ISBN 0 7340 2194 1

The Deficit at Ruin in the Stationary Renewal Risk Model

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RESEARCH PAPER NUMBER 99

June 2002

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The deficit at ruin in the stationary renewal risk model

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Abstract

Properties of the distribution of the deficit at ruin in the stationary renewal risk model are studied. A mixture representation for the conditional distribution of the deficit at ruin (given that ruin occurs) is derived, as well as a stochastic decomposition involving the residual lifetime associated with the maximal aggregate loss. When the individual claims have a phase-type distribution, the deficit at ruin is also of phase-type.

Keywords: Sparre Andersen model, ladder height, maximal aggregate loss, compound geometric convolution, defective renewal equation, phase-type distribution, DFR.

1. Introduction and notation

In this paper we are interested in studying the distribution of the deficit at ruin in the stationary renewal risk model, or equivalently the stationary Sparre Andersen model.

We begin by describing the (ordinary) renewal risk model. The number of claims process $\{N_t : t \geq 0\}$ is a renewal process whereby the interclaim times $\{W_1, W_2, \ldots\}$ are assumed to be independent and identically distributed positive random variables. Thus, W_1 is the time until the first claim occurs, and for $i = 2, 3, \ldots,$ W_i is the time between the $(i-1)$-th and i-th claim. Let W_i have distribution function (df) $K(t) = 1 - \bar{K}(t) = Pr\{W \leq t\}$ where W is an arbitrary W_i. Let $E\{W\} = \int_0^\infty t dK(t) < \infty$, and $\tilde{k}(s) = E\{e^{-sW}\} = \int_0^\infty e^{-st} dK(t)$. Anticipating what follows, we introduce the integrated tail or equilibrium df $K_1(t) = 1 - \bar{K}_1(t) = \int_0^t \bar{K}(x)dx / E\{W\}$ and (e.g. Feller, 1971, p. 435), $\tilde{k}_1(s) = \int_0^\infty e^{-st} dK_1(t) = \{1 - \tilde{k}(s)\}/\{sE\{W\}\}$.

The individual claim amounts $\{X_1, X_2, \ldots\}$ are iid positive random variables with df $P(x) = 1 - \bar{P}(x) = Pr\{X \leq x\}$ where X is an arbitrary X_i, which itself represents the amount of the i-th claim. Let $E\{X\} = \int_0^\infty x dP(x) < \infty$, and $\tilde{p}(s) = E\{e^{-sX}\} = \int_0^\infty e^{-sx} dP(x)$. Also, we introduce

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the integrated tail distribution as $P_1(x) = 1 - \bar{P}_1(x) = \int_0^x \bar{P}(y)dy/E\{X\}$, with $\tilde{p}_1(s) = \int_0^\infty e^{-sx}dP_1(s) = \{1-\tilde{p}(s)\}/\{sE\{X\}\}$.

Premiums are assumed to be payable continuously at rate ξ per unit time where $\xi = (1+\theta)E\{X\}/E\{W\}$ where $\theta > 0$ is the relative security loading. The insurer's surplus at time t is defined as $U_t = u + \xi t - \sum_{i=1}^{N_t} X_i$, where $u \geq 0$ is the initial surplus. The time of ruin is $T = \inf\{t : U_t < 0\}$ and $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$. The deficit at ruin (if ruin occurs) is $|U_T|$. The probability of ultimate ruin, as a function of the initial surplus, is $\psi(u) = Pr\{T < \infty\} = 1 - \delta(u)$, and it is well known (e.g. Embrechts et al, 1997, pp. 26-7, or Rolski et al, 1999, section 6.5) that

$$\psi(u) = Pr\{L > u\} = \sum_{n=1}^{\infty}(1-\rho)\rho^n \bar{F}^{*n}(u), \; u \geq 0, \tag{1.1}$$

where $\rho = \psi(0)$ is the probability of a drop in surplus below its initial level, L is the well-known maximal aggregate loss in the Sparre Andersen model, and $F(y) = 1 - \bar{F}(y)$ is the so-called ladder height df, i.e. the df of the amount of a drop in surplus, given that a drop below its initial level occurs. Let $\tilde{f}(s) = \int_0^\infty e^{-sy}dF(y)$, and $F^{*n}(u) = 1 - \bar{F}^{*n}(u)$ is the df of the n-fold convolution of F with itself, i.e. $\int_0^\infty e^{-su}dF^{*n}(u) = \{\tilde{f}(s)\}^n$. Then from (1.1) and Feller (1971, p. 435),

$$\tilde{\psi}(s) = \int_0^\infty e^{-su}\psi(u)du = \frac{1 - E\{e^{-sL}\}}{s}, \tag{1.2}$$

where

$$E\{e^{-sL}\} = \int_{0-}^\infty e^{-su}d\delta(u) = \frac{1-\rho}{1-\rho\tilde{f}(s)}. \tag{1.3}$$

A generalization of the ruin function is the defective deficit tail $\bar{G}(u,y) = Pr\{T < \infty, |U_T| > y\}$, with $\psi(u) = \bar{G}(u,0)$. By conditioning on the amount of the first drop in surplus, one obtains the defective renewal equation

$$\bar{G}(u,y) = \rho\int_0^u \bar{G}(u-t,y)dF(t) + \rho\bar{F}(u+y), \tag{1.4}$$

and it can be shown (e.g. Willmot, 2002) that

$$\bar{G}(u,y) = \frac{\rho}{1-\rho}\int_{0-}^u \bar{F}(u+y-t)d\delta(t). \tag{1.5}$$

In this paper we are interested in the stationary or equilibrium renewal risk model, which is identical in all aspects to the (ordinary) renewal risk model except that the time until the first claim occurs has df $K_1(t)$ rather than $K(t)$. See Grandell (1991, pp. 67-9) or Willmot and Lin (2001, p.

