$R_k = \frac{\lambda/c - 2\beta \pm \sqrt{(\lambda/c)^2 + 4\lambda\beta/c}}{2} \qquad \text{for } k = 1,2$$

Using the same approach as in Example 2 we find that

$$a(t) = (1-\gamma(u))\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\frac{(\beta c)^{2n+1}}{(2n+1)!} t^{2n} e^{-(\lambda+c\beta)t} + \gamma(u)\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} 2\frac{(\beta c)^{2n+2}}{(2n+2)!} t^{2n+1} e^{-(\lambda+c\beta)t}$$

and hence

$$\begin{aligned} A(t) &= (1-\gamma(u))\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\frac{(\beta c)^{2n+1}}{(2n+1)!}\frac{(3n)!}{(\lambda+c\beta)^{3n+1}}\Gamma((\lambda+c\beta)t, 3n+1) \\ &\quad + \gamma(u)\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} 2\frac{(\beta c)^{2n+2}}{(2n+2)!}\frac{(3n+1)!}{(\lambda+c\beta)^{3n+2}}\Gamma((\lambda+c\beta)t, 3n+2) \end{aligned}$$

Table 2 shows values of $A(t)$, and Table 3 shows values of $\tilde{K}(t)$ when $\beta = 2$, $\theta = 0.3$ and $\lambda = 1$ for $u = 0, 1, 2$ and 3. These latter values have been calculated by approximating $K(t)$ for $t > 0$ as the average of the lower and upper bounds, with both $A(t)$ and $D(t)$ being bounded exactly as in Example 1 in the calculation of the bounds for $K(t)$.

Table 2

t	$u=0$	$u=1$	$u=2$	$u=3$
2	0.7679	0.7966	0.7986	0.7988
4	0.8648	0.8818	0.8830	0.8831
6	0.9059	0.9178	0.9187	0.9187
8	0.9292	0.9382	0.9389	0.9389
10	0.9444	0.9515	0.9520	0.9520

Table 3

t	$u=0$	$u=1$	$u=2$	$u=3$
10	0.6909	0.6994	0.7000	0.7000
20	0.8374	0.8420	0.8423	0.8423
30	0.9048	0.9075	0.9077	0.9077
40	0.9412	0.9428	0.9430	0.9430
50	0.9624	0.9635	0.9635	0.9635

We can see that for this individual claim amount distribution the initial surplus has little effect on the conditional distribution of TT. The reason for this is that the function $\gamma(u)$ approaches its limiting value quickly, see dos Reis (1993, Section 7), and so given that ruin occurs, the initial surplus has little effect on the duration of the first period of negative surplus, and consequently on the total duration of negative surplus.

4. FURTHER RESULTS FOR THE CLASSICAL MODEL

In this section we will consider $\tilde{K}(t)$, $A(t)$ and $a(t)$ as functions of u, and will denote them as $\tilde{K}(t;u)$, $A(t;u)$ and $a(t;u)$ respectively.

Tables 2 and 3 suggest that for the parameters chosen for Example 3 both $A(t;u)$ and $\tilde{K}(t;u)$ are increasing functions of u. We can apply the following theorem to show that this is true.

THEOREM 2. *If $\tilde{G}(u,y)$ is an increasing (decreasing) function of u (for a fixed value of y), both $A(t;u)$ and $\tilde{K}(t;u)$ are increasing (decreasing) functions of u.*

Proof. Let $\tilde{p}_n = \Pr(N = n | N > 0)$ for $n = 1,2,3,\ldots$. Let $u > w \geq 0$ and let $\tilde{G}(u,y)$ be an increasing function of u. Then

$$\tilde{G}(u,ct) = \int_0^{ct} \tilde{g}(u,y)dy > \int_0^{ct} \tilde{g}(w,y)dy = \tilde{G}(w,ct)$$

$$\Rightarrow \int_0^{ct} \tilde{g}(u,y)H(t,y)dy > \int_0^{ct} \tilde{g}(w,y)H(t,y)dy$$

$$\Rightarrow A(t;u) > A(t;w)$$

Hence

$$\int_0^t a(r;u)dr > \int_0^t a(r;w)dr$$

$$\Rightarrow \int_0^t D^{(n-1)*}(t-r)a(r;u)dr > \int_0^t D^{(n-1)*}(t-r)a(r;w)dr \qquad \text{for } n = 2,3,4,\ldots$$

i.e., in an obvious notation, $A * D^{(n-1)*}(t;u) > A * D^{(n-1)*}(t;w)$. Since

$$\tilde{K}(t;u) = \tilde{p}_1 A(t;u) + \sum_{n=2}^{\infty} \tilde{p}_n A * D^{(n-1)*}(t;u)$$

the result follows.

When $\tilde{G}(u,y)$ is a decreasing function of u the results follow by reversing the above inequalities.

Dos Reis (1993) shows that when the individual claim amount distribution is Gamma(2,β)

$$\tilde{G}(u,y) = 1 - e^{-\beta y} - \gamma(u)\beta y e^{-\beta y}$$

and as $\gamma(u)$ is a decreasing function of u, $\tilde{G}(u,y)$ is an increasing function of u, as are $A(t;u)$ and $\tilde{K}(t;u)$.

A natural question to ask is under what circumstances is $\tilde{G}(u,y)$ an increasing (decreasing) function of u. We cannot provide a comprehensive answer to this question. However, we can indicate sufficient conditions for $\tilde{G}(u,y)$ to be a non-decreasing (non-increasing) function of u.

If the distribution $P(x)$ has a decreasing failure rate (DFR) then

$$\frac{1-P(x+y)}{1-P(x)} \geq 1 - P(y)$$

If the inequality is reversed, the distribution has an increasing failure rate (IFR). We can find results for $\tilde{G}(u,y)$ when $P(x)$ has either a DFR or an IFR.

