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KALMAN FILTERS WITH APPLICATIONS TO LOSS RESERVING

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APPLICATIONS TO LOSS RESERVING

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Preface

These notes have grown out of a number of sets of lecture notes prepared for statistical courses and actuarial courses at Macquarie University and The University of Copenhagen. The notes are far from perfect and far from complete.

The notes are primarily intended to provide an introductory set of lectures on the subject of Kalman filtering and least squares estimation and its intimate connection to Bayesian estimation and recursive estimation. Applications to loss reserving as a way of overcoming multicollinearity problems are also given.

0. Prologue and Introduction

Much of the earliest stimulus for the development of estimation theory was provided by astronomical studies. The problems they addressed involved making inference as to the location of a 'heavenly body', from a sequence of imperfect observations.

The concept of least squares estimation is inextricably linked to Karl Friedrick Gauss, one of the "giants" of mathematics.

Gauss showed how it is possible *"to find the changes which the most likely values of the unknowns undergo when a new equation (observation) is adjoined and to determine the weights of these new observations"*. To use contemporary terminology, he developed an algorithm for sequentially or recursively updating the least squares parameter estimates on receipt of additional data.

Gauss originated <u>recursive</u> least squares estimation theory. He also used Maximum likelihood estimation techniques.

Plackett (1950) re-discovered <u>recursive</u> least squares estimation for the general linear regression model. Plackett's paper went almost unnoticed.

Kalman (1960) re-discovered <u>recursive</u> estimation in a more sophisticated form, as the core of the linear filtering and prediction theory evolved by control and systems theorists.

Kalman's results were obtained without the knowledge of Gauss and Plackett. Kalman used an argument based on orthogonal projection.

The Kalman filter recursive estimation algorithm can now be derived in various ways:

- orthogonal projection theory
- maximum likelihood
- Gauss - Markov (fixed) parameter regression
- Bayesian estimation

Kalman filter type algorithms have had an incredibly profound effect on data processing in the last 30 years.

Kalman (1960) extends the theory to allow for estimation of time-variable (varying) parameters or states, and to handle the analysis of non-stationary time series. So Kalman's major contribution is recursive least squares estimation in the context of varying parameter models.

In the econometrics literature Goldberger and Thiel(1961) also present a least squares procedure for recursively updating regression estimators.

The first paper on Kalman Filters that appeared in the statistical literature is by Duncan and Horn (1972), which essentially presents the Kalman Filter equations from the viewpoint of fixed parameter least squares theory, even though the filter is applicable to varying parameter models.

Kalman filtering and state space models now form a mature area of statistics. Kalman filter algorithms are now rightfully regarded as efficient computational solutions of the least squares method.

There have been a number of generalizations (extensions) of the Kalman filter to non-Gaussian observation error terms. West, Harrison, and Mignon (1985) consider the case where observations are from a general exponential sampling distribution and develop algorithms that update the conditional error covariance matrices. Zehnwirth (1988) develops algorithms for models with state dependent observation variances. Here the filter updates the unconditional error covariance matrix. Naik-Nimbalkar and Rajaishi (1995) consider a number of extended filters and smoothers and test their relative performances.

In the second part of this paper we consider the loss reserving problem from the point of view of trends in the three directions, **development year**, **accident year** and **payment year**. Given, the nonorthogonality of the payment year direction to the other two directions, one is faced with multi-collinearity problems when trying to estimate "too many" parameters in the accident year and payment year directions. A neat way of overcoming this problem is by introducing varying parameters in one of the directions. This is tantamount to using exponential smoothing with changing weights. The Kalman filter is used as a computational tool for computing the least squares estimates.

The paper tries to avoid where possible undue rigour and complexity.

1. Introduction to Least Squares

1.1 Least Squares and Gauss (1809)

The concept of least squares is inextricably linked to Gauss. Gauss showed how *"to find changes which the most likely values of the unknowns undergo when a new equation (observation) is adjoined and to determine the weights of these new determinations"*.

1.2 Simplest Statistical (Regression) Model

Gauss considered the simplest regression model

$$Y_1,\ldots,Y_n \overset{iid}{\sim} N(\mu,\sigma^2)$$

Equivalently,

$$Y_i = \mu + \varepsilon_i \tag{1.2.1}$$

where $\varepsilon_1,\ldots,\varepsilon_n \overset{iid}{\sim} N(\mu,\sigma^2)$.

The least squares estimator (lse) of μ minimizes the sum of squares J,

$$J = \sum_{i=1}^{n}(Y_i - \mu)^2 . \tag{1.2.2}$$

Now,

$$\frac{dJ}{d\mu} = -2\sum_{i-1}^{n}(Y_i - \mu) = 0 ,$$

so,

$$\hat{\mu} = \sum_{i=1}^{n} Y_i \Big/ n . \tag{1.2.3}$$

That is, the sample mean $\overline{Y}_n = \sum_{i=1}^{n} Y_i \Big/ n$ is the lse of μ.

The variance $V(\overline{Y}_n) = \dfrac{\sigma^2}{n}$.

1.2.1 Recursive Estimation

How is the estimator $\overline{Y}_n$ modified or updated on receipt of an additional observation Y_{n+1}?

We have:

$$\hat{\mu}_n = \overline{Y}_n = \sum_{i=1}^{n} Y_i \Big/ n$$

$$\hat{\mu}_{n+1} = \overline{Y}_{n+1} = \sum_{i=1}^{n+1} Y_i \Big/ (n+1)$$

So,

$$(n+1)\hat{\mu}_{n+1} = \sum_{i=1}^{n+1} Y_i$$

$$= \sum_{i=1}^{n} Y_i + Y_{n+1}$$

$$= n\hat{\mu}_n + Y_{n+1}$$

$$\therefore \quad \hat{\mu}_{n+1} = \frac{n}{n+1}\hat{\mu}_n + \frac{1}{n+1} Y_{n+1}$$

ie, $$\hat{\mu}_{n+1} = \hat{\mu}_n + \frac{1}{n+1}\left(Y_{n+1} - \hat{\mu}_n\right) \qquad (1.2.1.1)$$

(Relate to Apollo mission)

"$\left(Y_{n+1} - \hat{\mu}_n\right)$" is called the innovation or one step ahead prediction error. The revised estimate is the "old" + "weight" times "prediction error".

Moreover,

$$Var\left(\hat{\mu}_{n+1}\right) = \frac{\sigma^2}{n+1}$$

$$= \left(1 - \frac{1}{n+1}\right)\frac{\sigma^2}{n}$$

$$= \left(1 - \frac{1}{n+1}\right) Var\left(\hat{\mu}_n\right)$$

Put $Z_n = \frac{1}{n}$ and $C_n = \frac{\sigma^2}{n} = \sigma^2 Z_n$.

We have

$$(n+1) = (n) + 1$$

So,

$$Z_{n+1}^{-1} = Z_n^{-1} + 1$$

or

$$Z_{n+1} = \frac{Z_n}{1+Z_n} \qquad (1.2.1.2)$$

Equivalently,

$$Z_{n+1} = \frac{\sigma^{-2}}{\sigma^{-2} + C_n^{-2}} \qquad (1.2.1.3)$$

Therefore, the "weight" or "credibility" assigned to the new observation or information Y_{n+1} is proportional to its relative precision vis a vis the estimator $\hat{\mu}_n$, for

$$Var(Y_{n+1}) = \sigma^2$$

and

$$Var(\hat{\mu}_n) = C_n.$$

We can also write

$$C_{n+1} = (1 - Z_{n+1})C_n. \qquad (1.2.1.4)$$

The reader should note the computational differences between

$$\hat{\mu}_{n+1} = \sum_{i=1}^{n+1} Y_i \Big/ (n+1)$$

and

$$\hat{\mu}_{n+1} = \hat{\mu}_n + \frac{1}{n+1}(Y_{n+1} - \hat{\mu}_n).$$

The former requires more computer storage capacity and more arithmetic operations than the latter. This is the main reason that recursive formulae like (1.2.1.1) were used in the Apollo moon mission and are today common place in every guidance navigational system.

SUMMARY OF RESULTS

1. $\hat{\mu}_{n+1} = \hat{\mu}_n + Z_{n+1}(Y_{n+1} - \hat{\mu}_n)$

2. $Z_{n+1} = C_n(C_n + \sigma^2)^{-1}$

$$= \frac{\sigma^{-2}}{\sigma^{-2} + C_n^{-1}} \qquad (1.2.1.5)$$

3. $C_{n+1} = (1 - Z_{n+1})C_n$

4. $C_{n+1}^{-1} = C_n^{-1} + (\sigma^2)^{-1}$

where, $Z_n = \frac{1}{n}$ and $C_n = Var(\hat{\mu}_n) = \sigma^2 Z_n$. Equations 1 to 4 represent the Kalman filter algorithm for the simple regression model.

1.3 Recursive Regression Estimation

In this section we develop recursive formulae for the multiple regression model. These were first developed by Plackett (1950)

Consider the linear model:

$$\mathbf{Y}_i = \beta_0 + \beta_1 x_{1_i} + \ldots + \beta_p x_{p_i} + \varepsilon_i \qquad i = 1, \ldots, n$$

Let the vector

$$\mathbf{X}_i = (1, x_{1_i}, \ldots, x_{p_i})'.$$

That is $\mathbf{X}_i'$ is the 'design' for the ith observation. The sum of squares of deviation J is given by

$$J = \sum_{i=1}^{n} (\mathbf{Y}_i - \mathbf{X}_i'\boldsymbol{\beta})^2 . \tag{1.3.1}$$

The OLSE of $\boldsymbol{\beta}$ is obtained by minimizing (1.3.1) with respect to $\boldsymbol{\beta}$.

$$\frac{dJ}{d\boldsymbol{\beta}} = -2\sum_{i=1}^{n} \mathbf{X}_i(\mathbf{Y}_i - \mathbf{X}_i'\boldsymbol{\beta}) = 0$$

$$\therefore \quad \sum_{i=1}^{n} \mathbf{X}_i\mathbf{X}_i'\hat{\boldsymbol{\beta}}_n = \sum_{i=1}^{n} \mathbf{X}_i\mathbf{Y}_i \tag{1.3.2}$$

Note that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1' \\ \vdots \\ \mathbf{X}_n' \end{pmatrix}.$$

The quantity $\hat{\boldsymbol{\beta}}_n$ is the OLSE of $\boldsymbol{\beta}$ based on the observations $\left(y_1, x_{1_1}, \ldots, x_{p_1}\right)$, ..., $\left(y_n, x_{1_n}, \ldots, x_{p_n}\right)$, that is, at 'time n'.

How is the estimator $\hat{\boldsymbol{\beta}}_n$ updated on receipt of additional information $\left(y_{n+1}, x_{1_{n+1}}, \ldots, x_{p_{n+1}}\right)$?

Consider (1.3.2), viz.,

$$\sum_{i=1}^{n} \mathbf{X}_i\mathbf{X}_i'\hat{\boldsymbol{\beta}}_n = \sum_{i=1}^{n} \mathbf{X}_i\mathbf{Y}_i \tag{1.3.3}$$

Note,

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} \mathbf{X}_i\mathbf{X}_i', \qquad \mathbf{X}'\mathbf{Y} = \sum_{i=1}^{n} \mathbf{X}_i\mathbf{Y}_i .$$

Put

$$Var\left(\hat{\boldsymbol{\beta}}_n\right) = \boldsymbol{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{C}_n = \boldsymbol{\sigma}^2\mathbf{P}_n, \text{ say.}$$

Also, let $\mathbf{b}_n = \sum_{i=1}^{n} \mathbf{X}_i\mathbf{Y}_i$.

$$\mathbf{P}_{n+1}^{-1} = \sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'$$

$$= \mathbf{P}_n^{-1} + \mathbf{X}_{n+1}\mathbf{X}_{n+1}' \qquad (1.3.4)$$

and similarly

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \mathbf{X}_{n+1}\mathbf{Y}_{n+1} \qquad (1.3.5)$$

But,

$$\mathbf{P}_{n+1}^{-1}\hat{\boldsymbol{\beta}}_{n+1} = \mathbf{b}_{n+1}$$

$$= \mathbf{b}_n + \mathbf{X}_{n+1}\mathbf{Y}_{n+1} \qquad (1.3.6)$$

from (1.3.3)

So,

$$\mathbf{P}_{n+1}^{-1}\hat{\boldsymbol{\beta}}_{n+1} = \mathbf{P}_n^{-1}\hat{\boldsymbol{\beta}}_n + \mathbf{X}_{n+1}\mathbf{Y}_{n+1} \qquad (1.3.7)$$

Alternatively,

$$\hat{\boldsymbol{\beta}}_{n+1} = \mathbf{P}_{n+1}\mathbf{P}_n^{-1}\hat{\boldsymbol{\beta}}_n + \mathbf{P}_{n+1}\mathbf{X}_{n+1}\mathbf{Y}_{n+1} \qquad (1.3.8)$$

Let

$$\mathbf{K}_{n+1} = \mathbf{P}_{n+1}\mathbf{X}_{n+1} \qquad (1.3.9)$$

Since,

$$\mathbf{P}_{n+1}^{-1} = \mathbf{P}_n^{-1} + \mathbf{X}_{n+1}\mathbf{X}_{n+1}',$$

it follows that

$$\mathbf{I} = \mathbf{P}_{n+1}\mathbf{P}_n^{-1} + \mathbf{P}_{n+1}\mathbf{X}_{n+1}\mathbf{X}_{n+1}'$$

$$\therefore \quad \mathbf{P}_n = \mathbf{P}_{n+1} + \mathbf{P}_{n+1}\mathbf{X}_{n+1}\mathbf{X}_{n+1}'\mathbf{P}_n$$

so,

$$\mathbf{P}_n = \mathbf{P}_{n+1} + \mathbf{K}_{n+1}\mathbf{X}_{n+1}'\mathbf{P}_n$$

$$\therefore \quad \mathbf{P}_{n+1} = (\mathbf{I} - \mathbf{K}_{n+1}\mathbf{X}_{n+1}')\mathbf{P}_n \qquad (1.3.10)$$

If we let

$$\mathbf{Z}_{n+1} = \mathbf{K}_{n+1}\mathbf{X}'_{n+1}$$

then

$$\mathbf{C}_{n+1} = (\mathbf{I} - \mathbf{Z}_{n+1})\mathbf{C}_n \qquad (1.3.11)$$

which is identical to (1.3.5)!