231), where it is shown that the ruin probability $\psi^e(u) = 1 - \delta^e(u)$ in the stationary renewal risk model is given by

$$\psi^e(u) = \frac{1}{1+\theta}\int_0^u \psi(u-t)dP_1(t) + \frac{1}{1+\theta}\bar{P}_1(u), \; u \geq 0. \tag{1.6}$$

See also Asmussen (2000, p. 142), who discusses the fact that $\psi^e(u)$ may also be interpreted as the tail of the limiting distribution of the virtual waiting time in the $G/G/1$ queue.

In connection with (1.6), it is useful to define the compound geometric convolution df $C(y) = 1 - \bar{C}(y) = \delta * P_1(y)$, i.e. $C(y) = Pr\{L + Y \leq y\}$ where Y has df $P_1(y) = Pr\{Y \leq y\}$ and is statistically independent of the maximal aggregate loss L in the renewal risk model. Then

$$\tilde{c}(s) = \int_0^\infty e^{-sy}dC(y) = \tilde{p}_1(s)E\{e^{-sL}\}, \tag{1.7}$$

with $E\{e^{-sL}\}$ given by (1.3), and (1.6) may be expressed as

$$\psi^e(u) = \frac{\bar{C}(u)}{1+\theta}. \tag{1.8}$$

Let L^e denote the maximal aggregate loss in the stationary renewal risk model, so that one has $\psi^e(u) = Pr\{L^e > u\}$, and from (1.6),

$$\psi^e(0) = Pr\{L^e > 0\} = \frac{1}{1+\theta}. \tag{1.9}$$

Therefore from (1.8), one has the stochastic law

$$Pr\{L^e > u | L^e > 0\} = \bar{C}(u) = Pr\{L + Y > u\}. \tag{1.10}$$

Now that these preliminary quantities have been defined, the remainder of the paper is arranged as follows. In the next section, a number of results referring to the deficit at the time of ruin are established in the stationary renewal risk model. In section 3, we focus our attention on the special case when the claim amounts follow a distribution belonging to the phase-type family of distributions. In section 4, illustrations are presented for both the stationary and the ordinary renewal risk models, in the case of a particular non-Poisson claim number process.

2. Basic properties of the deficit

The following straightforward proposition is needed in the subsequent analysis.

Proposition 2.1 In the stationary renewal risk model, $P_1(y)$ is the df of the first drop in surplus, given that the surplus ever drops below its initial level.

Proof: It follows from (1.9) that $\psi^e(0) = 1/(1+\theta)$ may be interpreted as the probability that the surplus ever drops below its initial level u. Now let $A(y) = 1 - \bar{A}(y)$ be the df of the amount of the first drop in surplus below its initial level, given that the surplus ever drops, and let $\tilde{a}(s) = \int_0^\infty e^{-sy} dA(y)$. Conditioning on the first drop in surplus yields, by the law of total probability,

$$\psi^e(u) = \psi^e(0) \int_0^u \psi(u-t)dA(t) + \psi^e(0)\bar{A}(u),$$

using the fact that the process reverts to the ordinary renewal risk process upon occurrence of the first claim, and a drop can only occur when a claim occurs. Since $\psi^e(0) = 1/(1+\theta)$ from (1.9), taking Laplace-Stieltjes transforms yields (with $\tilde{\psi}^e(s) = \int_0^\infty e^{-su}\psi^e(u)du$) the relationship $(1+\theta)\tilde{\psi}^e(s) = \tilde{\psi}(s)\tilde{a}(s) + \{1-\tilde{a}(s)\}/s$. But from (1.6), one also has $(1+\theta)\tilde{\psi}^e(s) = \tilde{\psi}(s)\tilde{p}_1(s) + \{1-\tilde{p}_1(s)\}/s$. Equating these transforms yields $\{\tilde{\psi}(s) - 1/s\}\{\tilde{a}(s) - \tilde{p}_1(s)\} = 0$. However, if $\tilde{\psi}(s) = 1/s$, then $\psi(u) = 1$ for all u which is impossible since $\theta > 0$. Therefore, we must have that $\tilde{a}(s) = \tilde{p}_1(s)$, which implies that $A(y) = P_1(y)$. □

We are now in a position to consider the deficit at ruin in the stationary renewal risk model. Let the defective deficit tail be $\bar{G}^e(u,y) = Pr\{$ruin occurs and the deficit at ruin exceeds $y\}$, where again the dependency on the initial surplus u is explicitly noted. By conditioning on the amount of the first drop in surplus, it follows immediately that

$$\bar{G}^e(u,y) = \frac{1}{1+\theta}\int_0^u \bar{G}(u-t,y)dP_1(t) + \frac{1}{1+\theta}\bar{P}_1(u+y), \tag{2.1}$$

where $\bar{G}(u,y)$ as given by (1.4) is the corresponding quantity in the ordinary renewal risk model. By probabilistic considerations or (1.6), it is clear that

$$\psi^e(u) = \bar{G}^e(u,0). \tag{2.2}$$

The following alternative representation for $\bar{G}^e(u,y)$ is of use in what follows.

Proposition 2.2 The defective deficit tail satisfies

$$\bar{G}^e(u,y) = \frac{\rho}{(1+\theta)(1-\rho)}\int_0^u \bar{F}(u+y-t)dC(t) + \frac{1}{1+\theta}\bar{P}_1(u+y). \tag{2.3}$$

Proof: It follows from (1.3) and the convolution representation (1.5) that

$$\begin{aligned}
\int_0^\infty e^{-su}\bar{G}(u,y)du &= \frac{\rho}{1-\rho}\int_0^\infty e^{-su}\int_{0-}^u \bar{F}(u+y-t)d\delta(t)du \\
&= \frac{\rho}{1-\rho}\left\{\int_0^\infty e^{-su}\bar{F}(u+y)du\right\}\int_0^\infty e^{-su}d\delta(u) \\
&= \frac{\rho}{1-\rho}E\{e^{-sL}\}\int_0^\infty e^{-su}\bar{F}(u+y)du,
\end{aligned}$$

and thus from (1.7),

$$\begin{aligned}
\int_0^\infty e^{-su}\int_0^u \bar{G}(u-t,y)dP_1(t)du &= \tilde{p}_1(s)\int_0^\infty e^{-su}\bar{G}(u,y)du \\
&= \frac{\rho}{1-\rho}\tilde{c}(s)\int_0^\infty e^{-su}\bar{F}(u+y)du,
\end{aligned}$$

which implies by inversion of the Laplace transform that

$$\int_0^u \bar{G}(u-t,y)dP_1(t) = \frac{\rho}{1-\rho}\int_0^u \bar{F}(u+y-t)dC(t).$$