Let $\tilde{f}(u,x)$ denote the conditional density of the surplus immediately prior to ruin given that ruin occurs. The conditioning makes $\tilde{f}(u,x)$ a proper density. We can rewrite equation (30) of Gerber (1973) as

$$\tilde{G}(u,y) = 1 - \int_0^\infty \tilde{f}(u,x)\frac{1-P(x+y)}{1-P(x)}dx$$

Since Dickson (1992) shows that the unconditional density of the surplus immediately prior to ruin, denoted $f(u,x)$, depends on the relationship between u and x, the above integral should really be written as the sum of two integrals, but we ignore this for ease of presentation. We can use Gerber's formula to prove the following two results.

THEOREM 3. *If $P(x)$ has a DFR, $\tilde{G}(u,y) \leq P(y)$. If $P(x)$ has an IFR, the inequality is reversed.*

Proof. If $P(x)$ has a DFR then

$$\tilde{G}(u,y) \leq 1 - (1-P(y))\int_0^\infty \tilde{f}(u,x)dx = P(y)$$

The inequality is reversed when $P(x)$ has an IFR.

COROLLARY. *If $P(x)$ has a DFR, $E(Y(u)) \geq p_1$. If $P(x)$ has an IFR, the inequality is reversed.*

Let us now assume that $\tilde{f}(u,x)$ is an increasing function of u.

THEOREM 4. *If $P(x)$ has a DFR, $\tilde{G}(u,y)$ is a non-increasing function of u. If $P(x)$ has an IFR, $\tilde{G}(u,y)$ is a non-decreasing function of u.*

Proof. Let $u > w \geq 0$. If $P(x)$ has a DFR then

$$\begin{aligned}\tilde{G}(w,y) - \tilde{G}(u,y) &= \int_0^\infty (\tilde{f}(u,x) - \tilde{f}(w,x))\frac{1-P(x+y)}{1-P(x)}dx \\ &\geq (1-P(y))\int_0^\infty (\tilde{f}(u,x) - \tilde{f}(w,x))dx \\ &= 0\end{aligned}$$

Hence $\tilde{G}(u,y)$ is a non-increasing function of u when $P(x)$ has a DFR. When $P(x)$ has an IFR, the result is similarly proved.

COROLLARY. *If $P(x)$ has a DFR then $E(Y(u))$ is a non-decreasing function of u. If $P(x)$ has an IFR then $E(Y(u))$ is a non-increasing function of u.*

Since $f(u,x)$ depends on $\psi(u)$ (see Dickson (1992) for formulae for $f(u,x)$) we can determine analytically whether $\tilde{f}(u,x)$ is an increasing or decreasing function of u only in a limited number of cases. However, as numerical algorithms exist which allow accurate calculation of $\psi(u)$ - see, for example, Dufresne and Gerber (1989) or Dickson and Waters (1991) - it is at least possible to determine numerically how $\tilde{f}(u,x)$ behaves as a function of u.

Note that $\tilde{f}(u,x)$ cannot be a decreasing function of u, since

$$\tilde{f}(u,x) = f(0,x)(\psi(u)^{-1} - 1)/\delta(0) \qquad \text{for } 0 \le u < x$$

5. A DISCRETE TIME RISK MODEL

In this section we will use the discrete time compound Poisson risk model discussed by Dickson and Waters (1991 and 1992) to obtain approximations to $K(t)$. The essential features of this model are as follows:

- individual claim amounts are distributed on the non-negative integers with mean β, where $\beta > 1$;
- the premium income per unit time is 1;
- the Poisson parameter for the expected number of claims per unit time is $1/[(1+\theta)\beta]$, so that θ is the premium loading factor.

For notational convenience we will use a subscript or superscript d to indicate quantities in this model that are analogues of quantities in the continuous model.

The insurer's surplus at time t, $t = 1, 2, 3, \ldots$, given initial surplus u (which we always assume to be a non-negative valued integer), is denoted $U_d(t)$ and defined by

$$U_d(t) = u + t - S_d(t)$$

$S_d(t)$ denotes aggregate claims up to time t, and has distribution function $F_d(x,t)$ and probability function $f_d(x,t)$. The time until ruin for this model is denoted T_d and defined as

$$T_d = \begin{cases} \min\{n : U_d(n) \le 0, n = 1, 2, 3, \ldots\} \\ \infty \text{ if } U_d(n) > 0 \text{ for } n = 1, 2, 3, \ldots \end{cases}$$

The ultimate ruin probability given initial surplus u is denoted $\psi_d(u)$ and defined as

$$\psi_d(u) = \Pr(T_d < \infty)$$

and the survival probability is defined as $\delta_d(u) = 1 - \psi_d(u)$. When $u = 0$ we have $\psi_d(0) = 1/(1+\theta)$ (see, for example, Dickson and Waters (1992)).

Define $g_d(u, y)$ to be the probability that ruin occurs from initial surplus u and that the deficit at the time of ruin is y, for $y = 0, 1, 2, \ldots$. Then

$$g_d(u, y) = \Pr(T_d < \infty \text{ and } -U_d(T_d) = y)$$

From equation (3.5) of Dickson and Waters (1992) we find that

$$g_d(0, y) = 1 - F_d(y, 1) \qquad \text{for } y = 0, 1, 2, \ldots$$

We can calculate $g_d(u, y)$ for $y = 1, 2, 3, \ldots$ from a recursion formula by adapting equation (1) of Dickson (1989). We can write

$$\delta_d(u+y) = \delta_d(u) + \sum_{j=0}^{y-1} g_d(u, j)\delta_d(y-j) \qquad \text{for } y = 1, 2, 3, \ldots$$

so that

$$g_d(u, y) = \frac{1}{\delta_d(1)}(\delta_d(u+y+1) - \delta_d(u) - \sum_{j=0}^{y-1} g_d(u, j)\delta_d(y+1-j))$$

with

$$g_d(u, 0) = \frac{1}{\delta_d(1)}(\delta_d(u+1) - \delta_d(u))$$

We use these formulae rather than adapt the algorithm of Dickson and Waters (1992) as that algorithm would be recursive in u whereas we require an algorithm that is recursive in y. We define $\tilde{g}_d(u, y) = g_d(u, y)/\psi_d(u)$.