Let's push on. Since

$$\mathbf{P}_{n+1} = \mathbf{P}_n - \mathbf{K}_{n+1}\mathbf{X}'_{n+1}\mathbf{P}_n$$

$$\Rightarrow \qquad \mathbf{P}_{n+1}\mathbf{X}_{n+1} = \mathbf{P}_n\mathbf{X}_{n+1} - \mathbf{K}_{n+1}\mathbf{X}'_{n+1}\mathbf{P}_n\mathbf{X}_{n+1}$$

i.e., $\mathbf{K}_{n+1} = \mathbf{P}_n\mathbf{X}_{n+1} - \mathbf{K}_{n+1}\mathbf{X}'_{n+1}\mathbf{P}_n\mathbf{X}_{n+1}$

$$\therefore \qquad \mathbf{K}_{n+1}\left(\mathbf{I} + \mathbf{X}'_{n+1}\mathbf{P}_n\mathbf{X}_{n+1}\right) = \mathbf{P}_n\mathbf{X}_{n+1}$$

So,

$$\mathbf{K}_{n+1} = \mathbf{P}_n\mathbf{X}_{n+1}\left(\mathbf{I} + \mathbf{X}'_{n+1}\mathbf{P}_n\mathbf{X}_{n+1}\right)^{-1} \qquad (1.3.12)$$

Equivalently,

$$\mathbf{K}_{n+1} = \mathbf{C}_n\mathbf{X}_{n+1}\left(\sigma^2\mathbf{I} + \mathbf{X}'_{n+1}\mathbf{C}_n\mathbf{X}_{n+1}\right)^{-1} \qquad (1.3.13)$$

and

$$\mathbf{Z}_{n+1} = \mathbf{C}_n\mathbf{X}_{n+1}\left(\sigma^2\mathbf{I} + \mathbf{C}_n\mathbf{X}_{n+1}\right)^{-1}\mathbf{X}'_{n+1} \qquad (1.3.14)$$

It can be shown using matrix manipulators that

$$\mathbf{Z}_{n+1} = \mathbf{C}_n\left(\mathbf{X}'_{n+1}\sigma^{-2}\mathbf{X}_{n+1} + \mathbf{C}_n^{-1}\right)^{-1}\mathbf{X}_{n+1}\sigma^{-2}\mathbf{X}'_{n+1} \qquad (1.3.15)$$

This formulae bears strong resemblance to 2 of (1.2.1.5).

From (1.3.8), (1.3.9) and (1.3.10) we have

$$\hat{\boldsymbol{\beta}}_{n+1} = (\mathbf{I} - \mathbf{K}_{n+1}\mathbf{X}'_{n+1})\hat{\boldsymbol{\beta}}_n + \mathbf{K}_{n+1}\mathbf{Y}_{n+1}$$

$$= \hat{\boldsymbol{\beta}}_n + \mathbf{K}_{n+1}\left(\mathbf{Y}_{n+1} - \mathbf{X}'_{n+1}\hat{\boldsymbol{\beta}}_n\right)$$

SUMMARY OF RECURSIONS

1. $$\hat{\boldsymbol{\beta}}_{n+1} = \hat{\boldsymbol{\beta}}_n + \mathbf{K}_{n+1}\left(\mathbf{Y}_{n+1} - \mathbf{X}'_{n+1}\hat{\boldsymbol{\beta}}_n\right)$$

2. $$\mathbf{K}_{n+1} = \mathbf{C}_n\mathbf{X}_{n+1}\left(\sigma^2\mathbf{I} + \mathbf{X}'_{n+1}\mathbf{C}_n\mathbf{X}_{n+1}\right)^{-1}$$

3. $$\mathbf{C}_{n+1} = \left(\mathbf{I} - \mathbf{K}\mathbf{X}'_{n+1}\right)\mathbf{C}_n \qquad (1.3.16)$$

4. $$\mathbf{C}^{-1}_{n+1} = \mathbf{C}^{-1}_n + \sigma^{-2}\mathbf{X}_{n+1}\mathbf{X}'_{n+1}$$

Note the similarities between (1.2.1.5) and (1.3.16).

2. Time Series

Consider the sequence of random variables $\{y_t \quad : t = 1,2,\ldots\}$. The index t denotes time. We study how past values $(y_n, y_{n-1}, \ldots, y_1)$ of y can be used to forecast future values y_{n+1}, y_{n+2}, ...

2.1 Constant mean model

Assume observations y_t are generated by

$$y_t = \mu + \varepsilon_t \tag{2.1.1}$$

where μ is a constant mean level and ε_t is a sequence of uncorrelated errors with variance σ_ε^2. If μ is known, the minimum mean square error forecast of a future observation $\hat{y}_{(n)+l}$ at time n is

$$\hat{y}_{(n)+l} = \mu .$$

Minimum

$$E\left(y_{n+1} - \hat{y}_{(n)+l}\right)^2 = E\left(y_{n+1} - \mu\right)^2 ,$$

so the best forecast is

$$\hat{y}_{(n)+l} = \mu .$$

If μ is unknown we estimate it from the data $D_n = (y_1, \ldots, y_n)$ by $\bar{y}_n$,

$$\hat{\mu}_n = \bar{y}_n = \sum_{i=1}^{n} y_i \Big/ n .$$

The 1-step-ahead forecast of y_{n+l} is

$$\hat{y}_{(n)+1} = \hat{\mu}_n = \bar{y}_n .$$

$$E\left(\left(y_{n+1} - \bar{y}_n\right)^2\right) = \sigma_\varepsilon^2\left(1 + \frac{1}{n}\right)$$

where

$$\hat{\sigma}^2_{\varepsilon_n} = \frac{1}{n-1}\sum_{t=1}^{n}(y_t - \bar{y}_n)^2 .$$

We have already developed a recursive formula for $\hat{\mu}_n$.

There is also a recursion for the residual sum of squares, $SSE(n)$,

$$SSE(n) = SSE(n-1) + (n-1)(\hat{\mu}_n - \hat{\mu}_{n-1})^2 + (y_n - \hat{\mu}_n)^2 .$$

This recursion is due to Plackett (1950). Indeed, Plackett (1950) developed a recursion for the estimate of σ^2 in the context of the multiple regression model of Section 1.3.

A recursive formula for one step ahead forecasts is given by

$$\hat{y}_{(n+1)+1} = \hat{y}_{(n)+1} + \frac{1}{n+1}\left(y_{n+1} - \hat{y}_{(n)+1}\right)$$

The new forecast is the previous forecast, corrected by a fixed fraction of the most recent forecast error, viz., $y_{n+1} - \hat{y}_{(n)+1}$.

2.2 Locally constant mean level

The model (2.1.1) assumes that the mean μ is constant over all time periods. This means that each observation carries the same weight in forecasting y_{n+1}.

Heuristically, it may make more sense to give more weight to the more recent observations especially if the mean moves slowly over time.

Let

$$\hat{y}_{(n)+1} = c\sum_{j=0}^{n-1} w^j y_{n-j}$$

$$= c\left(y_n + wy_{n-1} + \ldots + w^{n-1}y_1\right)$$

Here the weights decrease geometrically with the age of the observations. The quantity w: $|w| < 1$ is called the **discount** coefficient.

The quantity w should depend on how fast the mean level is moving.

$$\text{Sum of weights} = c + cw + \ldots + cw^{n-1}$$

$$= \frac{c\left(1-w^n\right)}{1-w}$$

In order that the sum of weights is 1, we have

$$c = \left(\frac{\left(1-w^n\right)}{1-w}\right)^{-1}$$

Indeed, if n is large $c \to 1-w$.

So,

$$\hat{y}_{(n)+1} = (1-w)\left(y_n + wy_{n-1} + \ldots + w^{n-1}y_1\right)$$

$$= (1-w)y_n + w\left[(1-w)\left(y_{n-1} + wy_{n-2} + \ldots + w^{n-2}y_1\right)\right]$$

$$= (1-w)y_n + w\hat{y}_{(n-1)+1}$$

Let $\alpha = 1-w$. The coefficient α is called the smoothing constant of exponential smoothing.

Let $S_n = \hat{y}_{(n)+1}$.

We have,

$$S_n = (1-w)y_n + w\hat{y}_{(n-1)+1}$$

$$= (1-w)y_n + wS_{n-1}$$

$$= y_n + (1-w)(y_n - S_{n-1})$$

$$\therefore \quad S_n = y_n + \alpha(y_n - S_{n-1})$$

Implementation

1. Choose an initial value S_0.

2. $\alpha = 1 - w$, determines the extent to which past observations influence the forecast.

The smoothing constant α is determined as follows:

The values S_t; $t = 1,\ldots,n$ are generated for various α's. For each α, compute the one-step-ahead forecast error

$$e_t = y_t - S_{t-1}$$

and Sum of Squares

$$SSE(\alpha) = \sum_{t=1}^{n} e_t^2 .$$

Choose $\alpha = \alpha_0$: $SSE(\alpha_0) = \min_{\alpha} SSE(\alpha)$.

Choose S_0 in conjunction with α

ie. if $\alpha = 0 \Rightarrow$ choose $S_0 = \bar{y}_n$.

if $\alpha = 1 \Rightarrow$ choose $S_0 = y_1$.

Note that α_0 is the maximum likelihood estimator of α in the Muth (1960) model described in Section 4.

2.3 Discount Least Squares

Suppose we minimize,

$$SSE = \sum_{j=0}^{n-1} w^j \left(y_{n-j} - \mu\right)^2$$

$$\frac{dSSE}{d\mu} = -2\sum_{j=0}^{n-1} w^j \left(y_{n-j} - \mu\right) = 0$$

$$\Rightarrow \quad \frac{1-w^n}{1-w}\hat{\mu} = y_n + wy_{n-1} + \ldots + w^{n-1} y_1$$

So, the forecast $\hat{y}_{(n)+1}$ is the optimal forecast based on a discount (weighted) least squares criterion.

3. Updating Least Squares Estimators

3.1 Updating Sample Based Estimators

Suppose X_1 is an estimate of μ with variance σ_0^2.

Suppose X_2 is an estimate of μ with variance σ_1^2.

The optimal weighted least squares estimate of μ, $\hat{\mu}$, is given by

$$\hat{\mu} = (1-z)X_1 + zX_2 \tag{3.1.1}$$

where

$$z = \frac{\sigma_1^{-2}}{\sigma_1^{-2} + \sigma_0^{-2}}$$

estimator $\hat{\mu}$ minimizes

$$\sum_{i=1}^{2} \sigma_{i-1}^{-2}(X_i - \mu)^2 .$$

We also have

$$\sigma^2 = Var(\hat{\mu}) = \left(\sigma_1^{-2} + \sigma_2^{-2}\right)^{-1}. \tag{3.1.2}$$

<u>Example</u>: Based on $Y_1,\ldots,Y_n \overset{iid}{\sim} N(\mu,\sigma^2)$, the lse of μ is $\overline{Y}_n$ with variance equal to $\frac{\sigma^2}{n}$.

If we have another observation Y_{n+1}, then $Y_{n+1} \sim N(\mu,\sigma^2)$.

Here

$$X_1 = \overline{Y}_n, \qquad \sigma_0^2 = \frac{\sigma^2}{n}$$

$$X_2 = Y_{n+1}, \qquad \sigma_1^2 = \sigma^2$$

$\therefore \qquad \hat{\mu} = (1-z)Y_{n+1} + z\overline{Y}$

where

$$z = \frac{n/\sigma^2}{1/\sigma^2 + n/\sigma^2} = \frac{n}{1+n}$$

So,

$$\hat{\mu} = \frac{1}{n+1} Y_{n+1} + \frac{n}{n+1} \overline{Y}_n$$

Also

$$\sigma^2 = \frac{\sigma^2}{n+1}.$$

These are identical to the results obtained in Section 1.2.1.

3.2 Bayesian Updating

Suppose

$$X|\mu \sim N(\mu, \sigma_1^2) \qquad \text{and} \qquad \mu|\mu_0 \sim N(\mu_0, \sigma_0^2)$$

then

$$\mu|X \sim N(m, \sigma^2)$$

where

$$m = E(\mu|X)$$

$$= (1-z)\mu_0 + zX$$

$$\therefore \qquad m = \mu_0 + z(X - \mu_0) \qquad (3.2.1)$$

where

$$z = \frac{\sigma_1^{-2}}{\sigma_1^{-2} + \sigma_0^{-2}}$$

and

$$\sigma^2 = Var(\mu|X)$$

$$= \left(\sigma_0^{-2} + \sigma_1^{-2}\right)^{-1} \qquad (3.2.2)$$

Case (i)

$$\sigma_0^2 \to \infty \quad \Rightarrow \quad w \to 1,\ m \to X \text{ and } \sigma^2 \to \sigma_1^2$$

Here we are completely ignorant about the prior estimate μ_0 and accordingly assign full credibility to X.

Case (ii)

$$\sigma_0^2 \to 0 \quad \Rightarrow \quad w \to 0,\ m \to \mu_0 \text{ and } \sigma^2 \to \sigma_0^2$$

Here we are absolutely certain about the prior estimate μ_0 and assign zero credibility to the sample information.

Interpretation

a) μ_0 is the 'prior' estimate of μ. The uncertainty associated with μ_0 is σ_0^2

b) X is the sampling information about μ. X is the estimate of μ based on the sample. The precision of X is determined by the value of σ_1^2.

c) m is the estimate of μ based on the two sources of information. It is a weighted average of X and μ_0. The weights are inversely proportional to the relative precisions.

d) σ^2 is the uncertainty associated with m.

Equivalence of Bayesian Updating and Sample based updating

It is important to recognise that Bayesian updating is equivalent to sample based updating provided the prior information (sample) are the same.

Equations (3.1.1) is the same as equation (3.2.1) and equation (3.1.2) is the same as (3.2.2) provided $\mu_0 = X_1$.

The Bayesian formulation can be written

$$X = \mu + \varepsilon \quad : \quad Var(\varepsilon) = \sigma_1^2$$

$$\mu = \mu_0 + \delta \quad : \quad Var(\delta) = \sigma_0^2$$

and the equivalent sampling formulation is

$$X = \mu + \varepsilon \quad : \quad Var(\varepsilon) = \sigma_1^2$$

$$\mu = \mu_0 + \delta \quad : \quad Var(\delta) = \sigma_0^2$$

In the Bayes formulation μ is random with μ_0 fixed, whereas in the sampling formulation μ_0 is random and μ is fixed.

