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Chain Ladder Bias

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CHAIN LADDER BIAS

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Summary. The chain ladder forecast of outstanding losses is known to be unbiased under suitable assumptions. According to these assumptions, claim payments in any cell of a payment triangle are dependent on those from preceding development years of the same accident year. If all cells are assumed stochastically independent, the forecast is no longer unbiased. Section 6 shows that, under very general assumptions, it is biased upward. This result is linked to earlier work on some stochastic versions of the chain ladder.

Keywords. Chain ladder, IBNR.

1. Introduction

The chain ladder (CL) approach to estimation of a loss reserve is well known. It is described, for example, by Taylor (2000).

Its origins are not altogether clear, but it seems likely that it originated as a heuristic device. As such, it may be viewed as a non-parametric estimator. The precise definition is given in Sections 2 and 3.

Kremer (1982) recognised that the CL involved a log-linear cross-classification structure. A number of parametric stochastic versions of the CL developed from this, eg Hertig (1985), Renshaw (1989), Verrall (1989, 1990, 1991).

Mack (1994) pointed out that these stochastic models gave mean estimates of liability that differed from the "classical" CL estimate. While the form of stochastic model underlying the classical CL was speculative, due to the latter's heuristic nature, Mack suggested one. It is distribution free. Details are given in Section 2. Mack also identified the differences between this and the other stochastic models.

Whereas the cross-classified models typically assume stochastic independence of all cells in the data set, the CL (in Mack's formulation) does not. It was shown by Mack (1993) that the algorithm of the classical CL produced unbiased forecasts of liability under its own assumptions.

However, it does not necessarily do so under the alternative assumption of independence between all cells. Some papers have studied the bias in estimates of liability in the parametric cross-classified models mentioned above, but little is known of the bias in the classical CL forecast when all cells are independent.

The purpose of the present paper is to investigate the direction of bias in this case.

2. Framework and notation

Consider a square array X of stochastic quantities $X(i,j) \geq 0$, $i = 0,1,\ldots,I$; $j = 0,1,\ldots,I$.

Denote row sums and column sums as follows:

$$R(i,j) = \sum_{h=0}^{j} X(i,h) \tag{2.1}$$

$$C(i,j) = \sum_{g=0}^{i} X(g,j). \tag{2.2}$$

In addition introduce the following notation for the total sum over a rectangular subset of X:

$$\begin{aligned} T(i,j) &= \sum_{g=0}^{i}\sum_{h=0}^{j} X(g,h) \\ &= \sum_{g=0}^{i} R(g,j) \\ &= \sum_{h=0}^{j} C(i,h). \end{aligned} \tag{2.3}$$

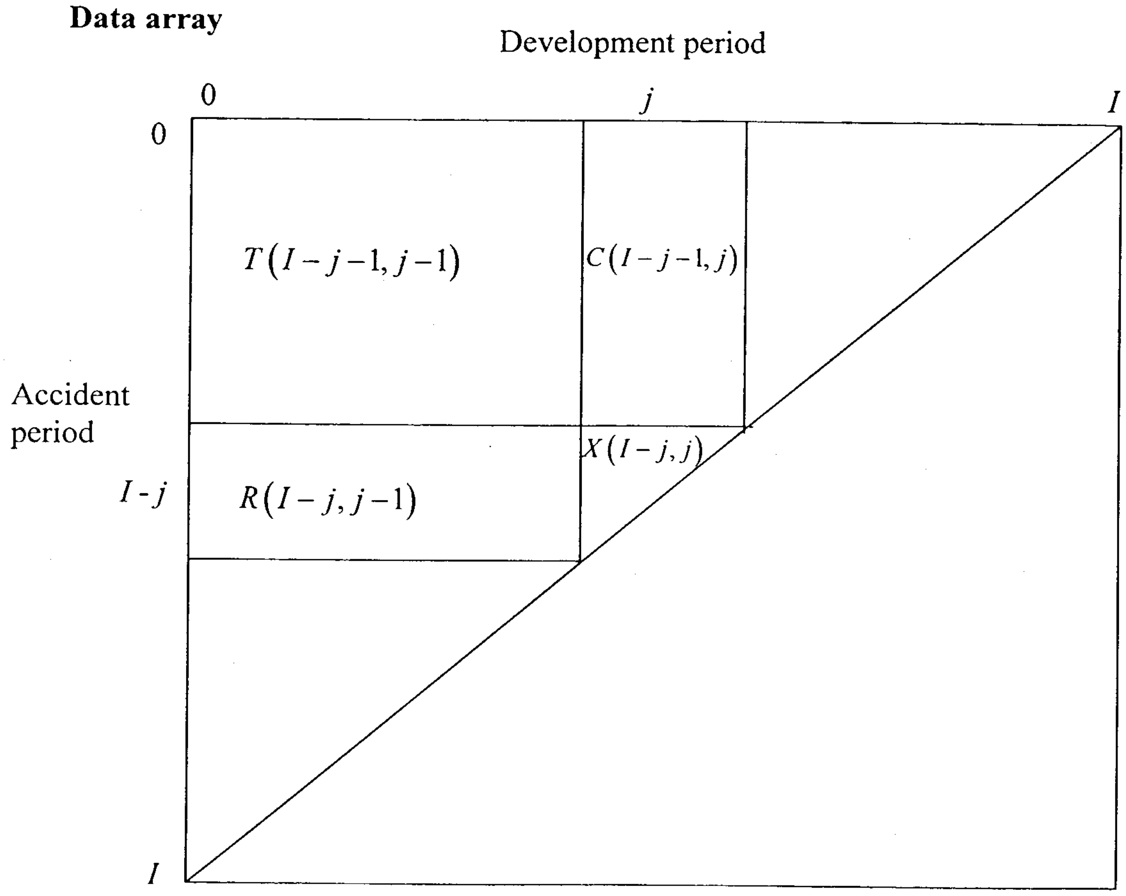
Generally, in the following, any summation of the form $\sum_{a}^{b}$ with $b < a$ will be taken to be zero.