As in Section 1, we allow the surplus process to continue if it falls below zero. The total duration of negative surplus is denoted TT^d and defined as

$$TT^d = \sum_{i=1}^{N_d} T_i^d \qquad (= 0 \text{ if } N_d = 0)$$

where T_i^d denotes the duration of the i-th period of negative surplus, and N_d denotes the number of periods of negative surplus. N_d has probability function

$$p_n^d = \Pr(N_d = n) = \begin{cases} \delta_d(u) & \text{for } n = 0 \\ \psi_d(u)[\psi_d(0)]^{n-1}\delta_d(0) & \text{for } n = 1, 2, 3, \ldots \end{cases}$$

In our examples, values of $\delta_d(u)$ will be calculated from the algorithm described by Dickson and Waters (1991).

In the continuous time model it is straightforward to define the duration of a single period of negative surplus as the time from which the surplus falls below zero to the time at which the surplus next attains zero. For the discrete time model our definition of $\psi_d(u)$ causes some problems in defining the duration of a single period of negative surplus. This is because a surplus of 0 (at $t > 0$) implies ruin. We resolve this problem by calculating approximate lower and upper values for $K(t)$.

We calculate upper values by defining the duration of a single period of negative surplus to be the time the surplus process takes to recover to 0 from the time at which it first falls below zero. If the surplus process falls to zero, we say that this results in a single period of negative surplus of duration 0. Thus, if the surplus level at successive time periods is $4, 0, 1, -1, 0, 0, 1$ we would say that there are three periods of negative surplus of durations $0, 1$ and 0 respectively. Corresponding to definitions in Section 1 we denote the relevant distribution functions by $K_h(t)$, $A_h(t)$ and $D_h(t)$, with respective probability functions $k_h(t)$, $a_h(t)$ and $d_h(t)$.

We calculate lower values by defining the duration of a single period of negative surplus to be 1 plus the time the surplus process takes to recover to level 0 from the time at which it first fell below 0. Thus, in the example in the previous paragraph, the three periods of negative surplus would have durations $1, 2$ and 1 respectively. In an obvious notation we have distribution functions $K_l(t)$, $A_l(t)$ and $D_l(t)$, with respective probability functions $k_l(t)$, $a_l(t)$ and $d_l(t)$.

Then

$$a_h(t) = a_l(t+1) \qquad \text{and} \qquad d_h(t) = d_l(t+1) \qquad \text{for } t = 0, 1, 2, \ldots \tag{5.1}$$

and $K_l(t) \leq K_h(t)$. For reasons given by Dickson and Waters (1991, Section 2) both $K_l((1+\theta)\beta t)$ and $K_h((1+\theta)\beta t)$, where $(1+\theta)\beta t$ is an integer, are approximations to $K(t)$. Although they are not lower and upper bounds for $K(t)$ it is reasonable to expect that $K(t)$ lies between these two values. The reason for this is that if the surplus in the continuous time model is positive at time $t-1$ and 0 at time t, then the duration of (possibly multiple periods of) negative surplus is greater than 0 and less than 1. Thus, our definitions understate and overstate the true duration of negative surplus in the continuous time model. However, we cannot say how the other approximations at work in the discrete time model affect our calculations and so we cannot say that $K_l((1+\theta)\beta t)$ and $K_h((1+\theta)\beta t)$ are lower and upper bounds for $K(t)$. We shall see, however, in our examples that with a suitably large value of β, $K_l((1+\theta)\beta t)$ and $K_h((1+\theta)\beta t)$ are very close together, effectively giving us an approximation to $K(t)$ as the bounds in Sections 2 and 3 did.

6. FORMULAE FOR THE DISCRETE TIME MODEL

Using the same reasoning as in Section 3 we can easily write down formulae for $a_h(t)$ for $t = 0, 1, 2, \ldots$. If the deficit at the time of ruin is j, $j = 1, 2, 3, \ldots$, the probability that the surplus returns to 0 at time $T_d + t$ (where $t \geq j$) is

$$\frac{j}{t}\Pr(S_d(t) = t - j) = \frac{j}{t} f_d(t-j,t)$$

(See Gerber (1979, p.21).) Hence

$$a_h(0) = \tilde{g}_d(u,0)$$

and for $t = 1,2,3,...$

$$a_h(t) = \sum_{j=1}^{t} \tilde{g}_d(u,j)\frac{j}{t} f_d(t-j,t) \tag{6.1}$$

We find $d_h(t)$, $t = 0,1,2,\ldots$, by setting $u = 0$. Values for $a_l(t)$ and $d_l(t)$ come from the relationships in (5.1).