In the Bayes formulation μ_0 represents a subjective guess at μ. In the sampling formulation μ_0 is an estimator of μ. based on a prior sample.

4. Exponential Smoothing Model - Muth (1960) and Locally Constant Mean

Muth (1960) presented a model for which exponential smoothing forecasts are optimal, that is, are minimum mean square error.

$$y_t = \mu_t + \varepsilon_t \quad t = 1,2,\ldots \tag{4.1}$$

$$\mu_t = \mu_{t-1} + \delta_t \tag{4.2}$$

$$Var(\varepsilon_t) = \sigma_\varepsilon^2$$

$$Var(\delta_t) = \sigma_\delta^2$$

The mean levels $\{\mu_t\}$ follow a random walk process. Since $\mu_t - \mu_{t-1}$ represents the mean trend, the model is also commonly known as a stochastic trend model or random walk plus noise model.

Note that if $\sigma_\delta^2 \equiv 0$, then this model is equivalent to the constant mean level model (2.1.1). If $\sigma_\delta^2 > 0$ then this model is equivalent as a locally constant model described in Sections 2.2 and 2.3.

At the other extreme, if $\sigma_\delta^2 \to \infty$, then this would suggest that the smoothing constant $\alpha \to 1$, $w \to 1$. We now use the Bayesians updating formula of Section 3.2 to develop a recursive algorithm for the estimate of μ_t.

Suppose we have reached time *n*-1, so that we have

$$\hat{\mu}_{n-1} \quad \text{(based on } y_1, \ldots, y_{n-1})$$

an estimate of μ_{n-1}, with variance σ_{n-1}^2.

According to equation (4.2) our estimate of μ_n, at time *n*-1, is

$$\hat{\mu}_{n|n-1} = \hat{\mu}_{n-1}$$

with variance

$$\sigma_{n|n-1}^2 = \sigma_{n-1}^2 + \sigma_\delta^2, \qquad \text{from equation (4.2).}$$

So before we observe y_n, we have the prior estimate $\hat{\mu}_{n|n-1}$ of μ_n with variance $\sigma_{n|n-1}^2$.

Based on the datum y_n, our estimate of μ_n is y_n, with variance σ_ε^2.

Therefore our updated estimate of μ_n, based on equation (3.2.1), is,

$$\hat{\mu}_n = (1-z)\hat{\mu}_{n|n-1} + zy_n$$
$$= \hat{\mu}_{n|n-1} + z\left(y_n - \hat{\mu}_{n|n-1}\right)$$

where,

$$1 - z = \frac{\sigma_{n|n-1}^{-2}}{\sigma_{n|n-1}^{-2} + \sigma_\varepsilon^{-2}}; \quad z = \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_{n|n-1}^{-2}}.$$

That is,

$$z = \sigma_{n|n-1}^2\left(\sigma_\varepsilon^2 + \sigma_{n|n-1}^2\right)^{-1}$$

Moreover,

$$\sigma_n^2 = (1-z)\sigma_{n|n-1}^2.$$

So we have

$$\hat{\mu}_n = \hat{\mu}_{n-1} + \kappa_n\left(y_n - \hat{\mu}_{n-1}\right) \qquad (1)$$

$$\kappa_n = \sigma_{n|n-1}^2\left(\sigma_{n|n-1}^2 + \sigma_\varepsilon^2\right)^{-1} = z_n \qquad (2)$$

$$\sigma_{n|n-1}^2 = \sigma_{n-1}^2 + \sigma_\varepsilon^2 \qquad (3) \qquad\qquad (4.3)$$

$$\sigma_n^2 = (1-\kappa_n)\sigma_{n|n-1}^2 \qquad (4)$$

These represent the Kalman filter algorithm for updating $\{\mu_n\}$ for the model described by (4.1) and (4.2).

The equations

$$y_t = \mu_t + \varepsilon_t \quad : \qquad Var(\varepsilon_t) = \sigma_\varepsilon^2, \qquad t = 1,2,\ldots$$

are called the <u>measurement</u> or <u>observation</u> equations.

We also have

$$\mu_t = \mu_{t-1} + \delta_t : \qquad Var(\delta_t) = \sigma_\delta^2, \qquad t = 2,3\ldots$$

These are called the system or state equations.

The two equations (measurement and system) describe a Dynamic Linear Model (DLM).

Implementation of Recursions

Set

$$\hat{\mu}_{1|0} = \mu_0 \qquad \text{and} \qquad \sigma_{1|0}^2 = Var(\delta_1)$$

Then,

$$\kappa_1 = \sigma_{1|0}^2\left(\sigma_{1|0}^2 + \sigma_\varepsilon^2\right)^{-1}$$

$$\hat{\mu}_1 = \mu_0 + \kappa_1(y_1 - \mu_0),$$

and

$$\sigma_1^2 = (1-\kappa_1)\sigma_{1|0}^2.$$

Go to (3) → (2) → (1) → (4) → (3) → (2) → (1) etc

Special Cases:

Case (i): $\sigma_{1|0}^2 \to \infty \quad \Rightarrow \quad \kappa_1 \to 1$ and $\hat{\mu}_1 = y_1$

So if we are completely uncertain about a prior estimate of μ_1 it does not matter what μ_0 is !

Case (ii) If $\sigma_\delta^2 = 0$, then by induction we can show that

$$\hat{\mu}_n = \frac{\sigma_{1|0}^{-2}}{\sigma_{1|0}^{-2} + \left(\sigma_\varepsilon/\sqrt{n}\right)^{-2}}\mu_0 + \frac{\left(\sigma_\varepsilon/\sqrt{n}\right)^{-2}}{\sigma_{1|0}^{-2} + \left(\sigma_\varepsilon/\sqrt{n}\right)^{-2}}\bar{y}_n$$

This is a weighted average of the 'prior' estimate μ_0 and the sample based estimate $\bar{y}_n$.

The variance of the estimate $\hat{\mu}_n$ is

$$\sigma_n^2 = \left(n\sigma_\varepsilon^{-2} + \sigma_{1|0}^{-2}\right)^{-1}.$$

Also,

$$\kappa_n = z_n = \frac{\sigma_{1|0}^2}{n\sigma_{1|0}^2 + \sigma_\varepsilon^2}$$

and of course

$$\sigma_{n|n-1}^2 = \sigma_{n-1}^2 \quad \text{(since } \sigma_\delta^2 = 0\text{)}$$

If in addition to $\sigma_\delta^2 = 0$, we let $\sigma_{1|0}^2 \to \infty$ (that is, our prior estimate is extremely 'vague') we have

$$\left.\begin{aligned} \hat{\mu}_n &= \bar{y}_n \\ \sigma_n^2 &= \frac{\sigma_\varepsilon^2}{n} \end{aligned}\right\} \text{ as expected.}$$

This is the constant mean level model (2.1.1)

Case (iii): Exponential Smoothing - Locally Constant Mean

$$\hat{\mu}_n = \hat{\mu}_{n-1} + \kappa_n\left(y_n - \hat{\mu}_{n-1}\right) \qquad (1)$$

$$\kappa_n = \sigma_{n|n-1}^2\left(\sigma_{n|n-1}^2 + \sigma_\varepsilon^2\right)^{-1} \qquad (2)$$

$$\sigma_{n|n-1}^2 = \sigma_{n-1}^2 + \sigma_\delta^2 \qquad (3)$$

$$\sigma_n^2 = \left(1 - \kappa_n\right)\sigma_{n|n-1}^2 \qquad (4)$$

With

$$\mu_1 = \mu_0 + \delta_1 \quad \text{and} \quad \sigma_{1|0}^2 = Var(\delta_1)$$

From (4) we have

$$\sigma_n^2 = \frac{\sigma_\varepsilon^2}{\sigma_{n|n-1}^2 + \sigma_\varepsilon^2} . \sigma_{n|n-1}^2$$

$$= \frac{\sigma_\varepsilon^2}{\sigma_{n-1}^2 + \sigma_\delta^2 + \sigma_\varepsilon^2}\left(\sigma_{n-1}^2 + \sigma_\delta^2\right)$$

Given that σ_ε^2, σ_δ^2 are constants, we would 'expect' $\sigma_n^2 \to c$ (say).

So

$$c = \frac{\sigma_\varepsilon^2}{c + \sigma_\delta^2 + \sigma_\varepsilon^2}\left(c + \sigma_\delta^2\right)$$

$$\therefore \quad c^2 + c\left(\sigma_\delta^2 + \sigma_\varepsilon^2\right) = c\sigma_\varepsilon^2 + \sigma_\varepsilon^2\sigma_\delta^2$$

$$\Rightarrow \quad c^2 + c\sigma_\delta^2 - \sigma_\varepsilon^2\sigma_\delta^2 = 0$$

$$\therefore \quad c = \frac{-\sigma_\delta^2 \pm \sqrt{\sigma_\delta^4 + 4\sigma_\varepsilon^2\sigma_\delta^2}}{2}$$

$$= \frac{-\sigma_\delta^2 + \sqrt{\sigma_\delta^4 + 4\sigma_\varepsilon^2\sigma_\delta^2}}{2}$$

$$\therefore \quad \kappa_n \to \frac{c + \sigma_\delta^2}{c + \sigma_\partial^2 + \sigma_\varepsilon^2} = \kappa \text{, say}$$

$$\therefore \quad \hat{\mu}_n = \hat{\mu}_{n-1} + \kappa\left(y_n - \hat{\mu}_{n-1}\right)$$

$$\therefore \quad S_{n+1} = S_n + \kappa\left(y_n - S_n\right)$$

So κ is equivalent to α in exponential smoothing. The estimator α_0 in Section 2.2, is the maximum likelihood estimator in the Muth (1960) model.

<u>Implementation</u>

Data: $y_1, \ldots, y_n$

Estimate: $\hat{\sigma}_\varepsilon^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$.

Find $\sigma_\delta^2 : \sum\left(y_n - \hat{\mu}_{n-1}\right)^2$ is minimized - just like in exponential smoothing.

4.1 Varying parameters, weighted least squares and smoothing

Suppose

$$y_{11},\ldots,y_{1n_1} \overset{iid}{\sim} N(\alpha_1,\sigma^2) \qquad \text{and} \qquad y_{21},\ldots,y_{2n_2} \overset{iid}{\sim} N(\alpha_2,\sigma^2)$$

That is,

$$\bar{y}_1 \sim N\left(\alpha_1,\frac{\sigma^2}{n_1}\right) \qquad \text{and} \qquad \bar{y}_2 \sim N\left(\alpha_2,\frac{\sigma^2}{n_2}\right).$$

Equivalently,

$$\bar{y}_1 = \alpha_1 + \varepsilon_1 \qquad \text{and} \qquad \bar{y}_2 = \alpha_2 + \varepsilon_2$$

where $Var(\varepsilon_1) = \dfrac{\sigma^2}{n_1}$ and $Var(\varepsilon_2) = \dfrac{\sigma^2}{n_2}$.

Suppose further that,

$$\alpha_2 = \alpha_1 + \eta \quad : \quad Var(\eta) = \sigma_\eta^2.$$

We can now write

$$\bar{y}_1 = \alpha_1 + \varepsilon_1$$

$$\bar{y}_2 = \alpha_1 + \underbrace{\eta + \varepsilon_2}$$

$$\text{Variance} = \frac{\sigma^2}{n_2} + \sigma_\eta^2$$

We wish to find the weighted least squares estimator of α_1 that minimizes

$$W_1(\bar{y}_1 - \alpha_1)^2 + W_2(\bar{y}_2 - \alpha_1)^2$$

where

$$W_1^{-1} = \frac{\sigma^2}{n_1} \qquad \text{and} \qquad W_2^{-1} = \frac{\sigma^2}{n_2} + \sigma_\eta^2.$$

The augmented regression model can be written

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} \varepsilon_1 \\ \eta + \varepsilon_2 \end{pmatrix}$$

$$\text{Variance} = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix}.$$

So,

$$\hat{\alpha}_1 = (1 - z_1)\bar{y}_2 + z_1 \bar{y}_1 .$$

Similarly,

$$\hat{\alpha}_2 = (1 - z_2)\bar{y}_1 + z_2 \bar{y}_2 ,$$

where

$$z_1 = \frac{n_1/\sigma^2}{n_1/\sigma^2 + n_2/\left(\sigma^2 + n_2\sigma_\eta^2\right)}$$

$$z_2 = \frac{n_2/\sigma^2}{n_2/\sigma^2 + n_1/\left(\sigma^2 + n_1\sigma_\eta^2\right)}$$

Both $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are weighted least squares estimators or credibility estimators.

The estimator $\hat{\alpha}_1$ is a smoother not a filter. At 'time t=2' we also revise our estimator of α_1, the parameter at 'time t=1'. We have demonstrated that a varying parameter model can be formulated as a generalised linear regression model, or a weighted least squares model.

5. Gauss - Markov (Duncan and Horn (1972))

Consider the linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad : \qquad Var(\boldsymbol{\varepsilon}) = \Sigma \tag{5.1}$$

with prior sampling information

$$\boldsymbol{\beta}_0 = \boldsymbol{\beta} + \boldsymbol{\delta} \quad : \qquad Var(\boldsymbol{\delta}) = P_0 \tag{5.2}$$

The quantity $\boldsymbol{\beta}_0$ is an estimate of $\boldsymbol{\beta}$ based on past data. Specifically $\boldsymbol{\beta}_0$ is unbiased for $\boldsymbol{\beta}$ and has precision P_0^{-1}. We can re-cast (5.1) and (5.2) as an augmented model:

$$\begin{pmatrix} \mathbf{Y} \\ \boldsymbol{\beta}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{I} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \end{pmatrix} \tag{5.3}$$

The GLSE of $\boldsymbol{\beta}$ is given by

$$\tilde{\boldsymbol{\beta}} = \left(\mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{X}' + \mathbf{P}_0^{-1}\right)^{-1}\left(\mathbf{P}_0^{-1}\boldsymbol{\beta}_0 + \mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right).$$

Useful matrix identities are

(i) $$\left(\mathbf{A}^{-1} + \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}\right)^{-1} = \mathbf{A} - \mathbf{A}\mathbf{B}'\mathbf{D}^{-1}\mathbf{B}\mathbf{A}$$

where $\mathbf{D} = \mathbf{C} + \mathbf{B}\mathbf{A}\mathbf{B}'$

(ii) $$\mathbf{B}(\mathbf{I} + \mathbf{A}\mathbf{B})^{-1} = (\mathbf{I} + \mathbf{B}\mathbf{A})^{-1}\mathbf{B}$$

Using these identities $\tilde{\boldsymbol{\beta}}$ can be re-cast

$$\tilde{\boldsymbol{\beta}} = (\mathbf{I} - \mathbf{Z})\boldsymbol{\beta}_0 + \mathbf{Z}\boldsymbol{\beta}$$

where,

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$$

and

$$\mathbf{Z} = \left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X} + \mathbf{P}_0^{-1}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$$

Note that $\tilde{\boldsymbol{\beta}}$ is the weighted least squares estimator for model (5.1).