Then (2.3) follows from (2.1). □

When $y=0$, (2.3) becomes

$$\psi^e(u) = \frac{\rho}{(1+\theta)(1-\rho)}\int_0^u \bar{F}(u-t)dC(t) + \frac{1}{1+\theta}\bar{P}_1(u). \tag{2.4}$$

In what follows it is convenient to introduce the residual lifetime df $F_x(y) = 1 - \bar{F}_x(y)$ defined for $x \geq 0$ by $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$. Similarly, let $\bar{P}_{1,x}(y) = 1 - P_{1,x}(y) = \bar{P}_1(x+y)/\bar{P}_1(x)$. The proper distribution of the deficit, conditional on ruin occurring, is $G_u^e(y) = 1 - \bar{G}_u^e(y)$ where $\bar{G}_u^e(y) = \bar{G}^e(u,y)/\psi^e(u)$. We have the following theorem.

Theorem 2.1 The deficit df satisfies the mixture relationship

$$G_u^e(y) = \frac{\rho\int_0^u F_{u-t}(y)\bar{F}(u-t)dC(t) + (1-\rho)\bar{P}_1(u)P_{1,u}(y)}{\rho\int_0^u \bar{F}(u-t)dC(t) + (1-\rho)\bar{P}_1(u)}. \tag{2.5}$$

Proof: Division of (2.3) by (2.4) results in

$$\bar{G}_u^e(y) = \frac{\rho\int_0^u \bar{F}(u+y-t)dC(t) + (1-\rho)\bar{P}_1(u+y)}{\rho\int_0^u \bar{F}(u-t)dC(t) + (1-\rho)\bar{P}_1(u)}, \tag{2.6}$$

from which (2.5) follows. □

It is clear from (2.5) that $G_u^e(y)$ is a mixture of the residual lifetime df's $F_t(y)$ for $0 < t < u$, and of $P_{1,u}(y)$. Of course, when $u = 0$, one has from (2.5) that

$$G_0^e(y) = P_1(y), \tag{2.7}$$

the same as in the classical model. The next result follows immediately.

Corollary 2.1 If the claim size df is the exponential df $P(x) = 1 - e^{-x/E\{X\}}$, then $G_u^e(y) = 1 - e^{-y/E\{X\}}$.

Proof: One has easily that $P_{1,u}(y) = P(y)$, and also (e.g. Rolski et al, 1999, p. 248) $F(y) = P(y)$ and hence $F_t(y) = P(y)$. The result follows immediately from (2.5). □

We note that this result also follows easily from the argument given by Bowers et al (1997, p. 414) in the context of the classical risk model.

Other properties also follow from the mixture relationship (2.5). For example, if $P(y)$ is a combination of exponentials then $G_u^e(y)$ is a different combination of the same exponentials. Since this is a special case of the phase-type situation to be discussed later, we omit the details. As a second application, we consider the decreasing failure rate or DFR class of distributions. The df $P(y)$ is said to be DFR if $\bar{P}(x+y)/\bar{P}(x)$ is nondecreasing in x for all fixed $y \geq 0$ (e.g. Faguioli and Pellerey, 1994).

Corollary 2.2 If the claim size df $P(y)$ is DFR, then $G_u^e(y)$ is also DFR.

Proof: If $P(y)$ is DFR, then $P_1(y)$ is DFR (e.g. Faguioli and Pellerey, 1994) and $F(y)$ is also DFR (Szekli, 1986). Thus, $P_{1,u}(y)$ and $F_t(y)$ are DFR, and since the DFR property is preserved under mixing (e.g. Barlow and Proschan, 1975, p. 103), $G_u^e(y)$ is DFR since (2.5) holds. □

We now demonstrate that a stochastic decomposition of the residual lifetime of L^e involving the deficit holds, in much the same way as it does for the ordinary renewal risk model (e.g. Willmot, 2002).

Theorem 2.2 Let V_u^e be statistically independent of L with df $G_u^e(y)$. Then the tail of the residual lifetime distribution of L^e satisfies

$$\frac{\psi^e(u+y)}{\psi^e(u)} = Pr\{L + V_u^e > y\}, \; y \geq 0. \tag{2.8}$$

Proof: We employ a probabilistic proof along the lines of Dickson (1989). An analytic proof in the context of compound geometric convolutions may be found in Willmot and Cai (2002). It is convenient to define the defective deficit df $G^e(u,y) = \psi^e(u) - \bar{G}^e(u,y)$, representing the probability that ruin occurs (from initial surplus u) and the deficit at ruin is between 0 and y. Then $G_u^e(y) = G^e(u,y)/\psi^e(u)$.

Recall that L^e is the maximal aggregate loss in the stationary renewal risk model, or equivalently the total drop in surplus below its initial level, and L^e has df δ^e. Consider an initial surplus level of $u+y$. Then the event that the total drop in surplus $L^e \leq u+y$ can happen in two ways. The first possibility is that the total drop $L^e \leq u$ with probability $\delta^e(u)$, in which case the surplus always remains above y. For the second possibility, the first drop in surplus exceeds u but does not exceed $u+y$. For this to happen, the surplus must fall below y for the first time to a level $y-t \geq 0$ with probability $dG^e(u,t)$, and then remain above 0 thereafter with probability $\delta(y-t)$, since the process behaves from that point onward like the ordinary renewal risk process. Hence by the law of total probability, it follows that