In Section 2 we noted that $D(t) = \psi(0,t)/\psi(0)$. We can find an equivalent result for the discrete time model as follows. Let

$$\psi_d(u,t) = \Pr(T_d \leq t)$$

and let $\delta_d(u,t) = 1 - \psi_d(u,t)$. For brevity, let $f_j = f_d(j,1)$ for $j = 0,1,2,\ldots$. Since $S_d(t)$ is the sum of t i.i.d. random variables, each with probability function $\{f_j\}_{j=0}^{\infty}$, it follows that $f_d(j,t) = f_j^{t*}$. If we set $u = 0$ in formula (6.1) we can write for $t = 1,2,3,...$

$$\psi_d(0)d_h(t) = \sum_{j=1}^{t} g_d(0,j)\frac{j}{t} f_{t-j}^{t*} = \sum_{j=1}^{t}[1 - F_d(j,1)]\frac{j}{t} f_{t-j}^{t*}$$

Dickson and Waters (1991) show that

$$\delta_d(0,t) = \sum_{j=1}^{t} \frac{j}{t} f_{t-j}^{t*}$$

which means that

$$\psi_d(0)d_h(t) = \delta_d(0,t) - \sum_{j=1}^{t} F_d(j,1)\frac{j}{t} f_{t-j}^{t*}$$

Now

$$\begin{aligned}
\sum_{j=1}^{t} F_d(j,1)\frac{j}{t} f_{t-j}^{t*} &= \sum_{j=0}^{t}\sum_{r=0}^{j} f_r \frac{j}{t} f_{t-j}^{t*} \\
&= \sum_{j=0}^{t}\sum_{r=0}^{j} f_{j-r} \frac{j}{t} f_{t-j}^{t*} \\
&= \sum_{r=0}^{t}\sum_{j=r}^{t} f_{j-r} \frac{j}{t} f_{t-j}^{t*} \\
&= \sum_{r=0}^{t}\sum_{s=0}^{t-r} f_s \frac{r+s}{t} f_{t-r-s}^{t*}
\end{aligned}$$

The well-known convolution formula gives

$$\frac{r}{t}\sum_{s=0}^{t-r} f_s f_{t-r-s}^{t*} = \frac{r}{t} f_{t-r}^{(t+1)*}$$

and since

$$\sum_{s=0}^{t-r} \frac{s}{t-r} f_s f_{t-r-s}^{t*} = \frac{1}{t+1} f_{t-r}^{(t+1)*}$$

(see Panjer (1981)) it follows that

$$\begin{aligned}
\sum_{j=1}^{t} F_d(j,1)\frac{j}{t} f_{t-j}^{t*} &= \sum_{r=0}^{t}\left(\frac{r}{t}+\frac{t-r}{t}\frac{1}{t+1}\right) f_{t-r}^{(t+1)*} \\
&= \sum_{r=0}^{t} \frac{r+1}{t+1} f_{t-r}^{(t+1)*} \\
&= \sum_{r=1}^{t+1} \frac{r}{t+1} f_{t+1-r}^{(t+1)*} \\
&= \delta_d(0,t+1)
\end{aligned}$$

Hence

$$\psi_d(0)d_h(t) = \delta_d(0,t) - \delta_d(0,t+1) = \psi_d(0,t+1) - \psi_d(0,t)$$

Finally, since

$$\psi_d(0)d_h(0) = 1 - f_0 = \psi_d(0,1)$$

we find that

$$\frac{\psi_d(0,t)}{\psi_d(0)} = D_h(t-1) = D_l(t) \qquad \text{for } t = 1,2,3,\ldots \tag{6.2}$$

A more intuitive proof of this result comes from noting that for $t = 1,2,3,\ldots$, ruin occurs at time $t+1$ if

(i) the surplus at time t is j, where j can be any value from 1 to t, and the surplus has been greater than zero at all times prior to t, and

(ii) aggregate claims in the $(t+1)-th$ time period are greater than or equal to $j+1$.

From Gerber (1979) we know that the probability of (i) is $(j/t)f_d(t-j,t)$. Hence, summing over j gives

$$\Pr(T_d = t+1 \mid U_d(0) = 0) = \sum_{j=1}^{t} \frac{j}{t} f_d(t-j,t)[1 - F_d(j,1)] = \psi_d(0)d_h(t)$$

since $\tilde{g}_d(0,j) = (1 - F_d(j,1))/\psi_d(0)$, leading to

$$\psi_d(0,t+1) - \psi_d(0,t) = \psi_d(0)d_h(t)$$

for $t = 1,2,3,\ldots$, which is equivalent to (6.2). Like its continuous analogue, this result can also be explained by dual events.

The discrete versions of (3.1) and (3.2) are easily found giving

$$k_h(0) = K_h(0) = \delta_d(u) + \frac{a_h(0)\psi_d(u)\delta_d(0)}{1 - \psi_d(0)d_h(0)}$$

and for $t = 1,2,3,\ldots$

$$k_h(t) = \frac{\psi_d(u)\delta_d(0)a_h(t) + \psi_d(0)\sum_{j=1}^{t} d_h(j)k_h(t-j) - \psi_d(0)\delta_d(u)d_h(t)}{1 - \psi_d(0)d_h(0)}$$

and

$$K_h(t) = \frac{\delta_d(u) + \psi_d(u)\delta_d(0)A_h(t) + \psi_d(0)\sum_{j=1}^{t} d_h(j)K_h(t-j) - \psi_d(0)\delta_d(u)D_h(t)}{1 - \psi_d(0)d_h(0)}$$

with similar formulae for $k_l(t)$ and $K_l(t)$, $t = 0,1,2,\ldots$.

Rather than apply these formulae directly, we will modify them in order to cut down the amount of calculation required. We will adapt the truncation procedure proposed by De Vylder and Goovaerts (1988).