We let

$$\mathbf{P}_1 = Var\left(\tilde{\boldsymbol{\beta}}\right).$$

Alternatively,

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}_0 + \mathbf{K}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) \tag{5.4}$$

$$\mathbf{K} = \mathbf{P}_0\mathbf{X}'(\mathbf{X}\mathbf{P}_0\mathbf{X}' + \boldsymbol{\Sigma})^{-1} \tag{5.5}$$

$$\mathbf{P}_1 = (\mathbf{I} - \mathbf{K}\mathbf{X})\mathbf{P}_0 \tag{5.6}$$

$$\mathbf{K} = \mathbf{P}_1\mathbf{X}'\boldsymbol{\Sigma}^{-1} \tag{5.7}$$

$$\mathbf{Z} = \mathbf{K}\mathbf{X} \tag{5.8}$$

$$\mathbf{P}_1^{-1} = \mathbf{P}_0^{-1} + \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X} \tag{5.9}$$

We also obtain the same algorithms with the alternative Bayesian formulation

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad : \qquad Var(\boldsymbol{\varepsilon}) = \Sigma$$

and

$$E(\boldsymbol{\beta}) = \boldsymbol{\beta}_0 \qquad \text{and} \qquad Var(\boldsymbol{\beta}) = \mathbf{P}_0.$$

So, the algorithms do not distinguish between prior information based on a sample and prior 'subjective' information.

In the sampling formulation, the quantity $\boldsymbol{\beta}_0$ is a prior estimator based on a prior sample. So, $\boldsymbol{\beta}_0$ is a random variable and $\boldsymbol{\beta}$ is a constant.

In the Bayesian formulation $\boldsymbol{\beta}$ is a random variable and $\boldsymbol{\beta}_0$ a constant.

5.1 Kalman Filter Algorithm

There are two sets of model equations

1. **Observation Equations**

$$\mathbf{Y}(t) = \mathbf{X}(t)\boldsymbol{\beta}(t) + \boldsymbol{\varepsilon}(t) \qquad : \qquad Var(\boldsymbol{\varepsilon}(t)) = \Sigma(t) \qquad (5.1.1)$$

2. **System Equations**

$$\boldsymbol{\beta}(t) = \mathbf{H}(t)\boldsymbol{\beta}(t-1) + \boldsymbol{\delta}(t) \qquad : \qquad Var(\boldsymbol{\delta}(t)) = \Gamma(t) \qquad (5.1.2)$$

Derivation of recursive algorithm

At time $t-1$ we have, $\hat{\boldsymbol{\beta}}(t-1)$ with variance covariance $\mathbf{P}(t-1)$.

So,

$$\hat{\boldsymbol{\beta}}(t|t-1) = \mathbf{H}(t)\hat{\boldsymbol{\beta}}(t-1)$$

with covariance

$$\mathbf{P}(t|t-1) = \mathbf{H}(t)\mathbf{P}(t-1)\mathbf{H}'(t) + \Gamma(t) \qquad \text{from (5.1.2)}$$

Prior at time t of $\boldsymbol{\beta}(t)$ has mean

$$\hat{\boldsymbol{\beta}}(t|t-1)$$

and variance

$$\mathbf{P}(t|t-1).$$

Sample at time t, or observation equation at time t is

$$\mathbf{Y}(t) = \mathbf{X}(t)\boldsymbol{\beta}(t) + \boldsymbol{\varepsilon}(t) \qquad : \qquad Var(\boldsymbol{\varepsilon}(t)) = \Sigma(t)$$

Therefore, using Gauss - Markov (or Bayesian Updating)

$$\hat{\boldsymbol{\beta}}(t) = \hat{\boldsymbol{\beta}}(t|t-1) + \mathbf{K}(t)\left(\mathbf{Y}(t) - \mathbf{X}(t)\hat{\boldsymbol{\beta}}(t|t-1)\right) \qquad (1)$$

$$\mathbf{K}(t) = \mathbf{P}(t|t-1)\mathbf{X}'(t)\left(\mathbf{X}(t)\mathbf{P}(t|t-1)\mathbf{X}'(t) + \boldsymbol{\Sigma}(t)\right)^{-1} \qquad (2)$$

$$\mathbf{P}(t) = \left(\mathbf{I} - \mathbf{K}(t)\mathbf{X}(t)\right)\mathbf{P}(t|t-1) \qquad (3) \qquad\qquad (5.1.3)$$

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{X}'(t)\boldsymbol{\Sigma}^{-1}(t) \qquad (4)$$

$$\mathbf{P}^{-1}(t) = \mathbf{P}^{-1}(t|t-1) + \mathbf{X}'(t)\boldsymbol{\Sigma}^{-1}(t)\mathbf{X}(t) \qquad (5)$$

The initial or starting values are given by

$$\boldsymbol{\beta}(1) = \boldsymbol{\beta}(0) + \boldsymbol{\delta}(0).$$

That is $\boldsymbol{\beta}(1)$ has a prior has mean $\boldsymbol{\beta}(0)$ with variance/covariance matrix $\mathbf{P}(1|0)$.

6. Smoothers

Our results so far have been chiefly concerned with the filtering problem when an estimate of the signal $\boldsymbol{\beta}(k)$ (at time k) is based on the measurement $\mathbf{Y}(1)$, ..., $\mathbf{Y}(k)$. The estimate $\hat{\boldsymbol{\beta}}(k)$ based on $\mathbf{Y}(1)$, ..., $\mathbf{Y}(k)$ is called the filter. The best estimate of $\boldsymbol{\beta}(j)$ $(1 \le j < k)$, say $\tilde{\boldsymbol{\beta}}(j)$, based on $\mathbf{Y}(1)$, ..., $\mathbf{Y}(k)$ is called the smoother or smoothed estimate. In section 4.1, if we regard the first sample as that obtained at "time 1" and the second as that obtained at "time 2", then $\hat{\boldsymbol{\alpha}}_1$ is the smooth estimate, whereas $\hat{\boldsymbol{\alpha}}_2$ is both the smoother and the filter.

The smoothing problem is a filter problem in disguise, and therefore may be solved by direct application of the Kalman filtering results.

One way of solving the ssmoothing problem is to augment the state at time t, thus

$$\boldsymbol{\beta}^a(t) = \left(\boldsymbol{\beta}'(t), \boldsymbol{\beta}'(t-1), \ldots, \boldsymbol{\beta}'(1)\right)'$$

So,

$$\boldsymbol{\beta}^a(t) = \left(\boldsymbol{\beta}'(t), \boldsymbol{\beta}^{a\prime}(t-1)\right)'$$

The augmented observation equation at time t is

$$\mathbf{Y}(t) = \left(\mathbf{X}(t), 0\right)\boldsymbol{\beta}^a(t) + \boldsymbol{\varepsilon}(t) \tag{6.1}$$

with augmented state equation

$$\boldsymbol{\beta}^a(t) = \begin{bmatrix} \mathbf{H}(t) & 0 & \cdots & 0 \\ \mathbf{I} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{I} \end{bmatrix} \boldsymbol{\beta}^a(t-1) + \begin{bmatrix} \boldsymbol{\delta}(t) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{6.2}$$

The filter $\hat{\boldsymbol{\beta}}^a(t)$ at time t for (6.1) and (6.2) yields the smoothed estimates $\tilde{\boldsymbol{\beta}}(1)$, $\tilde{\boldsymbol{\beta}}(2)$, ..., $\tilde{\boldsymbol{\beta}}(t)$ at time t.

We now revisit section 4.1. We have

$$\bar{y}_1 = \alpha_1 + \varepsilon_1 \quad : \qquad Var(\varepsilon_1) = \frac{\sigma^2}{n_1}$$

$$\alpha_2 = \alpha_1 + \delta \quad : \qquad Var(\delta) = \sigma^2$$

$$\bar{y}_2 = \alpha_2 + \varepsilon_2 \quad : \qquad Var(\varepsilon_2) = \frac{\sigma^2}{n_2}$$

The augmented observation equation at "time t=2" is

$$\bar{y}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix} + \varepsilon_1 \quad : \qquad Var(\varepsilon_1) = \frac{\sigma^2}{n_1}$$

and

$$\begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

7. Trend Properties Of Loss Development Arrays

Since a model is suppose to capture the trends in the data, it behoves us to discuss the geometry of trends in the three directions, viz., **development year** (or delay), **accident year** and **payment** (or calendar) **year**.

The most important direction is the payment year. Payments, claim counts, etc. made in the same payment year (or period) are made in the same year. So any payment year effects economic inflation, superimposed inflation will manifest themselves from one diagonal to the next.

Development years are denoted by d; $d = 0, 1, 2, \ldots, s\text{-}1$; accident years by w; $w = 1, 2, \ldots, s$; and payment years by t; $t = 1, 2, \ldots, s$.

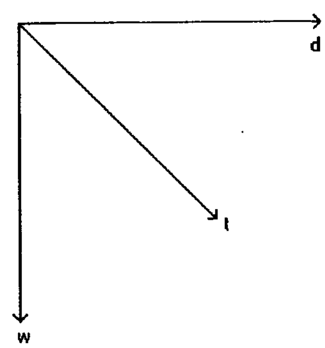
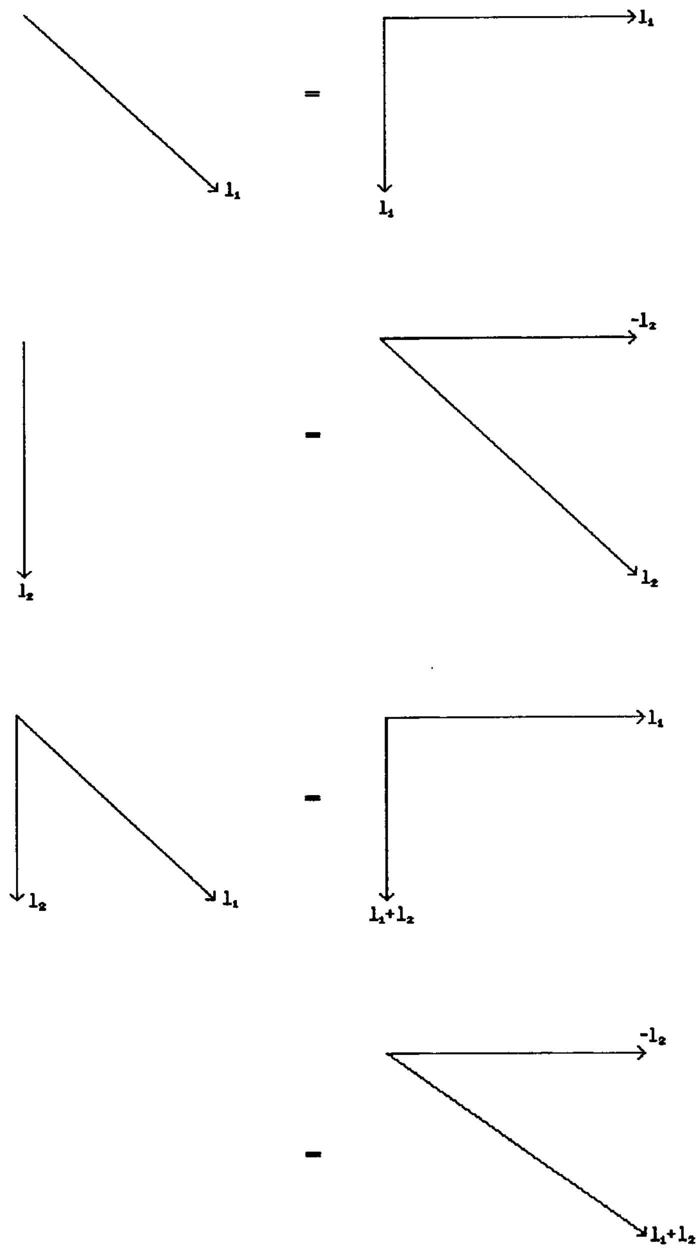


Figure 7.1

The payment year variable t can be expressed as $t = w + d$. This relationship between the three directions implies that there are only two 'independent' directions.

The two directions, development year and accident year, are orthogonal, equivalently, they have zero correlation. That is, trends in either direction are not projected onto the other. The payment year direction t however, is not orthogonal to either the development year or accident year directions. That is, a trend in the payment year direction is also projected onto the development year and accident year directions. Similarly, accident year trends are projected onto payment year trends.

The following displays demonstrate the equivalence of trends in general.



Trends on a log scale are additive and any trend in the payment year direction projects in the other two directions.

We can write a model that has parameters in the three directions as

$$y(w,d) = \alpha_w + \sum_{j=1}^{d} \gamma_j + \sum_{t=2}^{w+d} \iota_t + \varepsilon(w,d) \tag{7.1}$$

where

- $y(w,d) = \ln p(w,d)$
- α_w is "level" of accident year w
- γ_j is trend between development years j-1 and j
- ι_t is trend between payment years t-1 and t

The zero mean error terms $\varepsilon(w,d)$ are assumed to be independent from a normal distribution.

8. A Model With Three Inflation Parameters

In this section we simulate a triangle of icremental paid losses based on a model with three inflation parameters. We do this in order to illustrate properties of trends, and demonstrate that the three diagnostic tools provide the correct information.

The data in Appendix A1 to Appendix A9 are generated as follows.

First, we create payments based on the formula:

$$p(w,d) = \exp(\alpha - 0.2 * d) \tag{8.1}$$

That is, each accident year w is generated by the same exponential curve with γ (gamma) or decay factor equal to -0.2. The letter α (alpha) represents the intercept, level or (log) "exposure". Here $\alpha = 11.513$. See Appendix A1 for a display of the data.

