On the Vandermonde matrix and its role in mathematical finance

Ragnar Norberg
Univ. Copenhagen/London School of Economics/Univ. Melbourne

Summary: A short proof is given for the nonsingularity of the Vandermonde matrix (a classic) and of the matrix obtained by subtracting 1 from all of its elements (novel). The results are needed to establish completeness of certain zero coupon bond markets driven by more than one source of randomness. In quest for examples, a mixture of Vasiček models is proposed.

Key-words: Vandermonde matrix, bond markets, completeness, Ornstein-Uhlenbeck process.

1 Introduction

A. The Vandermonde matrix.
Let $\mathbf{A}_n$ denote the generic $n \times n$ matrix of the form

$$\mathbf{A}_n = \left(e^{\alpha_i \beta_j} \right)_{i=1,\ldots,n}^{j=1,\ldots,n} , \tag{1.1}$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are reals. This is a classic in matrix theory, known as the generalized *Vandermonde matrix* (usually its elements are written in the form $x_i^{\beta_j}$ with $x_i > 0$). It is well known that it is non-singular iff all α_i are different and all β_j are different, see Gantmacher [3] p. 87.

B. Purpose of the present study.
The matrix $\mathbf{A}_n$ in (1.1) and its close relative

$$\mathbf{A}_n - \mathbf{1}_n \mathbf{1}_n' = \left(e^{\alpha_i \beta_j} - 1 \right)_{i=1,\ldots,n}^{j=1,\ldots,n} , \tag{1.2}$$

arise naturally in zero coupon bond prices based on spot interest rates driven by certain homogeneous Markov processes. It turns out that, in such bond markets, the issue of completeness is closely related to the rank of the two archetype matrices. Roughly speaking, non-singularity of matrices of types (1.1) or (1.2) ensures that any simple T-claim can be duplicated by a portfolio consisting of the risk-free bank account and a sufficiently large number of zero coupon bonds. The non-singularity results are proved in Section 2, and applications to bond markets are presented in Section 3.

2 Two properties of the Vandermonde matrix

A. The main result.

We take the opportunity here to provide a short proof of the quoted result on nonsingularity of the Vandermonde matrix in (1.1), and will supply a similar result about its relative defined in (1.2).

Theorem

(i) *If the* α_i *are all different and the* β_j *are all different, then* $\mathbf{A}_n$ *is non-singular.*

(ii) *If, furthermore, the* α_i *and the* β_j *are all different from 0, then* $\mathbf{A}_n - \mathbf{1}_n\mathbf{1}_n'$ *is non-singular.*

Proof: The proof goes by induction. Let H_n be the hypothesis stated in the two items of the lemma. Trivially, H_1 is true. Assuming that H_{n-1} is true, we need to prove H_n.

Addressing first item (i) of the the hypothesis, it suffices to prove that $\det(\mathbf{A}_n) \neq 0$. Recast this determinant as

$$
\begin{aligned}
\det(\mathbf{A}_n) &= \left(\prod_{j=1}^{n} e^{\alpha_n\beta_j}\right) \det \begin{pmatrix} e^{(\alpha_1-\alpha_n)\beta_1} & \cdot\ \cdot\ \cdot & e^{(\alpha_1-\alpha_n)\beta_n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ e^{(\alpha_{n-1}-\alpha_n)\beta_1} & \cdot\ \cdot\ \cdot & e^{(\alpha_{n-1}-\alpha_n)\beta_n} \\ 1 & \cdot\ \cdot\ \cdot & 1 \end{pmatrix} \\
&= \left(\prod_{j=1}^{n} e^{\alpha_n\beta_j} \prod_{i=1}^{n-1} e^{(\alpha_i-\alpha_n)\beta_n}\right) \det \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{1}_{n-1} \\ \mathbf{1}_{n-1}' & 1 \end{pmatrix} \qquad (2.1)
\end{aligned}
$$

where

$$
\mathbf{A}_{n-1} = \left(e^{(\alpha_i-\alpha_n)(\beta_j-\beta_n)}\right)_{i=1,\dots,n-1}^{j=1,\dots,n-1}. \qquad (2.2)
$$