Define, for $u = 0,1,2,\ldots$ and $y = 0,1,2,\ldots$,

$$\tilde{G}_d(u,y) = G_d(u,y)/\psi_d(u)$$

to be the conditional distribution function of the severity of ruin given that ruin occurs from initial surplus u. Let ϵ be some small number and let k be the least integer such that

$$\tilde{G}_d(u,k) > 1 - \epsilon$$

Now define

$$\tilde{g}_d^{\epsilon}(u,y) = \begin{cases} \tilde{g}_d(u,y) & \text{for } y = 0,1,2,\ldots,k \\ 0 & \text{for } y = k+1,k+2,\ldots \end{cases}$$

$$a_h^{\epsilon}(0) = a_h(0)$$

$$a_h^{\epsilon}(t) = \sum_{j=1}^{t} \tilde{g}_d^{\epsilon}(u,j)\,\frac{j}{t}\,f_d(t-j,t) = \sum_{j=1}^{\min(k,t)} \tilde{g}_d(u,j)\,\frac{j}{t}\,f_d(t-j,t) \qquad \text{for } t = 1,2,3,\ldots$$

$$A_h^{\epsilon}(t) = \sum_{j=0}^{t} a_h^{\epsilon}(t)$$

Then, by noting that $(x/t)f_d(t-x,t)$ is the probability function of the first passage time of the surplus process to the (integer) level x from initial surplus zero, it follows that $A_h(t) - \epsilon \le A_h^\epsilon(t) \le A_h(t)$. Now define $d_h^\epsilon(t)$, $D_h^\epsilon(t)$ and ${}^\epsilon D_h^{n*}(t)$ in an obvious way to correspond to $d_h(t)$, $D_h(t)$ and $D_h^{n*}(t)$. The following result is easily proved.

RESULT. *For* $n = 1, 2, 3, \ldots$

$$A_h * D_h^{n*}(t) - (n+1)\epsilon \le A_h^\epsilon * {}^\epsilon D_h^{n*}(t) \le A_h * D_h^{n*}(t)$$

If we define

$$K_h^\epsilon(t) = p_0^d + p_1^d A_h^\epsilon(t) + \sum_{n=2}^{\infty} p_n^d \; A_h^\epsilon * {}^\epsilon D_h^{(n-1)*}(t)$$

then it follows that

$$K_h(t) - \epsilon E(N_d) \le K_h^\epsilon(t) \le K_h(t)$$

(See, for example, Dickson and Waters (1991, Section 6.)) In the next section we will calculate $K_h^\epsilon(t)$ and $K_l^\epsilon(t)$ rather than $K_h(t)$ and $K_l(t)$. Clearly the above arguments hold when the subscript h is replaced by l. In our calculations we set $\epsilon = 10^{-4}/E(N_d)$ so that the error in our computed values of $K_h(t)$ and $K_l(t)$ is at most 10^{-4}.

There is one other important point we wish to make about calculations. We will successively calculate $a_h(0), a_h(1), \ldots$. To calculate $a_h(t)$ where $t > 1$ we require values of $f_d(x,t)$ for $x = 0, 1, 2, \ldots, t-1$. Since we have already calculated $f_d(x, t-1)$, $x = 0, 1, 2, \ldots, t-2$, in the calculation of $a_h(t-1)$, it is computationally more efficient to calculate $f_d(x,t)$ for $x = 0, 1, 2, \ldots, t-2$ from the convolution formula

$$f_d(x,t) = \sum_{j=0}^{x} f_d(j,1) \; f_d(x-j, t-1)$$

as this requires fewer calculations than repeatedly applying Panjer's (1981) recursion formula. We do, however, apply the recursion formula to calculate $f_d(t-1,t)$ and, of course, to calculate $f_d(x,1)$ for $x = 0, 1, 2, \ldots, t$.

If we are interested in calculating $A_h(t)$ only, then it is possible to calculate this by employing a recursive calculation to find $a_h(t)$. For $t = 1, 2, 3, \ldots$, let

$$\alpha_h(t;u) = \psi_d(u) a_h(t) = \sum_{j=1}^{t} g_d(u,j) \frac{j}{t} f_d(t-j, t)$$

For a fixed value of t we can calculate $\alpha_h(t;u)$ recursively as follows. By conditioning on the surplus level at time 1, we have

$$g_d(1,j) = f_d(0,1)^{-1}(g_d(0,j) - f_d(j+1,1))$$

and for $u = 1, 2, 3, \ldots$

$$g_d(u+1,j) = f(0,1)^{-1}(g_d(u,j) - \sum_{i=1}^{u} f_d(i,1) g_d(u+1-i,j) - f_d(u+j+1,1))$$

Hence

$$\begin{aligned} \alpha_h(1,t) &= \sum_{j=1}^{t} g_d(1,j) \frac{j}{t} f_d(t-j,t) \\ &= f_d(0,1)^{-1} \sum_{j=1}^{t} [g_d(0,j) - f_d(j+1,1)] \frac{j}{t} f_d(t-j,t) \\ &= f_d(0,1)^{-1} \left(\alpha_h(0,t) - \sum_{j=1}^{t} f_d(j+1,1) \frac{j}{t} f_d(t-j,t) \right) \end{aligned}$$

and for $u = 1, 2, 3, \ldots$

$$\begin{aligned} \alpha_h(u+1,t) &= \sum_{j=1}^{t} g_d(u+1,j) \frac{j}{t} f_d(t-j,t) \\ &= f_d(0,1)^{-1} \left(\sum_{j=1}^{t} [g_d(u,j) - \sum_{i=1}^{u} f_d(i,1) g_d(u+1-i,j) \right. \\ &\qquad \left. - f_d(u+j+1,1)] \frac{j}{t} f_d(t-j,t) \right) \\ &= f_d(0,1)^{-1} \left(\alpha_h(u,t) - \sum_{j=1}^{t} \sum_{i=1}^{u} f_d(i,1) g_d(u+1-i,j) \frac{j}{t} f_d(t-j,t) \right. \\ &\qquad \left. - \sum_{j=1}^{t} f_d(u+j+1,1) \frac{j}{t} f_d(t-j,t) \right) \\ &= f_d(0,1)^{-1} \left(\alpha_h(u,t) - \sum_{i=1}^{u} f_d(i,1) \sum_{j=1}^{t} g_d(u+1-i,j) \frac{j}{t} f_d(t-j,t) \right. \\ &\qquad \left. - \sum_{j=1}^{t} f_d(u+j+1,1) \frac{j}{t} f_d(t-j,t) \right) \\ &= f_d(0,1)^{-1} \left(\alpha_h(u,t) - \sum_{i=1}^{u} f_d(i,1) \alpha_h(u+1-i,t) \right. \\ &\qquad \left. - \sum_{j=1}^{t} f_d(u+j+1,1) \frac{j}{t} f_d(t-j,t) \right) \end{aligned}$$

Since we can calculate $\psi_d(u)$ recursively, it is easy to find $a_h(t)$ and hence $A_h(t)$. Once again it is efficient to calculate values of $f_d(x,t)$ as described above.