$$\delta^e(u+y) = \delta^e(u) + \int_0^y \delta(y-t)dG^e(u,t). \tag{2.9}$$

Therefore, rearrangement of (2.9) and division by $\psi^e(u)$ yields

$$1 - \frac{\psi^e(u+y)}{\psi^e(u)} = \int_0^y \delta(y-t)dG_u^e(t),$$

or equivalently,

$$\frac{\psi^e(u+y)}{\psi^e(u)} = \bar{G}_u^e(y) + \int_0^y \psi(y-t)dG_u^e(t), \tag{2.10}$$

which is (2.8). □

We remark that from (1.8), $\psi^e(u+y)/\psi^e(u) = \bar{C}(u+y)/\bar{C}(u)$. Thus, (2.8) may be expressed as

$$Pr\{L^e > u+y | L^e > u\} = \frac{\bar{C}(u+y)}{\bar{C}(u)} = Pr\{L + V_u^e > y\},$$

which, in view of (2.7), implies that (2.8) is a generalization of (1.10), which itself may be recovered from (2.8) by setting $u=0$.

Moreover, moments of the deficit follow from Theorem 2.2.

Corollary 2.3 For $k = 1, 2, 3, \ldots$, one has

$$E\{(L+V_u^e)^k\} = k\int_0^\infty y^{k-1}\frac{\psi^e(u+y)}{\psi^e(u)}dy, \tag{2.11}$$

and in particular, the mean deficit satisfies

$$E\{V_u^e\} = \int_0^\infty \left\{\frac{\psi^e(u+y)}{\psi^e(u)} - \psi(y)\right\}dy. \tag{2.12}$$

Proof: It is easily shown that

$$E\{(L+V_u^e)^k\} = k\int_0^\infty y^{k-1}Pr\{L+V_u^e > y\}dy,$$

which yields (2.11) using (2.8). When $k=1$, (2.11) yields

$$E\{L\}+E\{V_u^e\} = \int_0^\infty \frac{\psi^e(u+y)}{\psi^e(u)}dy$$

which gives (2.12) since $E\{L\} = \int_0^\infty \psi(y)dy$. □

Higher moments of the deficit V_u^e may be obtained from (2.11). We remark that in the case when the individual claim amount distribution is of phase-type, a direct method to calculate moments of V_u^e exists. This is given by Corollary 3.2 in the next section.

Note that the role of $\bar{G}_u^e(y)$ and $\psi(y)$ in (2.10) may be interchanged, yielding

$$\frac{\psi^e(u+y)}{\psi^e(u)} = \psi(y) + \int_{0-}^y \bar{G}_u^e(y-t)d\delta(t), \tag{2.13}$$

which gives an alternative expression for the integrand in (2.12).

Let $\psi_u^e(y) = \psi^e(u+y)/\psi^e(u) = Pr\{L^e > u+y|L^e > u\}$. Using the right hand side of (2.8) and equation (9.3.2) of Willmot and Lin (2001), $\psi_u^e(y)$ satisfies the defective renewal equation (e.g. Willmot and Lin, 2001, p. 174)

$$\psi_u^e(y) = \rho\int_0^y \psi_u^e(y-t)dF(t) + \rho\bar{F}(y) + (1-\rho)\bar{G}_u^e(y), \tag{2.14}$$

which, when solved for $\bar{G}_u^e(y)$, provides an alternative to the mixture representation of (2.6). Moreover, defective renewal equations have various analytic implications for the solution, so that (2.14) is of interest in its own right. In particular, bounds for both $\psi_u^e(y)$ and $\bar{G}_u^e(y)$ can be found from equation (2.14) using the approaches of Willmot et al (2001), Willmot (2002), and Cai and Garrido (2002). We omit the details.

Asymptotics and bounds for many of the quantities of interest involve the Lundberg adjustment coefficient $\kappa > 0$ satisfying $\tilde{f}(-\kappa) = 1/\rho$, or equivalently (e.g. Rolski et al, 1999, pp. 255-9)

$$\tilde{k}(c\kappa)\tilde{p}(-\kappa) = 1. \tag{2.15}$$

The following theorem shows that the asymptotic (as the initial surplus $u \to \infty$) distribution of the deficit in the stationary renewal risk model is still of mixture form, and is the same as that under the (ordinary) renewal risk model.

Theorem 2.3 If $\kappa > 0$ satisfies (2.15) and $\tilde{f}(-\kappa-\epsilon) < \infty$ for some $\epsilon > 0$ with F non-arithmetic, then

$$\bar{G}^e_\infty(y) = \lim_{u\to\infty} \bar{G}^e_u(y) = \frac{\int_0^\infty e^{\kappa t}\bar{F}(y+t)dt}{\int_0^\infty e^{\kappa t}\bar{F}(t)dt}, \; y \geq 0. \tag{2.16}$$

Proof: First note that $\tilde{p}(-\kappa) < \infty$ since (2.15) holds. Thus

$$\tilde{p}_1(-\kappa) = \int_0^\infty e^{\kappa t}\{\bar{P}(t)/E\{X\}\}dt = \{\tilde{p}(-\kappa)-1\}/\{\kappa E\{X\}\} < \infty, \tag{2.17}$$

implying that $\lim_{t\to\infty} e^{\kappa t}\bar{P}(t) = 0$. Equation (2.17) implies that $\lim_{t\to\infty} e^{\kappa t}\bar{P}_1(t) = 0$ since $\int_0^\infty e^{\kappa t}\bar{P}_1(t)dt = \{\tilde{p}_1(-\kappa)-1\}/\kappa < \infty$. Also, by Lundberg's inequality (e.g. Rolski et al, 1999, Section 6.5.2) in the ordinary renewal risk model, $e^{\kappa u}\bar{G}(u,y) \leq e^{\kappa u}\psi(u) \leq 1$. Therefore, by dominated convergence and (2.1),

$$\begin{aligned}&\lim_{u\to\infty} e^{\kappa u}\bar{G}^e(u,y)\\ &= \frac{1}{1+\theta}\int_0^\infty \left\{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)\right\} e^{\kappa t}dP_1(t) + \frac{1}{1+\theta}\lim_{u\to\infty} e^{\kappa u}\bar{P}_1(u+y).\end{aligned}$$

But $0 \leq e^{\kappa u}\bar{P}_1(u+y) \leq e^{\kappa u}\bar{P}_1(u)$, implying that $\lim_{u\to\infty} e^{\kappa u}\bar{P}_1(u+y) = 0$.