The determinant appearing in (2.1) remains unchanged upon subtracting the n-th row of the matrix from all other rows, which gives

$$
\begin{aligned}
\det \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{1}_{n-1} \\ \mathbf{1}_{n-1}' & 1 \end{pmatrix} &= \det \begin{pmatrix} \mathbf{A}_{n-1} - \mathbf{1}_{n-1}\mathbf{1}_{n-1}' & \mathbf{0}_{n-1} \\ \mathbf{1}_{n-1}' & 1 \end{pmatrix} \\
&= \det\left(\mathbf{A}_{n-1} - \mathbf{1}_{n-1}\mathbf{1}_{n-1}'\right). \qquad (2.3)
\end{aligned}
$$

Now, since the α_i are all different and also the β_j are all different, the matrix $\mathbf{A}_{n-1}$ in (2.2) is of the form required in item (ii) of the lemma and so, by the assumed hypothesis H_{n-1}, $\det(\mathbf{A}_{n-1} - \mathbf{1}_{n-1}\mathbf{1}'_{n-1}) \neq 0$. It follows from (2.1) and (2.3) that $\det(\mathbf{A}_n) \neq 0$, hence item (i) of H_n holds true.

Next, we turn to item (ii) of H_n. Preparing for an ad absurdum argument, assume that $\mathbf{A}_n$ is as specified in item (ii) of the lemma and that $\mathbf{A}_n - \mathbf{1}_n\mathbf{1}'_n$ is singular. Then there exists a vector $\mathbf{c} = (c_1, \dots, c_n)' \neq \mathbf{0}_n$ such that

$$\mathbf{A}_n\mathbf{c} = \mathbf{1}_n\mathbf{1}'_n\mathbf{c}\,. \tag{2.4}$$

Introducing the function

$$f(\alpha) = \sum_{j=1}^{n} c_j e^{\alpha\beta_j}\,,$$

and putting $\alpha_0 = 0$, we can spell out (2.4) as

$$f(\alpha_0) = f(\alpha_1) = \cdots = f(\alpha_n)\,, \tag{2.5}$$

that is, f assumes the same value at $n+1$ distinct values of α. Since f is continuously differentiable, Rolle's theorem implies that the derivative f' of f is 0 at n distinct values $\alpha_1^*, \dots, \alpha_n^*$ (say) of α. Now,

$$f'(\alpha) = \sum_{j=1}^{n} c_j\beta_j e^{\alpha\beta_j}\,,$$

and since some c_j are different from 0 and all β_j are different from 0, it follows that the matrix $\mathbf{A}_n^* = \left(e^{\alpha_i^*\beta_j}\right)_{i=1,\dots,n}^{j=1,\dots,n}$ should be singular. This contradicts the previously established item (i) under H_n, showing that the assumed singularity of $\mathbf{A}_n - \mathbf{1}_n\mathbf{1}'_n$ is absurd. We conclude that also item (ii) of H_n holds true. $\square$

B. Remarks.
In fact, if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$, then $\det(\mathbf{A}_n) > 0$ (see [3]). If we take this fact for granted, (2.1) and (2.3) show that also $\det(\mathbf{A}_n - \mathbf{1}_n\mathbf{1}'_n) > 0$, implying that the latter is non-singular under the hypothesis of item (ii) in the theorem. The sign of a general Vandermonde determinant is, of course, the product of the signs of the row and column permutations needed to order the α_i and the β_j by their size.

3 Applications to finance

A. Zero coupon bond prices.
A zero coupon bond with maturity T, or just T-bond in short, is the simple

contingent claim of 1 at time T. Taking an arbitrage-free financial market for granted, the price process $\{p(t,T)\}_{t\in[0,T]}$ of the T-bond is

$$p(t,T) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r_u\,du} \middle| \mathcal{F}_t \right] , \tag{3.1}$$

where $\tilde{\mathbb{E}}$ denotes expectation under some martingale measure, and $\mathcal{F}_t$ is the information available at time t.

We will provide some examples where the results in Section 2 are instrumental for establishing linear independence of price processes of bonds with different maturities. The issue is non-trivial only in cases where the bond prices are governed by more than one source of randomness, of course, so we have to look into cases where the spot rate of interest is driven by more than one martingale.

B. Markov chain interest rate.
In a previous paper [4] the author proposed to model the spot rate of interest $\{r_t\}_{t\geq 0}$ as a continuous time, homogeneous, recurrent Markov chain with finite state space $\{r^1, \ldots, r^n\}$. The model has been further investigated in [2] and [6]. From the latter we quote the following results.