7. EXAMPLES

EXAMPLE 4. In Example 1 we considered the case when the parameters in the continuous time model were $\lambda = 1$, $u = 0$, $\theta = 0.3$ and $P(x) = 1 - e^{-x}$, and we calculated lower and upper bounds for $K(t)$. Table 4 shows lower and upper values for $K(t)$ calculated from the formulae of the previous section for three values of β, namely 20, 50 and 100. The table also shows the bounds from Table 1. In calculating $f_d(x, t)$ we require a discrete distribution and $P(x)$ was discretised using the method of De Vylder and Goovaerts (1988).

Table 4

t	Lower Value $\beta = 20$	Lower Value $\beta = 50$	Lower Value $\beta = 100$	Lower Bound (Table 1)	Upper Bound (Table 1)	Upper Value $\beta = 100$	Upper Value $\beta = 50$	Upper Value $\beta = 20$
0	0.2308	0.2308	0.2308	0.2308	0.2308	0.2325	0.2343	0.2396
10	0.7114	0.7122	0.7125	0.7124	0.7131	0.7131	0.7135	0.7147
20	0.8302	0.8307	0.8309	0.8308	0.8313	0.8313	0.8315	0.8321
30	0.8904	0.8908	0.8909	0.8908	0.8911	0.8911	0.8913	0.8917
40	0.9259	0.9261	0.9262	0.9262	0.9263	0.9263	0.9264	0.9267
50	0.9483	0.9485	0.9485	0.9485	0.9487	0.9487	0.9487	0.9489

We make the following comments about Table 4:

(i) The greater the value of β, the closer together the lower and upper values are. This is not surprising. As the value of β decreases, the discrete time model is intuitively a poorer approximation to the continuous time model.

(ii) When $\beta = 100$ the lower and upper values are very similar to the bounds in Table 1. At first sight this seems reasonable since discretisation, albeit of different distributions, is on the same span. However, it is perhaps surprising given the number of approximations involved when applying the discrete time model.

(iii) When $\beta = 100$, the lower and upper values effectively give an approximation to $K(t)$ to three decimal places, as do the lower and upper bounds.

EXAMPLE 5. Let the individual claim amount distribution be Pareto(4,3) and let $\lambda = 1$ and $\theta = 0.3$. Table 5 shows approximate values of $\tilde{K}(t)$ for $u = 0, 5, 10, 20, 40$. These have been calculated by approximating $K(t)$ for $t > 0$ by the average of the lower and upper values calculated with $\beta = 20$, with $P(x)$ discretised as in Example 4.

Table 5

t	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 40$
10	0.5572	0.4990	0.4771	0.4454	0.3768
20	0.7046	0.6578	0.6372	0.6043	0.5258
30	0.7866	0.7492	0.7312	0.7005	0.6212
40	0.8395	0.8093	0.7938	0.7659	0.6890
50	0.8761	0.8515	0.8382	0.8132	0.7397

In this example, $\tilde{K}(t)$ appears to be a decreasing function of u for a fixed value of t. Table 6 shows some approximate values of $\tilde{G}(u, y)$, calculated from the algorithms described earlier, which suggest that $\tilde{G}(u, y)$ is a decreasing function of u for a given value of y. Thus, by Theorem 2, we would expect $\tilde{K}(t)$ to be a decreasing function of u.

Table 6

y	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 40$
1	0.5879	0.4422	0.4086	0.3692	0.2979
3	0.8755	0.7564	0.7140	0.6569	0.5429
5	0.9472	0.8699	0.8337	0.7788	0.6581
10	0.9877	0.9581	0.9377	0.8987	0.7937
20	0.9978	0.9901	0.9826	0.9639	0.8968

From Theorem 4 we might expect $\tilde{f}(u, x)$ to be an increasing function of u for a fixed value of x, since this Pareto distribution has a decreasing failure rate. Figure 2 shows approximate values of $\tilde{f}(u, x)$ as a function of u when $x = 2$ and 3. These values have been calculated using the formulae for $f(u, x)$ given by Dickson (1992), and the function $\psi(u)$ in these formulae has been calculated using the algorithms described earlier. These graphs indicate that in this case $\tilde{f}(u, x)$ is not an increasing function of u for a given value of x, and this pattern can be observed for other values of x. Thus, these figures suggest that when the individual claim amount distribution has a decreasing failure rate, it is not necessary for $\tilde{f}(u, x)$ to be an increasing function of u in order for $\tilde{G}(u, y)$ and $\tilde{K}(t)$ to be decreasing functions of u.

8. REFERENCES

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1986) *Actuarial Mathematics.* Society of Actuaries, Itasca, IL.

De Vylder, F. and Goovaerts, M.J. (1988) Recursive calculation of finite time ruin probabilities. *Insurance: Mathematics and Economics* 7, 1-8.

Dickson, D.C.M. (1989) Recursive calculation of the probability and severity of ruin. *Insurance: Mathematics and Economics* 8, 145-148.

Dickson, D.C.M. (1992) On the distribution of the surplus prior to ruin. *Insurance: Mathematics and Economics* 11, 191-207.

Dickson, D.C.M. and Gray, J.R. (1984) Approximations to ruin probability in the presence of an upper absorbing barrier. *Scandinavian Actuarial Journal,* 105-115.

Dickson, D.C.M. and Waters, H.R. (1991) Recursive calculation of survival probabilities. *ASTIN Bulletin* 21, 199-221.