Thus,

$$\begin{aligned}\lim_{u\to\infty} e^{\kappa u}\bar{G}^e(u,y) &= \left\{\frac{1}{1+\theta}\int_0^\infty e^{\kappa t}dP_1(t)\right\}\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)\\ &= \frac{\tilde{p}_1(-\kappa)}{1+\theta}\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y).\end{aligned}$$

Since $\bar{G}^e_u(y) = \bar{G}^e(u,y)/\psi^e(u)$ and (2.2) holds, one has

$$\lim_{u\to\infty} \bar{G}^e_u(y) = \frac{\lim_{u\to\infty} e^{\kappa u}\bar{G}^e(u,y)}{\lim_{u\to\infty} e^{\kappa u}\bar{G}^e(u,0)} = \frac{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)}{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,0)} = \lim_{u\to\infty}\bar{G}_u(y),$$

i.e.

$$\bar{G}^e_\infty(y) = \bar{G}_\infty(y), \; y \geq 0. \tag{2.18}$$

Then (2.16) follows from (3.35) of Willmot (2002). □

Equation (2.18) expresses the equivalence between the limiting distributions in the stationary and ordinary cases. Also, (2.16) may be expressed in mixture form as

$$\bar{G}^e_\infty(y) = \frac{\int_0^\infty e^{\kappa t}\bar{F}(t)\bar{F}_t(y)dt}{\int_0^\infty e^{\kappa t}\bar{F}(t)dt}, \; y \geq 0, \tag{2.19}$$

which is considerably simpler than the mixture representation for finite u given in Theorem 2.1. Properties of this limiting distribution may be found in Willmot (2002, Section 3).

3. Phase-type claim amounts

In this section, we assume that the iid claim amounts $\{X_1, X_2, \ldots\}$ are *phase-type distributed.* Specifically, we write $X \sim PH_m(\alpha, S)$, i.e. X has a phase-type distribution with representation (α, S) of dimension m, having df

$$P(x) = 1 - \alpha \exp\{xS\}e^T, \ x \geq 0,$$

where the matrix exponential is defined by

$$\exp\{xS\} = \sum_{n=0}^{\infty} \frac{x^n}{n!} S^n.$$

When a random variable X is phase-type distributed, it can be viewed as the time to absorption in a continuous-time Markov Chain (CTMC) with m transient states (denoted without loss of generality $1, 2, \ldots, m$) and a single absorbing state (likewise denoted as state 0). Here, the row vector α contains the initial probabilities α_j of starting in the various transient states $j = 1, 2, \ldots, m$, while S is an $m \times m$ matrix containing the transition rates among the m transient states. In the foregoing, $\alpha \exp\{xS\}e^T$ represents the probability that absorption has *not* occurred by time x. In this case, the process must be in one of the transient states at that time. Defining J_t as the state of the underlying CTMC at time $t \geq 0$, one readily recognizes

$$Pr\{X > x, J_x = j; j = 1, \ldots, m\} = \alpha \exp\{xS\}.$$

Usually α satisfies $\alpha e^T = 1$, where e^T is a column vector of ones of length m. However, one can just as easily accommodate probability mass at $x = 0$ by allowing for a positive probability $\alpha_0 = 1 - \alpha e^T$ of starting in the absorbing state. We distinguish between these situations by referring in the latter case to a *defective* phase-type distribution. For a more detailed description of phase-type distributions, see Neuts (1981) and Latouche and Ramaswami (1999).

If we assume that $X \sim PH_m(\alpha, S)$, then several well-known ruin-theoretic results follow. In particular (e.g. Asmussen, 2000, pp. 229-30, Proposition 4.3), the ladder height df $F(y) = 1 - \alpha_+^* \exp\{xS\}e^T$ where $\alpha_+^* = \alpha_+/\rho$ and the row vector α_+ is the (defective) unique solution of the fixed-point problem $\alpha_+ = \varphi(\alpha_+)$ where

$$\varphi(\alpha_+) = \alpha\tilde{k}(-\xi(S - Se^T\alpha_+)) = \alpha \int_0^\infty \exp\{t\xi(S - Se^T\alpha_+)\}dK(t).$$

Asmussen also shows that both the maximal aggregate loss L and the stationary maximal aggregate loss L^e follow defective phase-type distributions (see Asmussen, 2000, Theorem 4.4, pp. 230-31). In particular, $L \sim PH_m(\alpha_+, B)$ and $L^e \sim PH_m(\alpha^{(s)}, B)$ where $B = S + D$, $D = -Se^T\alpha_+$, and $\alpha^{(s)} = -\alpha S^{-1}/\{(1+\theta)E\{X\}\}$. In other words, $\psi(u) = Pr\{L > u\} = \alpha_+ \exp\{uB\}e^T$ whereas $\psi^e(u) = Pr\{L^e > u\} = \alpha^{(s)} \exp\{uB\}e^T$.

We remark that examination of the consequences of Asmussen's result for the distribution of L^e leads to the observation that the distribution of the sum $Z = U + V$ of two independent random variables U, V where $U \sim PH_m(\gamma, S)$ and $V \sim PH_m(\alpha_+, B)$ for $B = S + D$ is again phase-type for *any* initial probability vector γ. In fact, if $\gamma e^T = 1$, then $Z \sim PH_m(\gamma, B)$. This is readily seen by observing that $Z \sim PH_{2m}((\gamma, 0), R)$ for

$$R = \begin{bmatrix} S & D \\ 0 & S+D \end{bmatrix}.$$

Working directly with these matrices one immediately obtains

$$Pr\{Z \le y\} = 1 - \gamma \exp\{yB\}e^T,$$

which is the df of a $PH_m(\gamma, B)$ random variable.