We are working under some martingale measure given by an infinitesimal matrix $\tilde{\mathbf{\Lambda}} = (\tilde{\lambda}^{jk})$ of the Markov chain, that is, the transition intensities are $\tilde{\lambda}^{jk}$, $j \neq k$, and $\tilde{\lambda}^{jj} = -\sum_{k;k\neq j} \tilde{\lambda}^{jk}$. The price at time $t \leq T$ of a zero coupon bond with maturity T is

$$p(t,T) = \sum_{j=1}^{n} I_t^j p^j(t,T)\,,$$

where $I_t^j = 1[r_t = r^j]$ and

$$p^j(t,T) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r_u\,du} \middle| r_t = r^j \right] .$$

The vector of state-wise prices,

$$\mathbf{p}(t,T) = (p^j(t,T))_{j=1,\ldots,n}\,,$$

is given by

$$\mathbf{p}(t,T) = \exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})(T-t)\}\mathbf{1} = \mathbf{\Phi}\,\mathrm{Diag}(e^{\rho_j(T-t)})\,\mathbf{\Psi}\mathbf{1}\,, \tag{3.2}$$

where $\mathbf{R} = \mathrm{Diag}(r^j)$ is the $n \times n$ diagonal matrix with the entries r^j down the principal diagonal, $\mathbf{1}$ is the n-vector with all entries equal to 1, $\tilde{\rho}^j$,

$j = 1, \ldots, n$, are the eigenvalues of $\tilde{\mathbf{\Lambda}} - \mathbf{R}$, and $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are the $n \times n$ matrices formed by the right and left eigenvectors, respectively.

It is proved in [6] that the price processes of m zero coupon bonds with maturities $T_1 < \cdots < T_m$ are linearly independent only if the matrix

$$(e^{\tilde{\rho}^j T_i})_{i=1,\ldots,m}^{j=1,\ldots,n}$$

has rank m. From item (i) in the theorem in Paragraph 2A we conclude that this is the case if there are at least m distinct eigenvalues $\tilde{\rho}^j$. It also follows that the market consisting of the bank account with price process $\exp\left(\int_0^t r_s\, ds\right)$ and the m zero coupon bonds is complete for the class of all $\mathcal{F}^r_{T_1}$-claims only if both the number of distinct eigenvalues and the number of bonds are no less than the maximum number of states that can be directly accessed from any single state of the Markov chain.

C. Mixed Vaciček interest rate.

The Vasiček model takes the spot rate of interest to be an Ornstein-Uhlenbeck process given by

$$dr_t = \alpha\,(\rho - r_t)\,dt + \sigma\,d\tilde{W}_t\,. \tag{3.3}$$

Here ρ is the stationary mean of the process, α is a positive mean reversion parameter, σ is a positive volatility parameter, and $\tilde{W}$ is a standard Brownian motion under a martingale measure. The dynamics of the discounted T-bond price,

$$\tilde{p}(t, T) = e^{-\int_0^t r_u\, du} p(t, T)\,, \tag{3.4}$$

is

$$d\tilde{p}(t, T) \;=\; \tilde{p}(t, T)\frac{\sigma}{\alpha}\left(e^{-\alpha\,(T-t)} - 1\right)\,d\tilde{W}_t\,, \tag{3.5}$$

confer e.g. [1] or [5]. Obviously, any $\mathcal{F}^{\tilde{W}}_T$ claim can be duplicated by a self-financing portfolio in the T-bond and the bank account, and so the completeness issue is trivial in this model.

To create an example where one bond is not sufficient to complete the market, let us concoct a mixed Vasiček model by putting

$$r_t = \sum_{j=1}^{n} r_t^j\,,$$

where the r^j are independent Ornstein-Uhlenbeck processes,

$$dr_t^j = \alpha^j\,(\rho^j - r_t^j)\,dt + \sigma^j\,d\tilde{W}_t^j\,,$$

$j = 1, \dots, n$, and the $\tilde{W}^j$ are independent standard Brownian motions. We assume that the α^j are all distinct (otherwise we could gather all processes r^j with coinciding mean reversion parameter into one Ornstein-Uhlenbeck process). The mixed Vasiček process is not mean-reverting in the same simple sense as the traditional Vasiček process. It is stationary, however, and is apt to describe interest that is subject to several random phenomena, each of mean-reverting type.