Dickson, D.C.M. and Waters, H.R. (1992) The probability and severity of ruin in finite and in infinite time. *ASTIN Bulletin* 22, 177-190.

Dos Reis, A.E. (1993) How long is the surplus below zero? *Insurance: Mathematics and Economics* 12, 23-38.

Dufresne F. and Gerber, H.U. (1989) Three methods to calculate the probability of ruin. *ASTIN Bulletin* 19, 71-90.

Gerber, H.U. (1973) Martingales in risk theory. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 73, 205-216.

Gerber, H.U. (1979) *An Introduction to Mathematical Risk Theory.* SS Huebner Foundation Monograph Series No. 8. Distributed by R.Irwin, Homewood, IL.

Gerber, H.U., Goovaerts, M.J. and Kaas, R. (1987) On the probability and severity of ruin. *ASTIN Bulletin* 17, 151-163.

Panjer, H.H. (1981) Recursive calculation of a family of compound distributions. *ASTIN Bulletin* 12, 22-26.

Seal, H.L. (1969) *Stochastic Theory of a Risk Business.* Wiley, New York.

Sundt, B. and Jewell, W.S. (1981) Further results on recursive evaluation of compound distributions. *ASTIN Bulletin* 12, 27-39.

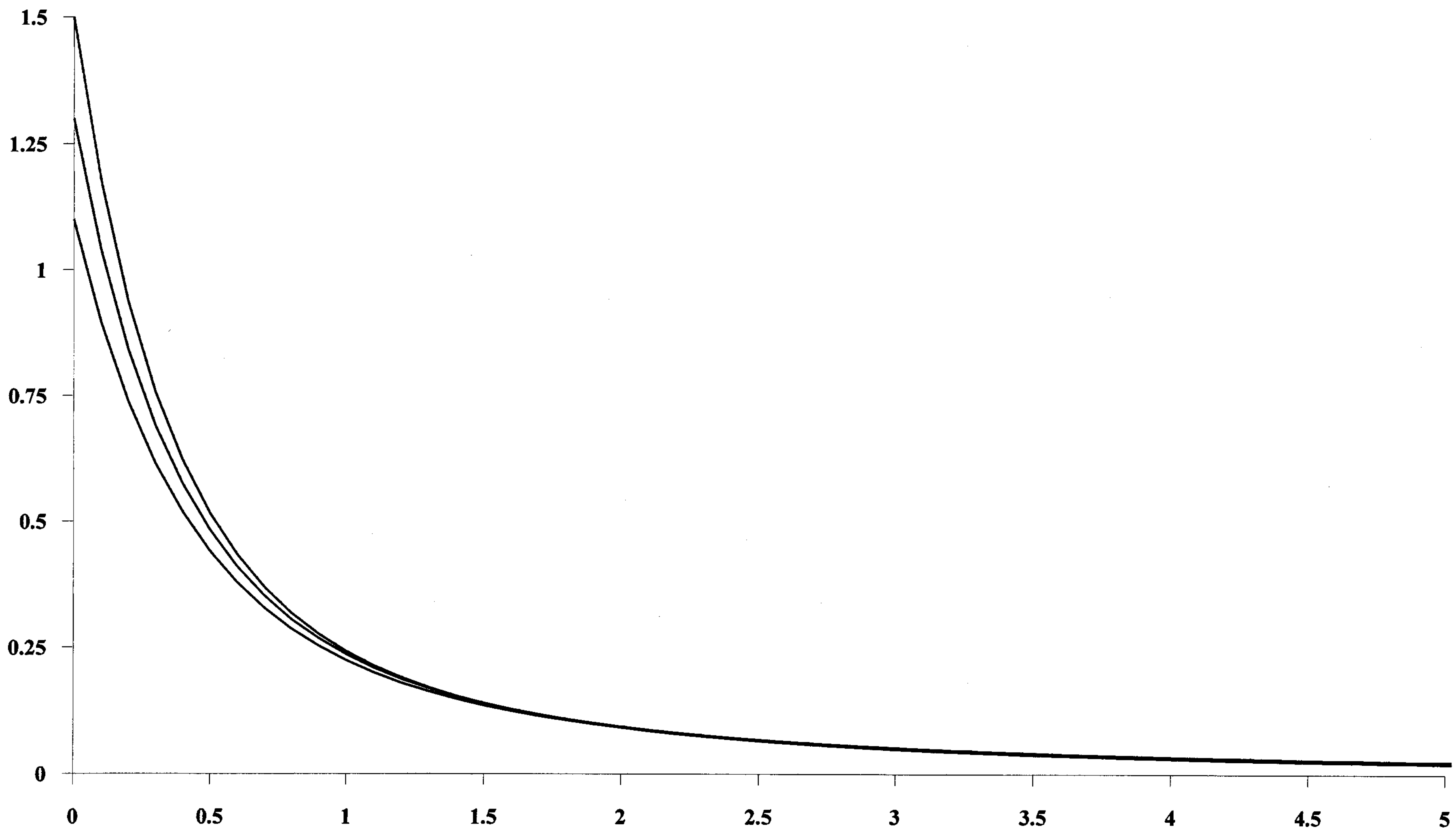

Figure 1. a(t) when $P(x) = 1- \exp\{-x\}$, $\lambda=1$ and $\theta=10\%$ (bottom), 30% (middle) and 50% (top).