TRENDS

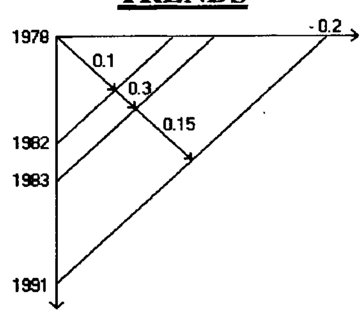


Figure 8.1

On a log scale we introduce payment/calendar year trends thus: 10% trend from 1978-82, 30% trend from 1982-83 and 15% trend from 1983-91. The logarithms of the payments with these trends are given in Appendix A2.

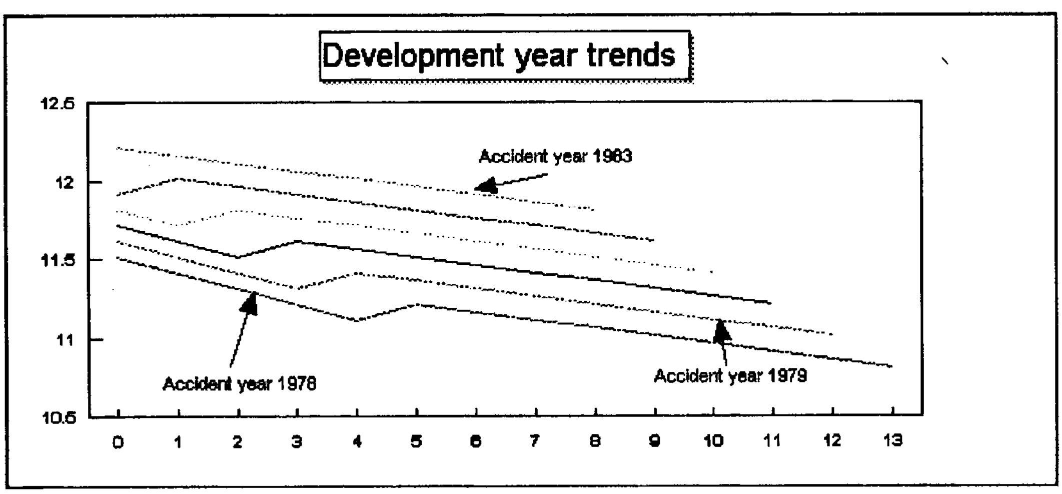


Figure 8.2

Figure 8.2 displays the graph of the log paid losses versus development year for the first six accident years. (The log paid losses are presented in Appendix A2).

Observe how payment/calendar year trends project onto development years and accident years. Each of the first six accident years has a different resultant run-off development.

Consider the first accident year 1978. The 10% calendar year trend projects onto the development year, so that the resultant trend from development year 0 to development year 4 is -0.2 (the gamma) + .1 (the payment year trend)= -.1. The 30% trend between calendar years 1982 and 1983 also projects onto the development year so that the trend between development year 4 and 5 is $-.2+.3=.1$. Thereafter the trend is $-.2+.15=-.05$. Since .15 is larger than .1, the resultant decay in the tail is less rapid ($-.05>-.1$).

Consider the next accident year 1979. First, up to development year 3, this accident year is 10% higher than the previous one since the 10% calendar year trend also projects onto the accident years. The 10% upward trend is one development year earlier than in the previous accident year since the 30% trend is a calendar year change.

So, changing payment/calendar year trends can cause some interesting development year patterns. The run-off pattern is different for each accident year. The payment year trends cannot be determined by the link ratios (age-to-age development factors) displayed in Appendix A4.

The patterns became much more complicated in the presence of random fluctuations superimposed on the trends.

The model describing the data we have constructed can be represented pictorially thus:

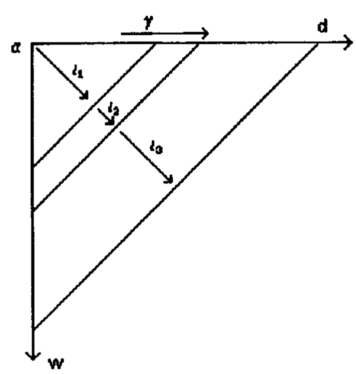


Figure 8.3

where $\gamma=-0.2$, $\iota_1=0.1$, $\iota_2=0.3$ and $\iota_3=0.15$.

Writing the equations explicitly is not necessary. Indeed, it is too complicated. It is understanding the trend structure that is important.

We note that the resultant trend (age-to-age development factor) between development years *j*-1 and *j* is the (base) development factor γ between the two development years plus the payment year trend ι (iota) between the two corresponding payment years.

We now introduce random fluctuations or deviations from trends.

To all the log "payments" in the triangle we add random numbers from a normal distribution with mean zero. Equivalently, to the trends depicted in Figure 8.2, we add random numbers from a normal distribution displayed in Appendix A5. The sum of trends (Appendix A2) plus random fluctuations (Appendix A5) is displayed in Appendix A6.

The graph of the first six accident years of the data in Appendix A6 is given in the Figure 8.4.

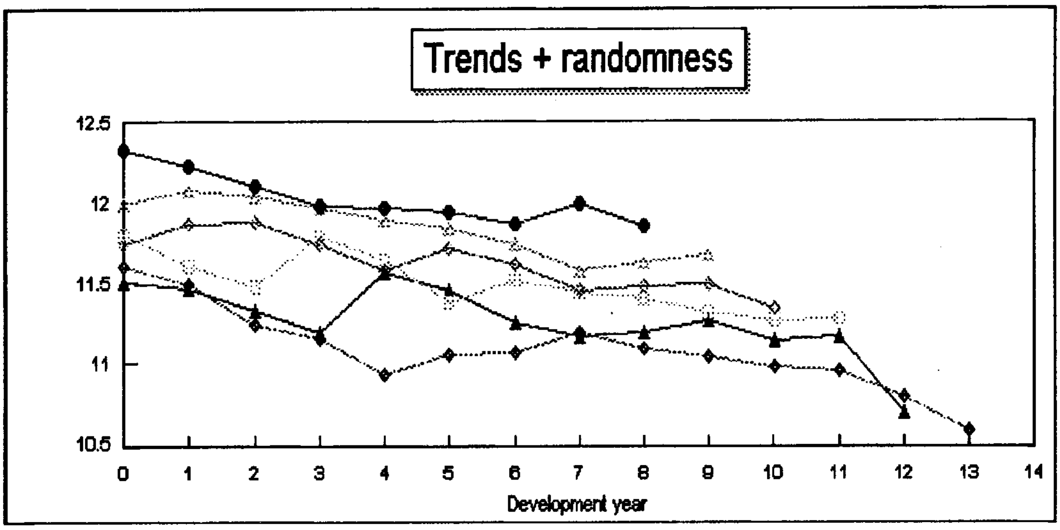


Figure 8.4

Note that it is not possible to determine the trends and/or changes in trends by eye or from the age-to-age link ratios of the cumulative payments (Appendix A9). See Appendices A7 - A9.

The incremental paid losses we have generated in Appendix A7 were generated by five trend parameters $(\alpha, \gamma, \iota_1, \iota_2, \iota_3)$ and one variance (noise, randomness) parameter $\sigma^2 = 0.01$.

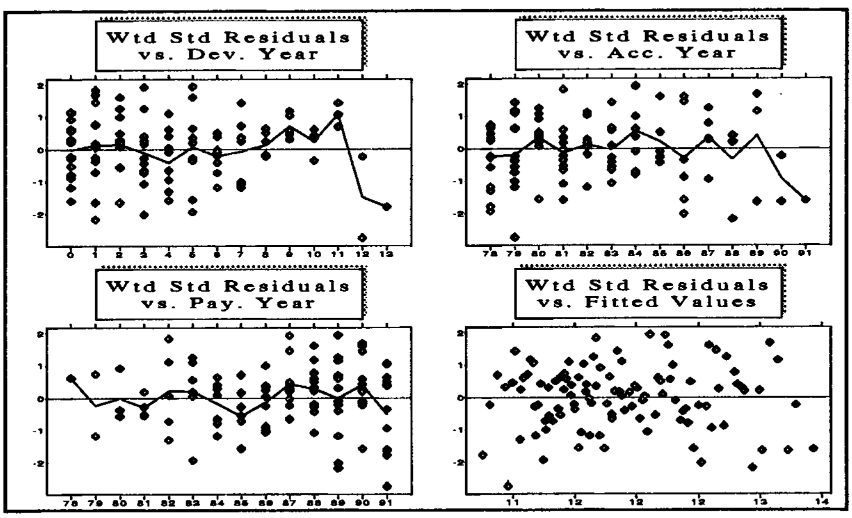
Since the incremental paid losses possess a stable trend (15%) along the payment years from 1983 to 1991 we would expect that the estimated model will validate well and be stable.

8.1 Best Estimated Model

We now estimate the parameters $\gamma = -0.2$, $\alpha = 11.513$, $\iota_1 = 0.1$, $\iota_2 = 0.3$, $\iota_3 = 0.15$ of the model.

We find that the estimate of gamma is -0.2062 ± 0.0033, which is close to the true value of -0.2. The iota estimates are 0.0873 ± 0.0209, relatively close in terms of the standard error to 0.1; 0.3927 ± 0.0442 which is within 3 standard errors of 0.3; and 0.1446 ± 0.0046 which is close to the true value 0.15.

Residuals in all three directions do not look great. There seems to be also a slight drop in the last couple of payment years. But this is a sample you obtain when you generate the errors randomly!



If you test for changing payment year trends from 87-88 or 89-90-91, even though there is a drop in inflation (due to sampling variation), the changes are <u>not</u> significant.

Here is some additional analysis including forecasts and stability analysis.

Forecasting for the estimated model using all the data,

gamma (in tail) = $-.2062 \pm .0033$

iota (83-91) $= .1446 \pm .0046$

So the model assumes future inflation that has an average of 14.46% and standard deviation of 0.46%.

Total Forecast = $23{,}426{,}542 \pm 927{,}810$. See Appendix B2 for the forecasting table.

Compare this with the true mean of \$24.8M ± \$292,746.

Validation of year 1991. Here we assign weight to the payment year 1991.

gamma (in tail) = -.2075 ± .0036

iota (83-90) = .1527 ± .0051

Note stability of gamma estimate but a slight increase in iota estimate.

The model assumes future inflation that has an average of 15.27% and standard deviation of .51%. So now the forecast is higher, as expected.

Total Forecast = 25,333,522 ± 1,191,129. See Appendix B3.

Validation of years 1991 and 1990. Here we assign zero weight to the last two payment years 1990 and 1991.

gamma (in tail) = -.2086 ± .0042

iota (83-89) = .1512 ± .0064

Since parameter estimates are the 'same' as when validating only 1991, the forecast is essentially the same.

Total Forecast = 24,850,972 ± 1,526,246. See Appendix B4.

Validation of years 1991, 1990 and 1989. We are now leaving much information out.

gamma (in tail) = -.2119 ± .0045

iota (83-88) = .1575 ± .0075

Forecast is slightly higher mainly as a result of increased iota (plus increased uncertainty).

Forecast = 26,296,366 ± 1,997,089. See Appendix B5.

Payment yrs in Estimation	**Estimate of gamma (in tail) %**	**Estimate of iota (since 1983) %**	**Forecast $M**
1978-91	-20.62±0.33	14.46±0.46	23±0.9
1978-90	-20.75±0.36	15.27±0.51	25±1.2
1978-89	-20.86±0.42	15.12±0.64	25±1.5
1978-88	-21.19±0.45	15.75±0.75	26±2.0
1978-87	-21.31±0.55	15.63±1.03	26±2.9

It is not amazing that answers do not change significantly as we leave out years, as the trend from 1983 is stable.

9. Varying Parameter, Dynamic Or Credibility Models

9.1 Multicollinearity

Many of the models within the family described by equation 7.1 cannot be estimated in a spreadsheet or any statistical package. Models that contain "many" iotas, alphas and gammas suffer from a problem known as multicollinearity. This problem is explained as follows:

To estimate the Ordinary Least Squares line for the simple linear regression:

$$y_i = \alpha + \beta x_i + \varepsilon, \tag{9.1.1}$$

we estimate the intercept α and slope β by minimising the error sum of squares,

$$SS = \sum (y_i - \alpha - \beta x_i)^2$$

Taking partial derivatives of the last equation with respect to α and β, and setting them to zero we obtain:

$$-2\sum (y_i - \alpha - \beta x_i) \tag{9.1.2}$$

and

$$-2\sum x_i (y_i - \alpha - \beta x_i) \tag{9.1.3}$$

Equivalently,

$$\bar{y} - \alpha - \beta \bar{x} = 0 \tag{9.1.4}$$

and

$$\sum x_i y_i - n\alpha\bar{x} - \beta \sum x_i^2 = 0 \tag{9.1.5}$$

The two linear homogeneous equations are known as the normal equations and their solution yields the least squares estimates of α and β.

For a model having *P* parameters in the *DFF* or *SCF* family, a spreadsheet (or a statistical package) sets up *P* linear homogeneous equations in order to solve for *P* unknowns. However, as a result of the non-orthogonality of the payment year direction with the other two directions, some of the equations in the normal equations are redundant, e.g.,

$$\alpha - \beta = 2$$

and

$$2\alpha - 2\beta = 4 .$$

So, there is no unique solution.

If there are some equations that are almost redundant, e.g.,

$$\alpha - \beta = 2$$

and

$$2\alpha - 2\beta = 4.00001 ,$$

then the estimates will have high standard errors, so that the resulting model will be unstable.

9.2 Overcoming Multicollinearity - Varying Or Stochastic Parameters

The phenomenon of multicollinearity associated with fixed parameter models can be interpreted in terms of information. There is not sufficient information in one loss development array to estimate many payment year parameters and accident year parameters (especially, for the more recent accident years). Another interpretation is that the independent variables in the regression are not really independent. We showed in Section 7 that calendar year trends are related to development year trends and accident year trends.

When we include another a parameter for the last accident in our model we are using one single datum to estimate that parameter. That is, we assign full credibility to the last accident year's datum and zero credibility to previous years in respect of the estimation of the additional a parameter. A better approach may be to assign some credibility to the previous years data and less than full credibility to the last year's datum.