By the assumed independence, the price of the T-bond is just

$$p(t,T) = \prod_{j=1}^{n} p^j(t,T)\,,$$

where $p^j(t,T) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r_u\,du} \,\middle|\, r_t^j \right]$, and the discounted price is

$$\tilde{p}(t,T) = \prod_{j=1}^{n} \tilde{p}^j(t,T)\,,$$

where $\tilde{p}^j(t,T)$ is the "j-analogue" to (3.4). By virtue of (3.5), we conclude that the discounted T-bond price has dynamics

$$d\tilde{p}(t,T) = \tilde{p}(t,T) \sum_{j=1}^{n} \frac{\sigma^j}{\alpha^j} \left(e^{-\alpha^j\,(T-t)} - 1 \right) d\tilde{W}_t^j\,. \tag{3.6}$$

Now, consider the market consisting of the bank account and m zero coupon bonds with maturities $T_1 < \cdots < T_m$. From (3.6) it is seen that this market is complete for the class of $\mathcal{F}_{T_1}^{\tilde{W}_1,\dots,\tilde{W}_n}$-claims if and only if the matrix

$$\left(e^{-\alpha^j\,(T_i-t)} - 1 \right)_{j=1,\dots,n}^{i=1,\dots,m} \tag{3.7}$$

has rank n. By virtue of item (ii) in the theorem in Paragraph 2A, we conclude that this is the case if $m \geq n$.

D. Mixed Poisson-driven Ornstein-Uhlenbeck interest rate. Referring to [5], let us replace the Brownian motions in Paragraph C above with independent compensated Poisson processes, that is,

$$d\tilde{W}_t^j = dN_t^j - \lambda^j\,dt\,,$$

where each N^j is a Poisson process with intensity λ^j. Instead of (3.6) we obtain

$$d\tilde{p}(t,T) = \tilde{p}(t-,T) \sum_{j=1}^{n} \left(\exp\left\{ \frac{\sigma^j}{\alpha^j} \left(e^{-\alpha^j\,(T-t)} - 1 \right) \right\} - 1 \right) d\tilde{W}_t^j\,. \tag{3.8}$$

It is seen from (3.8) that the market consisting of the bank account and m zero coupon bonds with maturities $T_1 < \cdots < T_m$ is complete for the class of $\mathcal{F}_{T_1}^{\tilde{N}_1,\ldots,\tilde{N}_n}$-claims if and only if the matrix

$$\left(\exp\left\{ \frac{\sigma^j}{\alpha^j} \left(e^{-\alpha^j\,(T-t)} - 1 \right) \right\} - 1 \right)_{j=1,\ldots,n}^{i=1,\ldots,m}$$

has rank n. By item (ii) in the theorem in Paragraph 2A, we know that the matrix (3.7) has full rank. Thus, completeness of a market consisting of the bank account and at least n bonds would be established – and we would be done – if we could prove that the $n \times m$ matrix $(e^{\gamma_{ji}} - 1)$ has full rank whenever (γ_{ji}) has full rank. With this conjecture our study of these problems will have to halt for the time being.

Acknowledgments
The present paper was drafted during the author's stay as GIO professorial fellow at the Centre for Actuarial Studies, University of Melbourne, December 98 – January 99, and finished during a visit at the Department of Mathematics, Stanford University, November 99 - March 00. The work was partly supported by the Mathematical Finance Network, Grant No. 9800335, under the Danish Social Science Research Council. My thanks are due to these institutions and to David Dickson, Melbourne, and George Papanicolaou, Stanford.

References

[1] Björk, T. (1998): *Arbitrage Theory in Continuous Time*, Oxford University Press.

[2] Elliott, R.J., Kopp, P.E. (1998): *Mathematics of financial markets*, Springer-Verlag.

[3] Gantmacher, F.R. (1959): *Matrizenrechnung II*, VEB Deutscher Verlag der Wissenschaften, Berlin.

[4] Norberg, R. (1995): A time-continuous Markov chain interest model with applications to insurance. *J. Appl. Stoch. Models and Data Anal.*, 245-256.

[5] Norberg, R. (1998): Vasiček beyond the normal. *Working paper No. 152*, Laboratory of Actuarial Math., Univ. Copenhagen.

[6] Norberg, R. (1999): *A Markov chain financial market.* (Forthcoming.)

Address for correspondence:
Ragnar Norberg
Department of Statistics
London School of Economics
Houghton Street
London WC2A 2AE
United Kingdom

R.Norberg@lse.ac.uk