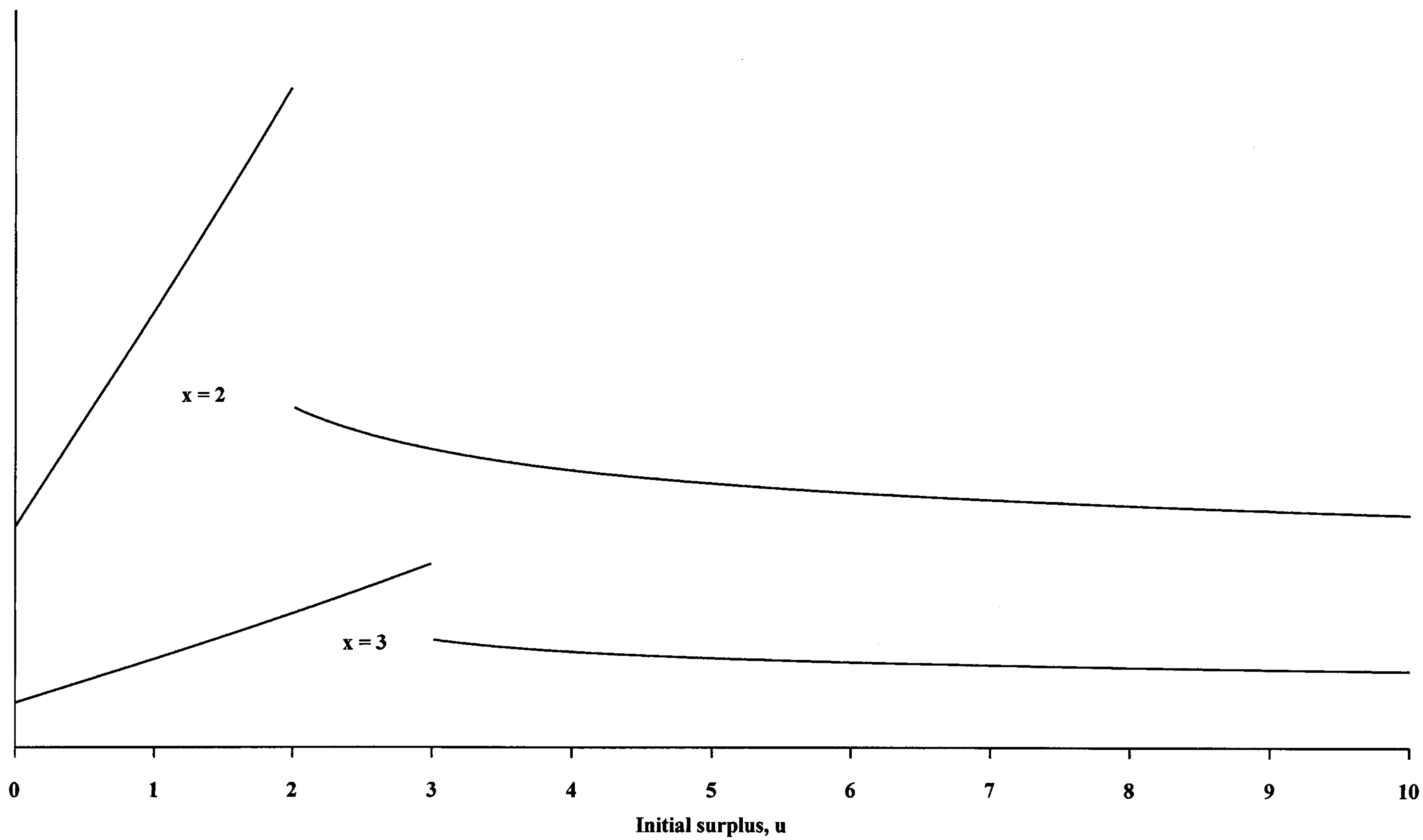

Figure 2. f(u,x) as a function of u when x = 2 and 3.

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30	MAR 96	THE EFFECT OF INTEREST ON NEGATIVE SURPLUS	David C M Dickson Alfred D Egídio dos Reis

31	MAR 96	RESERVING CONSECUTIVE LAYERS OF INWARDS EXCESS-OF-LOSS REINSURANCE	Greg Taylor
32	AUG 96	EFFECTIVE AND ETHICAL INSTITUTIONAL INVESTMENT	Anthony Asher
33	AUG 96	STOCHASTIC INVESTMENT MODELS: UNIT ROOTS, COINTEGRATION, STATE SPACE AND GARCH MODELS FOR AUSTRALIA	Michael Sherris Leanna Tedesco Ben Zehnwirth
34	AUG 96	THREE POWERFUL DIAGNOSTIC MODELS FOR LOSS RESERVING	Ben Zehnwirth
35	SEPT 96	KALMAN FILTERS WITH APPLICATIONS TO LOSS RESERVING	Ben Zehnwirth
36	OCT 96	RELATIVE REINSURANCE RETENTION LEVELS	David C M Dickson Howard R Waters
37	OCT 96	SMOOTHNESS CRITERIA FOR MULTI-DIMENSIONAL WHITTAKER GRADUATION	Greg Taylor
38	OCT 96	GEOGRAPHIC PREMIUM RATING BY WHITTAKER SPATIAL SMOOTHING	Greg Taylor
39	OCT 96	RISK, CAPITAL AND PROFIT IN INSURANCE	Greg Taylor
40	OCT 96	SETTING A BONUS-MALUS SCALE IN THE PRESENCE OF OTHER RATING FACTORS	Greg Taylor
41	NOV 96	CALCULATIONS AND DIAGNOSTICS FOR LINK RATION TECHNIQUES	Ben Zehnwirth Glen Barnett
42	DEC 96	VIDEO CONFERENCING IN ACTUARIAL STUDIES - A THREE YEAR CASE STUDY	David M Knox
43	DEC 96	ALTERNATIVE RETIREMENT INCOME ARRANGEMENTS AND LIFETIME INCOME INEQUALITY: LESSONS FROM AUSTRALIA	Margaret E Atkinson John Creedy David M Knox
44	JAN 97	AN ANALYSIS OF PENSIONER MORTALITY BY PRE-RETIREMENT INCOME	David M Knox Andrew Tomlin