Recently, Drekic et al (2001, p. 12) established the following results pertaining to the deficit at ruin in the ordinary renewal risk model when claim amounts are phase-type distributed:

$$\bar{G}(u,y) = \alpha_+ \exp\{uB\}\exp\{yS\}e^T \tag{3.1}$$

and

$$G_u(y) = Pr\{|U_T| \le y | T < \infty\} = 1 - \frac{\alpha_+ \exp\{uB\}}{\alpha_+ \exp\{uB\}e^T} \exp\{yS\}e^T. \tag{3.2}$$

We are now ready to state the following theorem.

Theorem 3.1 If the claim size random variable $X \sim PH_m(\alpha, S)$, then

$$\bar{G}^e(u,y) = \alpha^{(s)} \exp\{uB\} \exp\{yS\}e^T, \; u, y \ge 0. \tag{3.3}$$

Proof: First, we have that $\bar{P}_1(y) = \eta \exp\{yS\}e^T$ where $\eta = -\alpha S^{-1}/E\{X\}$ and $\eta e^T = 1$ (see Asmussen, 2000, p. 230, or Rolski et al, 1999, Lemma 8.3.1). Making use of this result and (3.1), we have via (2.1) that

$$\begin{aligned} \bar{G}^e(u,y) &= \frac{1}{1+\theta}\Big[\int_0^u \alpha_+ \exp\{(u-t)B\}\exp\{yS\}e^T dP_1(t) + \eta\exp\{(u+y)S\}e^T\Big] \\ &= \frac{1}{1+\theta}\Big[\int_0^u \alpha_+ \exp\{(u-t)B\} dP_1(t) \exp\{yS\}e^T + \eta\exp\{uS\}\exp\{yS\}e^T\Big] \\ &= \frac{1}{1+\theta}\Big[\int_0^u \alpha_+ \exp\{(u-t)B\} dP_1(t) + \eta\exp\{uS\}\Big]\exp\{yS\}e^T \\ &= \frac{1}{1+\theta}\Big[\int_0^u Pr\{L > (u-t), J_{(u-t)} = j; j = 1,\ldots,m\} dPr\{Y \le t\} \\ &\quad + Pr\{Y > u, J_u = j; j = 1,\ldots,m\}\Big]\exp\{yS\}e^T. \end{aligned}$$

The expression within brackets above is the convolution of Y and L, which by (1.10) is the tail probability of L^e conditional on $L^e > 0$. Therefore, this yields

$$\bar{G}^e(u,y) = Pr\{L^e > u, J_u = j; j = 1,\dots,m\} \exp\{yS\}e^T. \qquad (3.4)$$

Finally, making use of $L^e \sim PH_m(\alpha^{(s)}, B)$, we obtain (3.3). □

Corollary 3.1 Under the same conditions as Theorem 3.1, the conditional distribution of the deficit at ruin is given by

$$G_u^e(y) = 1 - \pi^e \exp\{yS\}e^T, \; y \geq 0, \qquad (3.5)$$

where

$$\pi^e = \frac{\alpha S^{-1} \exp\{uB\}}{\alpha S^{-1} \exp\{uB\}e^T}. \qquad (3.6)$$

Proof: The tail of the distribution of the deficit at ruin is given by

$$\begin{aligned} \bar{G}_u^e(y) &= \frac{\bar{G}^e(u,y)}{\psi^e(u)} \\ &= \frac{\alpha^{(s)} \exp\{uB\} \exp\{yS\}e^T}{\alpha^{(s)} \exp\{uB\}e^T} \\ &= \frac{-\{(1+\theta)E\{X\}\}^{-1} \alpha S^{-1} \exp\{uB\} \exp\{yS\}e^T}{-\{(1+\theta)E\{X\}\}^{-1} \alpha S^{-1} \exp\{uB\}e^T} \\ &= \pi^e \exp\{yS\}e^T. \end{aligned}$$

Taking the complement of $\bar{G}_u^e(y)$ yields the desired result. □

Corollary 3.2 Under the same conditions as Theorem 3.1, the conditional moments of the deficit at ruin are given by

$$E\{(V_u^e)^k\} = (-1)^k k! \pi^e S^{-k} e^T. \qquad (3.7)$$

Proof: This is a standard result for phase-type distributed quantities; see for instance Neuts (1981) or Asmussen (2000).

4. An application

In this section, we present some illustrations of the conditional distribution of the deficit at ruin. We consider first the df of V_u^e as a function of u, then make comparisons between the conditional distributions of the deficit at ruin in the ordinary and stationary renewal risk models.

To demonstrate ideas, we use the following distributions from Wikstad (1971):

$$K(t) = 1 - 0.25e^{-0.4t} - 0.75e^{-2t},\ t \geq 0,$$

and

$$P(x) = \sum_{i=1}^{3} \alpha_i \left(1 - e^{-\beta_i x}\right),\ x \geq 0,$$

where $\alpha_1 = 0.0039793$, $\alpha_2 = 0.1078392$, $\alpha_3 = 0.8881815$, $\beta_1 = 0.014631$, $\beta_2 = 0.190206$, and $\beta_3 = 5.514588$. For this choice of K in the stationary renewal risk model, we find that $E\{W_1\} = 1.75$ and $Var\{W_1\} = 4.9375$, whereas for $i > 1$, $E\{W_i\} = 1$ and $Var\{W_i\} = 2.5$. Since the characteristics underlying the distributions of W_1 and W_i, $i > 1$, are quite different, we would expect to see some differences in the distributions of the deficit at ruin in the ordinary and stationary renewal risk models. It is also clear that P is a member of the phase-type family with $E\{X\} = 1$ and $Var\{X\} = 42.2$, so that we anticipate the initial surplus u will be a major factor in the distribution of V_u^e. For each of our illustrations below, we have set $\xi = 1.2$.

Figure 1 displays the df of V_u^e for $u = 0, 5, 10, 25$, and 100. We can see that there is a clear difference between the df's for these values of u, but for $u > 100$ we found very little difference so that we anticipate the df of V_{100}^e is close to that of V_∞^e.