We are motivated to introduce exponential smoothing/varying parameter/credibility models/dynamic models, as a result of multicollinearity. Multicollinearity can lead to fixed parameter regression models that *(i)* are unstable and *(ii)* have large prediction errors.

If a model contains a number of iota parameters, instead of introducing more alpha parameters (in the accident year direction), in order to adjust for remaining trends in the accident year direction, we could exponentially smooth the alphas as described in sections 2.2, 2.3 and 4.

10. Analysis of real data

We now study a real life example given in Appendix C1. Note the incremental paid losses are quite volatile.

The three graphs below depict the data after only adjusting for trends in the development year direction.

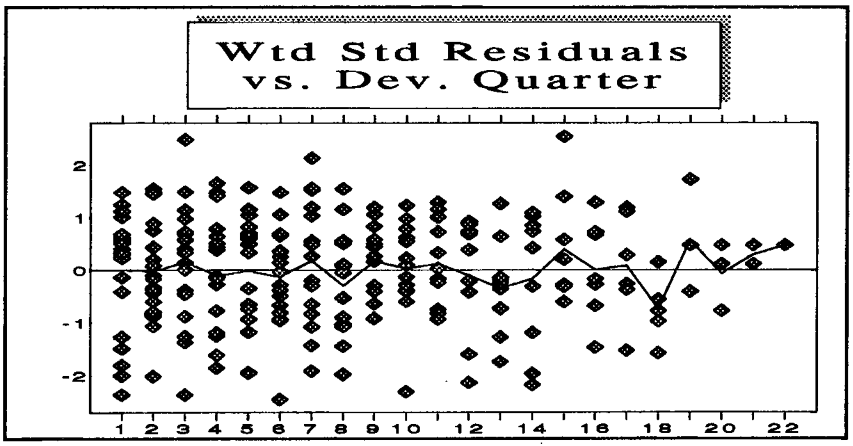


Figure 10.1

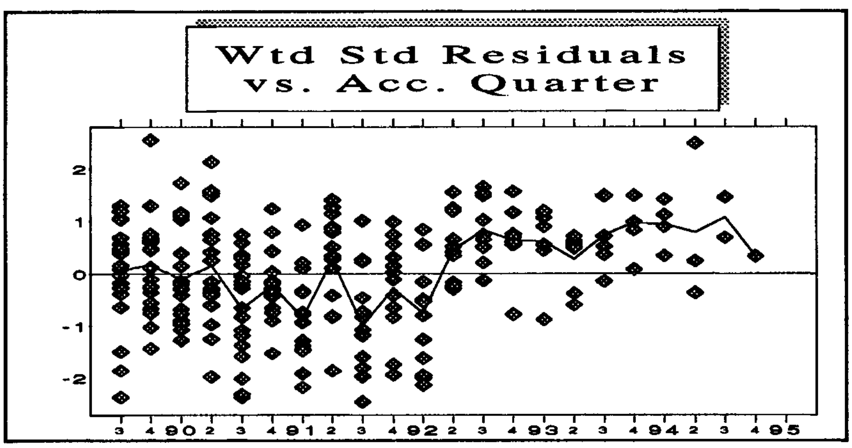


Figure 10.2

Note the minor shifts in accident quarters 2-90 to 3-90, 1-91 to 2-91 to 3-91 and a major shift from 1-92 to 2-92.

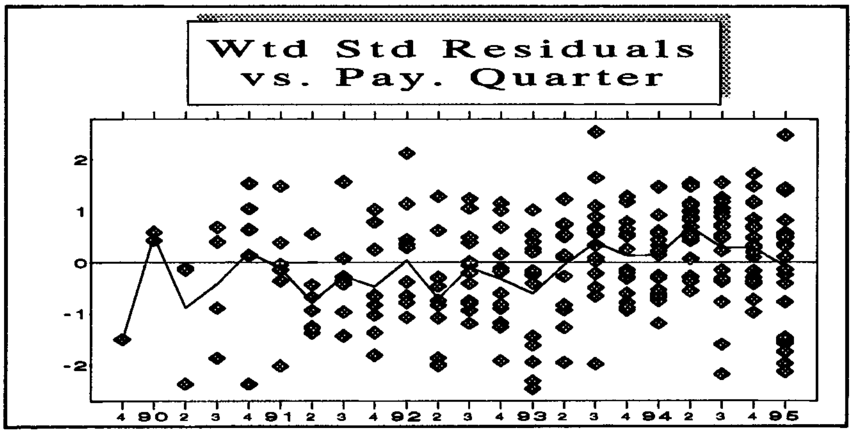


Figure 10.3

After making some exponential smoothing (varying parameter) adjustments to these accident quarters we obtain residual displays in figures 10.4, 10.5 and 10.6. The SSPE is 135.46. If instead we add new parameters for these years the SSPE is 135.68.

So we are better off with the varying parameter model, as the additional parameters also increase the standard errors and hence make the model less stable.

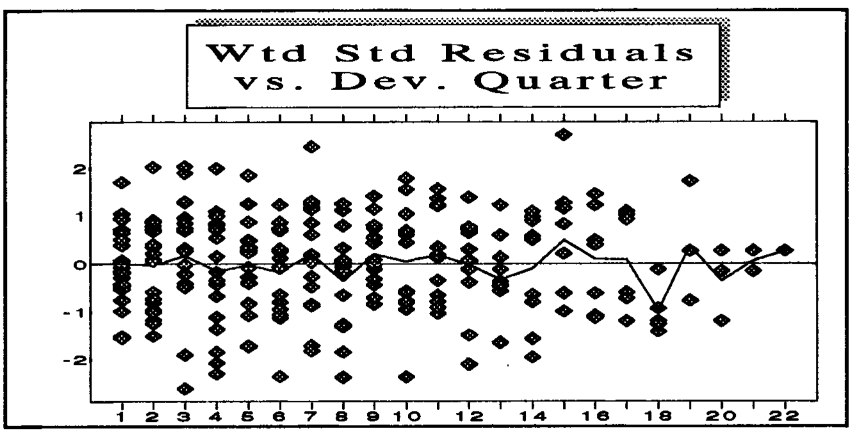


Figure 10.4

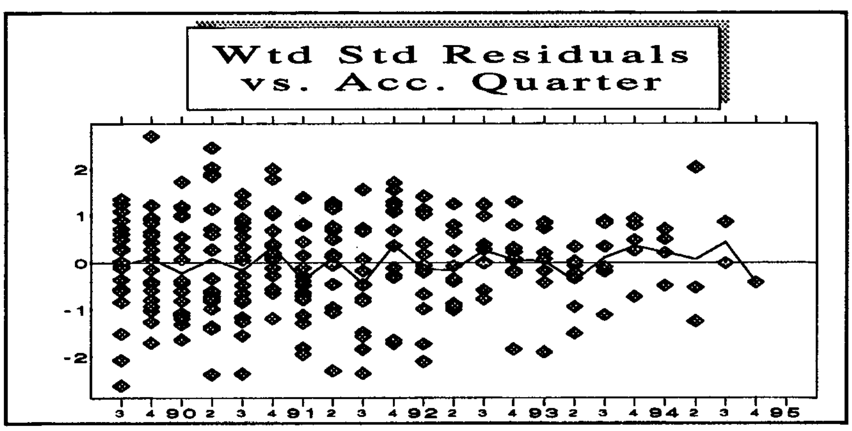


Figure 10.5

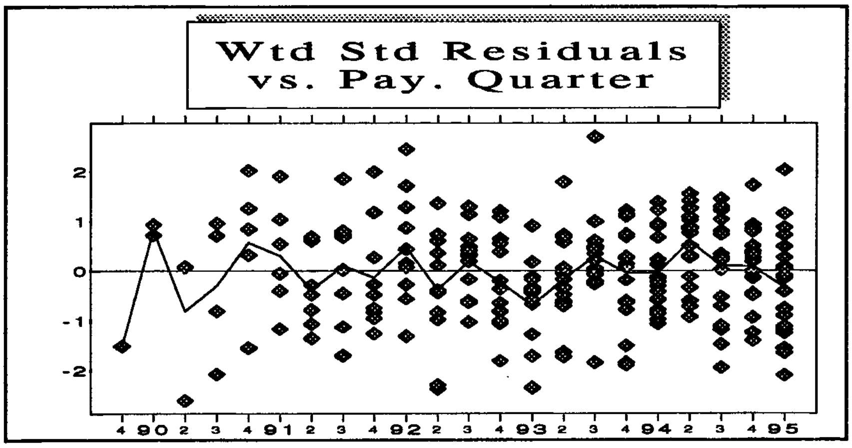


Figure 10.6

NOTE THE TREND STABILITY IN THE PAYMENT QUARTER YEAR DIRECTION

Forecasts of the total outstanding claim liabilities is

Forecast = 132,355 ± 13,053.

When we leave out the last nine payment years (68% of the data), the graphs below depict the prediction errors as at end 4-92 for the next 9 diagonals!

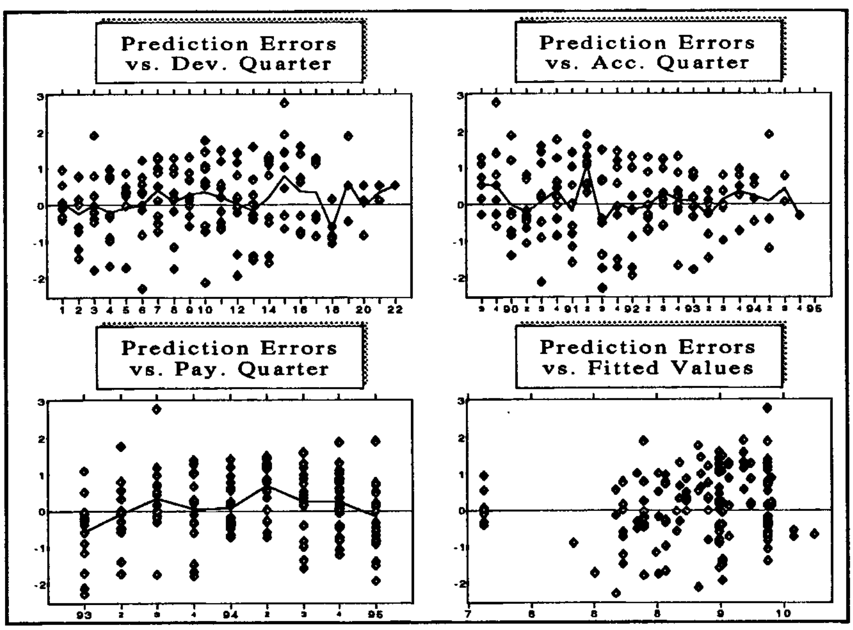


Figure 10.7

It is remarkable how the prediction errors for the last nine payment quarter years are centred around zero. Note also normality of prediction errors in Figure 10.8.

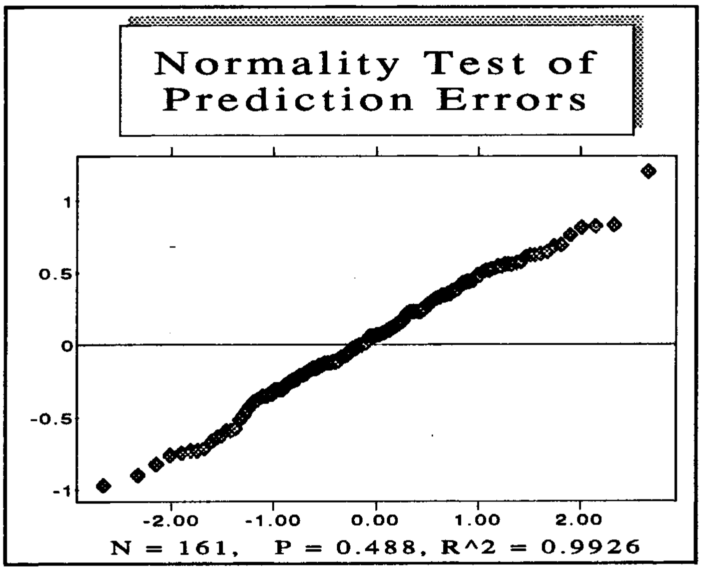


Figure 10.8

Note from Table 10.1 below, that had the model been estimated at payment (quarter) year end 4-92, the estimate of total outstanding of \$121,397T ± \$39,018T is not statistically different to \$132,355T ± \$13,053T, that obtained by the model using the experience to payment quarter year end 1-95. Indeed, the two answers are remarkably close, especially that \$121,397T ± \$39,018T is obtained after removing 68% of the most recent experience.

Table 10.1

Payment years in Estimation	Forecast \$000
3Q-89 to 1Q-95	132,355 ± 13,053
3Q-89 to 4Q-92	121,397 ± 39,018

11. References

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2. Goldberger, A.S. and Thiel, H. (1961). "On pure and mixed statistical estimation in Economics", International Economic Review 2, pp 65-78

3. Kalman, R.E. (1960). "A New Approach to Linear Filtering and Prediction Problems", Journal of Basic Engineering 82, pp35-44

4. Naik-Nimbalker and Rajashi, M.B. (1995). "Filtering and Smoothing via Estimating Functions", Journal of the American Statistical Association, 90, pp 301-306

5. Plackett, R.L. (1950). "Some theories in least squares", Biometrika 37, pp149-157

6. West, M.P., Harrison, P.J. and Migan, H.S. (1985). "Dynamic Generalized Linear Models and Bayesian Filtering", Journal of the American Association, 80, pp 73-96

7. Zehnwirth, B. (1988). "A generalization of the Kalman Filter for models with state - dependent observation variance", Journal of the American Association, 83, pp 164-167

Appendix A1

Model is p = exp(alpha-.2d) **no error or randomness**
alpha = 11.51293

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	7427
1979	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	
1980	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080		
1981	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534			
1982	100000	81873	67032	54881	44933	36788	30119	24660	20190	16530				
1983	100000	81873	67032	54881	44933	36788	30119	24660	20190					
1984	100000	81873	67032	54881	44933	36788	30119	24660						
1985	100000	81873	67032	54881	44933	36788	30119							
1986	100000	81873	67032	54881	44933	36788								
1987	100000	81873	67032	54881	44933									
1988	100000	81873	67032	54881										
1989	100000	81873	67032											
1990	100000	81873												
1991	100000													

Appendix A2

y=log(p) plus .1 inf. (per year) from 78-82, .3 inf. from 82-83 and .15 inf. from 83-91

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	11.5129	11.4129	11.3129	11.2129	11.1129	11.2129	11.1629	11.1129	11.0629	11.0129	10.9629	10.9129	10.8629	10.8129
1979	11.6129	11.5129	11.4129	11.3129	11.4129	11.3629	11.3129	11.2629	11.2129	11.1629	11.1129	11.0629	11.0129	
1980	11.7129	11.6129	11.5129	11.6129	11.5629	11.5129	11.4629	11.4129	11.3629	11.3129	11.2629	11.2129		
1981	11.8129	11.7129	11.8129	11.7629	11.7129	11.6629	11.6129	11.5629	11.5129	11.4629	11.4129			
1982	11.9129	12.0129	11.9629	11.9129	11.8629	11.8129	11.7629	11.7129	11.6629	11.6129				
1983	12.2129	12.1629	12.1129	12.0629	12.0129	11.9629	11.9129	11.8629	11.8129					
1984	12.3629	12.3129	12.2629	12.2129	12.1629	12.1129	12.0629	12.0129						
1985	12.5129	12.4629	12.4129	12.3629	12.3129	12.2629	12.2129							
1986	12.6629	12.6129	12.5629	12.5129	12.4629	12.4129								
1987	12.8129	12.7629	12.7129	12.6629	12.6129									
1988	12.9629	12.9129	12.8629	12.8129										
1989	13.1129	13.0629	13.0129											
1990	13.2629	13.2129												
1991	13.4129													

Appendix A3

Cumulative data (on a $ scale) derived from Appendix A2

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	100000	190484	272357	346439	413471	487552	558021	625053	688816	749469	807164	862045	914250	963908
1979	110517	210517	301001	382874	473358	559428	641302	719182	793263	863732	930764	994527	1055180	
1980	122140	232657	332657	443174	548302	648302	743425	833908	919979	1001852	1079732	1153814		
1981	134986	257126	392112	520515	642655	758838	869355	974482	1074482	1169605	1260089			
1982	149182	314055	470886	620068	761975	896961	1025363	1147504	1263687	1374204				
1983	201375	392929	575141	748467	913339	1070170	1219352	1361259	1496245					
1984	233965	456519	668219	869594	1061148	1243360	1416685	1581557						
1985	271828	530399	776359	1010324	1232878	1444578	1645954							
1986	315819	616236	902001	1173829	1432400	1678360								
1987	366930	715964	1047976	1363795	1664212									
1988	426311	831831	1217574	1584504										
1989	495303	966450	1414619											
1990	575460	1122855												
1991	668589													

Appendix A4

Age-to-age link ratios of the cumulative losses of Appendix A3

Year	Development Year												
	0:1	1:2	2:3	3:4	4:5	5:6	6:7	7:8	8:9	9:10	10:11	11:12	12:13
1978	1.904837	1.429817	1.272003	1.193489	1.179171	1.144536	1.120124	1.102012	1.088054	1.076981	1.067993	1.060559	1.054316
1979	1.904837	1.429817	1.272003	1.236328	1.181830	1.146351	1.121441	1.103009	1.088834	1.077607	1.068506	1.060987	
1980	1.904837	1.429817	1.332225	1.237214	1.182381	1.146726	1.121712	1.103214	1.088994	1.077736	1.068611		
1981	1.904837	1.524979	1.327464	1.234653	1.180787	1.145640	1.120925	1.102619	1.088529	1.077363			
1982	2.105171	1.499376	1.316812	1.228857	1.177153	1.143153	1.119119	1.101249	1.087456				
1983	1.951229	1.463727	1.301361	1.220280	1.171712	1.139401	1.116379	1.099163					
1984	1.951229	1.463727	1.301361	1.220280	1.171712	1.139401	1.116379						
1985	1.951229	1.463727	1.301361	1.220280	1.171712	1.139401							
1986	1.951229	1.463727	1.301361	1.220280	1.171712								
1987	1.951229	1.463727	1.301361	1.220280									
1988	1.951229	1.463727	1.301361										
1989	1.951229	1.463727											
1990	1.951229												
1991													

Appendix A5

Random error from Normal with mean 0, s-2=.01

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	0.083	0.075	-0.076	-0.065	-0.188	-0.164	-0.101	0.078	0.021	0.029	0.005	0.030	-0.073	-0.241
1979	-0.113	-0.049	-0.086	-0.123	0.148	0.090	-0.060	-0.099	-0.032	0.096	0.028	0.100	-0.331	
1980	0.086	-0.007	-0.037	0.170	0.071	-0.138	0.047	0.022	0.036	0.003	0.004	0.058		
1981	-0.071	0.147	0.067	-0.028	-0.132	0.049	0.000	-0.117	-0.042	0.026	-0.078			
1982	0.081	0.059	0.073	0.048	0.025	0.029	-0.023	-0.133	-0.044	0.066				
1983	0.117	0.059	-0.017	-0.081	-0.051	-0.024	-0.048	0.124	0.033					
1984	-0.024	-0.026	0.134	0.214	0.071	0.193	-0.022	0.012						
1985	0.022	0.015	0.076	-0.028	-0.004	0.155	0.032							
1986	-0.043	0.181	0.184	-0.192	-0.160	-0.048								
1987	0.070	0.106	0.144	0.032	-0.102									
1988	0.056	-0.195	0.032	0.041										
1989	0.145	0.187	-0.159											
1990	0.001	-0.153												
1991	-0.142													

Deterministic data (on log scale) with 3 infs from file mod3inf.wk1

Appendix A6

Sum of data in Appendices A2 and A5 to produce trends + randomness

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	11.5959	11.4879	11.2369	11.1479	10.9249	11.0489	11.0619	11.1909	11.0839	11.0419	10.9679	10.9429	10.7899	10.5719
1979	11.4999	11.4639	11.3269	11.1899	11.5609	11.4529	11.2529	11.1639	11.1809	11.2589	11.1409	11.1629	10.6819	
1980	11.7989	11.6059	11.4759	11.7829	11.6339	11.3749	11.5099	11.4349	11.3989	11.3159	11.2669	11.2709		
1981	11.7419	11.8599	11.8799	11.7349	11.5809	11.7119	11.6129	11.4459	11.4709	11.4889	11.3349			
1982	11.9939	12.0719	12.0359	11.9609	11.8879	11.8419	11.7399	11.5799	11.6189	11.6789				
1983	12.3299	12.2219	12.0959	11.9819	11.9619	11.9389	11.8649	11.9869	11.8459					
1984	12.3389	12.2869	12.3969	12.4269	12.2339	12.3059	12.0409	12.0249						
1985	12.5349	12.4779	12.4889	12.3349	12.3089	12.4179	12.2449							
1986	12.6199	12.7939	12.7469	12.3209	12.3029	12.3649								
1987	12.8829	12.8689	12.8569	12.6949	12.5109									
1988	13.0189	12.7179	12.8949	12.8539										
1989	13.2579	13.2499	12.8539											
1990	13.2639	13.0599												
1991	13.2709													

Appendix A7

Incremental paids derived from Appendix A6

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	108651	97529	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023
1979	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	
1980	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504		
1981	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692			
1982	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054				
1983	226364	203191	179136	159835	156670	153108	142187	160637	139511					
1984	228411	216837	242050	249422	205644	220996	169549	166858						
1985	277868	262472	265375	227499	221660	247187	207918							
1986	302519	360015	343485	224336	220334	234427								
1987	393525	388054	383425	326081	271278									
1988	450855	333667	398276	382277										
1989	572576	568013	382277											
1990	576021	469724												
1991	580068													

Appendix A8

Cumulative paids from Appendix A7

Year	Development Year													
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1978	108651	206180	282059	351477	407019	469894	533591	606059	671173	733609	791592	848143	896671	935694
1979	98706	193922	276947	349343	454257	548431	625534	696072	767819	845386	914320	984787	1028347	
1980	133106	242849	339214	470207	583067	670175	769873	862367	951591	1033708	1111898	1190402		
1981	125731	267209	411545	536399	643433	765448	875962	969479	1065364	1162990	1246682			
1982	161765	336653	505357	661871	807366	946320	1071800	1178727	1289906	1407960				
1983	226364	429555	608691	768526	925196	1078304	1220491	1381128	1520639					
1984	228411	445248	687298	936720	1142364	1363360	1532909	1699767						
1985	277868	540340	805715	1033214	1254874	1502061	1709979							
1986	302519	662534	1006019	1230355	1450689	1685116								
1987	393525	781579	1165004	1491085	1762363									
1988	450855	784522	1182798	1565075										
1989	572576	1140589	1522866											
1990	576021	1045745												
1991	580068													

Appendix A9

Age-to-age factors (link ratios) of the cumulative payments

Year	Development Year												
	0:1	**1:2**	**2:3**	**3:4**	**4:5**	**5:6**	**6:7**	**7:8**	**8:9**	**9:10**	**10:11**	**11:12**	**12:13**
1978	1.897636	1.368023	1.246112	1.158025	1.154477	1.135556	1.135812	1.107438	1.093025	1.079038	1.071440	1.057217	1.043520
1979	1.964642	1.428136	1.261407	1.300318	1.207314	1.140588	1.112764	1.103074	1.101023	1.081541	1.077070	1.044233	
1980	1.824478	1.396810	1.386166	1.240022	1.149396	1.148764	1.120142	1.103464	1.086294	1.075640	1.070604		
1981	2.125244	1.540161	1.303379	1.199542	1.189631	1.144378	1.106759	1.098904	1.091636	1.071963			
1982	2.081124	1.501121	1.309710	1.219824	1.172108	1.132598	1.099764	1.094321	1.091521				
1983	1.897629	1.417027	1.262588	1.203858	1.165487	1.131862	1.131617	1.101012					
1984	1.949328	1.543630	1.362902	1.219536	1.193455	1.124361	1.108851						
1985	1.944592	1.491126	1.282357	1.214534	1.196982	1.138422							
1986	2.190057	1.518441	1.222994	1.179082	1.161597								
1987	1.986097	1.490577	1.279897	1.181933									
1988	1.740076	1.507667	1.323197										
1989	1.992031	1.335158											
1990	1.815463												
1991													

one cannot determine changing calendar year trends from the age-to-age link ratios.

Appendix B1

Forecast results for true model

Year	Development Year 0	1	2	3	4	5	6	7	8	9	10	11	12	13	Accident Total
1978	100501	90937	82283	74453	67368	74453	70822	67368	64082	60957	57984	55156	52466	49907	0
	108651	97528	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023	0
1979	111071	100501	90937	82283	90937	86502	82283	78270	74453	70822	67368	64082	60957	57984	57984
	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	5813	5813
1980	122753	111071	100501	111071	105654	100501	95600	90937	86502	82283	78270	74453	70822	67368	138190
	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504	7100	6754	9799
1981	135663	122753	135663	129046	122753	116766	111071	105654	100501	95600	90937	86502	82283	78270	247056
	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692	8672	8249	7847	14311
1982	149930	165699	157617	149930	142618	135663	129046	122753	116766	111071	105654	100501	95600	90937	392692
	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054	10592	10075	9584	9117	19715
1983	202385	192514	183125	174194	165699	157617	149930	142618	135663	129046	122753	116766	111071	105654	585290
	226364	203191	179136	159835	156670	153108	142187	160637	139511	12937	12306	11706	11135	10592	26306
1984	235137	223670	212761	202385	192514	183125	174194	165699	157617	149930	142618	135663	129046	122753	837627
	228410	216837	242050	249422	205644	220996	169549	166858	15801	15031	14298	13600	12937	12306	34406
1985	273191	259867	247193	235137	223670	212761	202385	192514	183125	174194	165699	157617	149930	142618	1165698
	277867	262472	265375	227499	221660	247187	207918	19300	18358	17463	16611	15801	15031	14298	44389
1986	317402	301922	287198	273191	259867	247193	235137	223670	212761	202385	192514	183125	174194	165699	1589485
	302519	360015	343485	224336	220334	234427	23573	22423	21329	20289	19300	18358	17463	16611	56705
1987	368769	350784	333676	317402	301922	287198	273191	259867	247193	235137	223670	212761	202385	192514	2133916
	393525	388054	383425	326081	271278	28792	27388	26052	24781	23573	22423	21329	20289	19300	71898
1988	428448	407553	387676	368769	350784	333676	317402	301922	287198	273191	259867	247193	235137	223670	2830040
	450855	333667	398276	382277	35166	33451	31820	30268	28792	27388	26052	24781	23573	22423	90634
1989	497786	473509	450415	428448	407553	387676	368769	350784	333676	317402	301922	287198	273191	259867	3716486
	572576	568013	382277	42952	40857	38865	36969	35166	33451	31820	30268	28792	27388	26052	113725
1990	578345	550139	523308	497786	473509	450415	428448	407553	387676	368769	350784	333676	317402	301922	4841249
	576021	469724	52462	49903	47469	45154	42952	40857	38865	36969	35166	33451	31820	30268	142164
1991	671941	639170	607997	578345	550139	523308	497786	473509	450415	428448	407553	387676	368769	350784	6263898
	580068	64077	60952	57979	55152	52462	49903	47469	45154	42952	40857	38865	36969	35166	177165
Payment Total		3264185	3049833	2837008	2624193	2409707	2191683	1968027	1736383	1494081	1238089	964945	670691	350784	24799610
Payment Error		111290	105718	100356	95170	90112	85123	80121	74990	69561	63570	56583	47779	35166	292746