Figures 2 to 4 show a comparison between the df's $G_u(y)$ and $G_u^e(y)$ for $u = 0, 5$, and 10. In each case, we see that $G_u^e(y)$ lies below $G_u(y)$ for a given value of u, which implies that V_u^e is stochastically larger than its non-stationary counterpart, i.e. $\bar{G}_u^e(y) \geq \bar{G}_u(y)$. Furthermore, we note that the two distributions become more similar as u grows larger. This is to be expected as a result of (2.18), which states that the distributions of the deficit at ruin in the ordinary and stationary renewal risk models are asymptotically identical.

References

Asmussen, S. (2000). Ruin Probabilities. World Scientific, Singapore.

Barlow, R. and F. Proschan (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

Bowers, N., H. Gerber, J. Hickman, D. Jones, and C. Nesbitt (1997). Actuarial Mathematics, 2nd edition. The Society of Actuaries, Schaumburg, Illinois.

Cai, J. and J. Garrido (2002). "Asymptotic forms and bounds for tails of convolutions of compound geometric distributions, with applications." In *Proceedings of Statistics 2001 Canada*, (Y. Chaubey, editor), World Scientific, London, to appear.

Dickson, D. (1989). "Recursive calculation of the probability and severity of ruin." *Insurance: Mathematics and Economics*, 8, 145-148.

Drekic, S., D. Dickson, D. Stanford, and G. Willmot (2001). "On the distribution of the deficit at ruin when claims are phase-type." *Institute of Insurance and Pension Research Report*, 01-12, University of Waterloo, Waterloo, Ontario.

Embrechts, P., C. Klüppelberg, and T. Mikosch (1997). Modelling Extremal Events. Springer-Verlag, Berlin.

Fagiuoli, E. and F. Pellerey (1994). "Preservation of certain classes of life distributions under Poisson shock models." *Journal of Applied Probability* 31, 458-465.

Feller, W. (1971). An Introduction to Probability Theory and Its Applications, Vol. 2, 2nd edition. John Wiley, New York.

Grandell, J. (1991). Aspects of Risk Theory. Springer-Verlag, New York.

Latouche, G. and V. Ramaswami (1999). Introduction to Matrix Analytic Methods in Stochastic Modeling. ASA SIAM, Philadelphia.

Neuts, M. (1981). Matrix-geometric Solutions in Stochastic Models: An Algorithmic Approach. John Hopkins University Press, Baltimore.

Rolski, T., H. Schmidli, V. Schmidt, and J. Teugels (1999). Stochastic Processes for Insurance and Finance. John Wiley, Chichester.

Szekli, R. (1986). "On the concavity of the waiting time distribution in some GI/G/1 queues." *Journal of Applied Probability* 23, 555-561.

Wikstad, N. (1971). "Exemplification of ruin probabilities." *ASTIN Bulletin*, 6, 147-152.

Willmot, G. (2002). "Compound geometric residual lifetime distributions and the deficit at ruin." *Insurance: Mathematics and Economics*, to appear.

Willmot, G. and J. Cai (2002). "On applications of residual lifetimes of compound geometric convolutions." Submitted for publication.

Willmot, G., J. Cai, and X. Lin (2001). "Lundberg inequalities for renewal equations." *Advances in Applied Probability*, 33, 674-689.

Willmot, G. and X. Lin (2001). Lundberg Approximations for Compound Distributions with Insurance Applications. Springer-Verlag, New York.

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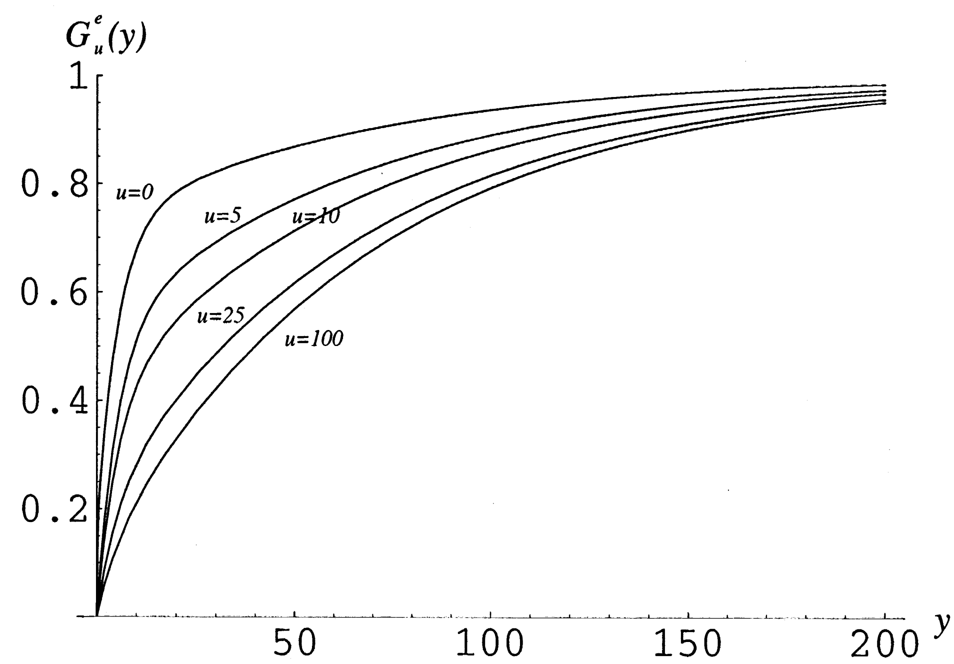

Figure 1: Comparison of $G^e_u(y)$ for various values of u.