Appendix B2

Forecast results

Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Accident Total
1978	102622	91034	80789	71729	63712	76742	72156	67845	63793	59984	56403	53038	49874	46900	0
	108651	97528	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023	0
1979	111881	99290	88154	78301	94314	88677	83378	78398	73717	69316	65180	61292	57637	54201	54201
	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	5559	5559
1980	122028	108341	96232	115911	108982	102470	96349	90595	85187	80103	75324	70832	66609	62639	129248
	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504	6791	6458	9736
1981	133153	118269	142454	133938	125934	118410	111339	104691	98443	92570	87048	81858	76979	72392	231229
	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692	8305	7894	7517	14752
1982	145356	175078	164611	154773	145526	136834	128664	120984	113765	106979	100600	94603	88966	83666	367834
	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054	10168	9660	9195	8766	21101
1983	215176	202310	190218	178852	168169	158127	148688	139815	131474	123634	116263	109334	102821	96697	548749
	226364	203191	179136	159835	156670	153108	142187	160637	139511	12461	11834	11259	10729	10241	29304
1984	248647	233783	219813	206682	194339	182737	171832	161580	151943	142884	134368	126363	118836	111760	786155
	228410	216837	242050	249422	205644	220996	169549	166858	15289	14512	13800	13145	12542	11983	40022
1985	287329	270158	254017	238847	224587	211182	198582	186737	175603	165136	155296	146046	137349	129173	1095340
	277867	262472	265375	227499	221660	247187	207918	18780	17816	16933	16122	15374	14684	14043	54123
1986	332037	312198	293551	276023	259547	244060	229502	215816	202951	190857	179487	168798	158749	149301	1495461
	302519	360015	343485	224336	220334	234427	23094	21896	20799	19793	18866	18010	17217	16479	72750
1987	383708	360788	339244	318993	299957	282062	265241	249428	234563	220588	207451	195099	183487	172570	2010491
	393525	388054	383425	326081	271278	28430	26939	25576	24325	23174	22113	21129	20215	19363	97406
1988	443431	416949	392057	368659	346664	325989	306553	288281	271105	254957	239776	225503	212085	199469	2670382
	450855	333667	398276	382277	35037	33181	31483	29927	28496	27176	25955	24821	23765	22777	130062
1989	512459	481862	453102	426067	400654	376764	354306	333193	313346	294686	277144	260651	245145	230566	3512521
	572576	568013	382277	43227	40913	38797	36858	35076	33433	31914	30505	29194	27968	26819	173284
1990	592245	556893	523663	492425	463061	435456	409506	385110	362175	340613	320342	301283	283364	266516	4583516
	576021	469724	53389	50500	47859	45440	43218	41171	39280	37526	35895	34373	32946	31606	230411
1991	684468	643621	605224	569130	535200	503303	473317	445126	418623	393707	370281	348256	327548	308078	5941415
	580068	66007	62398	59099	56078	53304	50750	48391	46206	44175	42281	40508	38844	37276	305762
Payment Total		3217162	2974321	2738084	2506809	2278761	2052087	1824785	1594672	1359354	1116178	862186	594065	308078	23426542
Payment Error		131248	128153	125427	122636	119405	115402	110321	103865	95719	85534	72880	57170	37276	927810

Appendix B3

Forecast results

Year	Development Year 0	1	2	3	4	5	6	7	8	9	10	11	12	13	Accident Total
1978	102525	90901	80625	71538	63499	74877	70878	67095	63515	60128	56923	53890	51020	48305	0
	108651	97528	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023	0
1979	111862	99216	88032	78139	92140	87219	82563	78157	73989	70044	66312	62781	59438	56276	56276
	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	5464	5464
1980	122094	108331	96156	113384	107328	101598	96176	91046	86192	81599	77252	73139	69247	65564	134811
	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504	6672	6426	9811
1981	133313	118329	139529	132075	125023	118350	112037	106063	100410	95061	90000	85210	80677	76387	242274
	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692	8159	7854	7578	15225
1982	145617	171705	162531	153851	145639	137869	130517	123560	116977	110748	104853	99275	93996	89000	387124
	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054	9990	9612	9270	8957	22288
1983	211302	200012	189330	179223	169660	160611	152050	143948	136282	129027	122162	115665	109516	103697	580067
	226364	203191	179136	159835	156670	153108	142187	160637	139511	12250	11779	11354	10966	10611	31656
1984	246140	232992	220553	208784	197647	187110	177139	167704	158776	150327	142331	134764	127603	120825	834625
	228410	216837	242050	249422	205644	220996	169549	166858	15042	14455	13924	13442	13001	12594	44175
1985	286728	271418	256933	243227	230258	217986	206374	195386	184987	175147	165834	157021	148680	140786	1167841
	277867	262472	265375	227499	221660	247187	207918	18497	17763	17100	16498	15948	15442	14975	60960
1986	334018	316190	299321	283359	268256	253964	240440	227643	215532	204071	193224	182959	173244	164048	1601160
	302519	360015	343485	224336	220334	234427	22778	21859	21029	20275	19588	18957	18374	17832	83484
1987	389118	368356	348711	330122	312533	295888	280137	265232	251126	237777	225144	213187	201870	191160	2161522
	393525	388054	383425	326081	271278	28092	26937	25895	24951	24090	23300	22572	21895	21261	113684
1988	453320	429141	406262	384613	364127	344742	326397	309036	292607	277058	262343	248416	235234	222758	2882718
	450855	333667	398276	382277	34694	33242	31931	30744	29663	28673	27760	26913	26122	25379	154096
1989	528128	499969	473323	448110	424250	401672	380306	360085	340949	322838	305697	289474	274119	259585	3807083
	572576	568013	382277	42907	41078	39428	37933	36572	35327	34181	33119	32129	31199	30320	208033
1990	615298	582502	551469	522103	494314	468016	443130	419577	397287	376191	356224	337327	319440	302509	4987587
	576021	469724	53137	50830	48748	46863	45147	43579	42136	40801	39558	38393	37294	36251	279803
1991	716873	678678	642534	608331	575964	545333	516345	488911	462947	438373	415114	393100	372264	352540	6490434
	580068	65892	62979	60349	57968	55801	53822	52003	50321	48757	47294	45915	44610	43366	375001
Payment Total		3377487	3144401	2915003	2687633	2460477	2231537	1998595	1759169	1510465	1249317	972125	674773	352540	25333522
Payment Error		141263	141030	140808	140091	138451	135507	130898	124257	115187	103231	87836	68281	43366	1191129

Appendix B4

Forecast results

Year	Development Year 0	1	2	3	4	5	6	7	8	9	10	11	12	13	Accident Total
1978	102523	90848	80534	71417	63357	75025	70837	66885	63156	59637	56317	53183	50227	47436	0
	108651	97528	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023	0
1979	111921	99213	87981	78050	92424	87264	82394	77800	73465	69374	65514	61871	58433	55188	55188
	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	5553	5553
1980	122226	108388	96153	113860	107502	101502	95842	90500	85460	80704	76216	71980	67982	64210	132192
	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504	6775	6570	10300
1981	133532	118457	140270	132436	125044	118069	111488	105278	99418	93888	88670	83744	79096	74709	237549
	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692	8278	8025	7799	16442
1982	145938	172809	163156	154048	145454	137346	129694	122474	115661	109231	103163	97436	92030	86928	379557
	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054	10133	9817	9537	9285	24681
1983	212900	201005	189783	179194	169203	159776	150880	142485	134563	127086	120030	113370	107084	101151	568721
	226364	203191	179136	159835	156670	153108	142187	160637	139511	12426	12031	11681	11366	11081	35837
1984	247639	233811	220764	208453	196837	185876	175533	165772	156560	147866	139660	131915	124605	117705	818311
	228410	216837	242050	249422	205644	220996	169549	166858	15266	14769	14329	13936	13580	13253	50977
1985	288059	271982	256814	242501	228995	216250	204223	192872	182160	172050	162508	153501	144999	136974	1145064
	277867	262472	265375	227499	221660	247187	207918	18789	18162	17608	17114	16667	16258	15878	71499
1986	335090	316398	298762	282120	266416	251597	237611	224412	211955	200198	189101	178626	168738	159404	1570045
	302519	360015	343485	224336	220334	234427	23169	22375	21674	21049	20486	19972	19496	19049	99241
1987	389815	368083	347577	328226	309966	292733	276470	261121	246634	232961	220054	207871	196371	185514	2119730
	393525	388054	383425	326081	271278	28623	27614	26724	25932	25219	24569	23970	23409	22878	136613
1988	453496	428227	404383	381883	360649	340610	321697	303847	286999	271096	256085	241915	228538	215910	2827346
	450855	333667	398276	382277	35426	34140	33006	31998	31092	30268	29510	28802	28134	27497	186759
1989	527601	498220	470494	444329	419636	396332	374337	353578	333983	315487	298028	281546	265986	251297	3734540
	572576	568013	382277	43924	42282	40834	39547	38391	37342	36379	35483	34640	33837	33066	253773
1990	613842	579676	547435	517008	488292	461190	435610	411465	388675	367162	346854	327682	309583	292496	4893450
	576021	469724	54554	52454	50601	48953	47475	46136	44907	43768	42700	41686	40715	39776	342972
1991	714208	674478	636984	601599	568203	536683	506932	478849	452341	427318	403696	381395	360340	340462	6369280
	580068	67868	65178	62802	60689	58795	57079	55509	54057	52697	51412	50185	49002	47855	461242
Payment Total		3343831	3105917	2872816	2642844	2414187	2184860	1952679	1715220	1469770	1213275	942275	652836	340462	24850972
Payment Error		162511	164507	165940	166263	165030	161853	156358	148156	136813	121823	102576	78299	47855	1526246

Appendix B5

Forecast results

Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Accident Total
1978	102417	90619	80205	71011	62890	73695	69785	66087	62588	59278	56146	53183	50378	47724	0
	108651	97528	75879	69418	55542	62875	63697	72468	65114	62436	57983	56551	48528	39023	0
1979	112003	99131	87766	77729	91082	86249	81678	77353	73261	69390	65726	62260	58980	55875	55875
	98706	95216	83025	72396	104914	94174	77103	70538	71747	77567	68934	70467	43560	5447	5447
1980	122526	108478	96071	112573	106600	100948	95602	90544	85758	81230	76945	72890	69053	65421	134475
	133106	109743	96365	130993	112860	87108	99698	92494	89224	82117	78190	78504	6652	6564	10667
1981	134080	118743	139139	131754	124768	118159	111906	105990	100393	95096	90084	85341	80852	76603	242796
	125731	141478	144336	124854	107034	122015	110514	93517	95885	97626	83692	8139	8029	7939	17775
1982	146769	171977	162847	154211	146041	138312	130998	124079	117531	111336	105472	99923	94672	89701	389768
	161765	174888	168704	156514	145495	138954	125480	106927	111179	118054	9980	9840	9727	9632	27629
1983	212569	201283	190606	180506	170951	161910	153356	145263	137603	130355	123496	117004	110860	105044	586759
	226364	203191	179136	159835	156670	153108	142187	160637	139511	12264	12085	11940	11819	11712	41290
1984	248795	235596	223110	211297	200121	189546	179540	170072	161112	152633	144608	137013	129823	123018	848207
	228410	216837	242050	249422	205644	220996	169549	166858	15105	14873	14685	14528	14391	14265	60147
1985	291210	275774	261170	247353	234281	221911	210206	199130	188648	178728	169338	160451	152039	144076	1192411
	277867	262472	265375	227499	221660	247187	207918	18647	18342	18096	17890	17712	17549	17394	86033
1986	340876	322821	305740	289579	274287	259817	246124	233166	220902	209295	198309	187910	178066	168748	1642521
	302519	360015	343485	224336	220334	234427	23075	22671	22344	22071	21836	21623	21422	21224	121358
1987	399034	377916	357936	339031	321143	304215	288195	273034	258686	245104	232249	220081	208561	197655	2227780
	393525	388054	383425	326081	271278	28620	28082	27644	27280	26966	26684	26420	26163	25904	169288
1988	467141	442439	419067	396951	376023	356219	337476	319738	302948	287056	272013	257773	244291	231528	2985066
	450855	333667	398276	382277	35579	34859	34271	33780	33358	32981	32631	32294	31959	31618	233969
1989	546903	518006	490664	464792	440307	417136	395206	374450	354804	336207	318603	301937	286159	271221	3960823
	572576	568013	382277	44327	43362	42569	41906	41335	40828	40361	39915	39477	39036	38584	320806
1990	640319	606515	574527	544257	515610	488498	462838	438550	415559	393796	373193	353688	335221	317736	5213472
	576021	469724	55342	54048	52978	52079	51306	50620	49993	49400	48822	48247	47665	47067	436839
1991	749733	710185	672760	637343	603825	572100	542073	513650	486745	461275	437161	414332	392716	372249	6816414
	580068	69230	67496	66053	64835	63786	62857	62012	61219	60454	59700	58943	58174	57385	591207
Payment Total		3482205	3245878	3013117	2782159	2551068	2317699	2079653	1834225	1578352	1308536	1020773	710452	372249	26296366
Payment Error		190959	196679	201036	203484	203570	200874	194962	185356	171493	152700	128154	96835	57385	1997089

Appendix C1

INSURER "A" WRITING SMALL EXPOSURE NSW CTP

PAID LOSSES IN "$000"

Development Year

Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
4Q-89	222	18	126	40	60	480	194	412	304	176	270	1646	678	388	782	402	654	852	498	592	592	592	592
1Q-90	222	48	98	168	194	120	104	58	102	518	722	220	910	364	678	1726	898	404	348	592	490	490	
2Q-90	222	34	68	130	172	68	78	656	98	248	262	1444	382	132	812	400	326	822	310	1124	310		
3Q-90	222	44	212	244	80	796	256	1932	40	944	218	394	960	870	394	338	400	382	280	374			
4Q-90	222	12	40	64	126	90	284	82	266	390	40	206	374	402	250	624	674	536	204				
1Q-91	222	34	84	102	208	90	152	196	116	222	1314	478	382	222	574	392	424	210					
2Q-91	222	20	88	64	100	82	80	36	68	134	430	184	1148	324	150	514	216						
3Q-91	222	30	70	208	60	278	90	310	310	8388	850	640	1096	1598	716	950							
4Q-91	222	14	56	88	74	48	16	94	36	382	438	1118	86	204	150								
1Q-92	222	30	50	88	98	18	166	90	348	126	734	680	220	60									
2Q-92	222	8	40	38	38	14	70	230	94	424	376	222	32										
3Q-92	222	30	44	68	84	110	176	82	596	494	136	170											
4Q-92	222	30	58	88	160	166	442	546	236	276	176												
1Q-93	222	30	94	102	64	198	256	708	564	376													
2Q-93	222	36	120	62	136	408	440	586	370														
3Q-93	222	40	64	130	150	278	114	314															
4Q-93	222	38	168	200	110	280	234																
1Q-94	222	56	82	148	222	294																	
2Q-94	222	44	112	102	200																		
3Q-94	222	28	60	270																			
4Q-94	222	32	130																				
1Q-95	222	26																					
2Q-95	222																						

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6	SEP 93	AN ANALYSIS OF THE EQUITY INVESTMENTS OF AUSTRALIAN SUPERANNUATION FUNDS	David M Knox
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