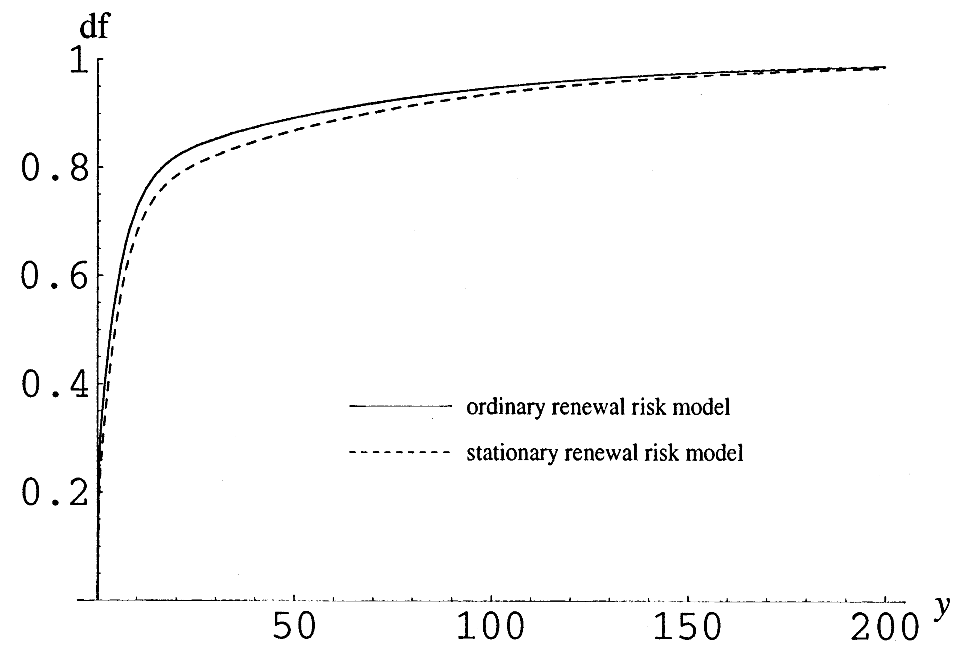

Figure 2: Comparison of $G_u(y)$ and $G_u^e(y)$ for $u = 0$.

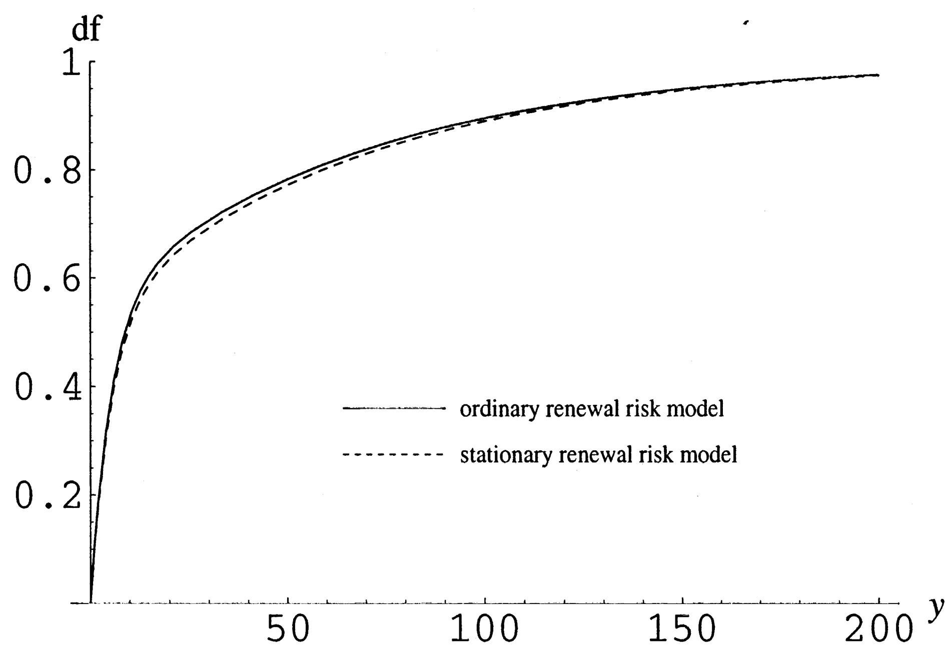

Figure 3: Comparison of $G_u(y)$ and $G_u^e(y)$ for $u = 5$.

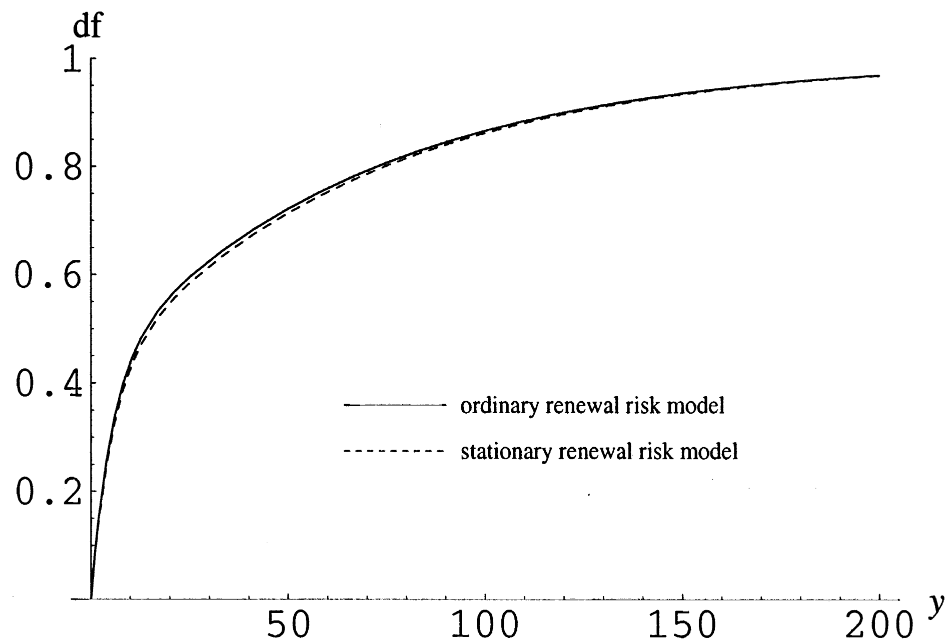

Figure 4: Comparison of $G_u(y)$ and $G_u^e(y)$ for $u = 10$.

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