In a typical loss reserving framework, i denotes accident period, j development period, and available data will consists of observations on the triangular subset Δ of X:

$$\Delta = \{X(i,j), i = 0,1,\ldots,I; j = 0,1,\ldots,I-i\} \tag{2.4}$$

Figure 2.1 illustrates the situation.

Figure 2.1
Data array



Still in a loss reserving context, Δ would represent some form of claims experience, eg claim counts or claim amounts. The loss reserving problem consists of forecasting the lower triangle in Figure 2.1, conditional on Δ. There is particular interest in forecasting $R(i,I) \mid \Delta, \quad i=1,\dots,I$.

3. Chain ladder forecast

Define

$$\begin{aligned}\hat{v}(j) &= T(I-j-1,j+1)/T(I-j-1,j) \\ &= 1+C(I-j-1,j+1)/T(I-j-1,j)\end{aligned} \tag{3.1}$$

and

$$\hat{R}(i,I) = R(i,I-i)\prod_{k=I-i}^{I-1}\hat{v}(k) \tag{3.2}$$

The value of $\hat{R}(i,I)$ calculated in this way will be referred to as the **chain ladder forecast** (CLF) of $R(i,I)$.

4. Chain ladder models

4.1 Dependent increments

The CLF has been formulated in Section 3 just as an algorithm. No model for the data X has yet been stated.

It is evident that the properties of the CLF will depend on the model. This and the next sub-section consider two alternative models. The first is represented by the following two assumptions.

Assumption 1. $E\left[R(i,j+1) \mid X(i,0), X(i,1),\dots,X(i,j)\right] = v(j)R(i,j)$. (4.1)

Assumption 2. $X(i_1,j_1)$ and $X(i_2,j_2)$ are stochastically independent for $i_1 \neq i_2$.

Remark 1. It follows from Assumptions 1 and 2 that

$$E\left[R(i,j+1) \mid \Delta\right] = v(j)R(i,j) \tag{4.2}$$

for any $j \geq I-i$ (ie future j).

Since $R(i,j+1) = R(i,j) + X(i,j+1)$, one may re-write (4.2) in the form:

$$E\left[X(i,j+1) \mid \Delta\right] = \left[v(j)-1\right]R(i,j). \tag{4.3}$$

Remark 2. It is clear from (4.3) that $R(i,j)$ and $X(i,j+1)$ are **not** independent. Generally, under Assumptions 1 and 2, the $X(i,j)$ for fixed i are **not** independent.

Theorem 1 (Mack). Under Assumptions 1 and 2,

(1) $\hat{\nu}(j)$ is an unbiased estimator of $\nu(j)$ for $j = 0,1,\dots, I-1$; and

(2) the CLF $\hat{R}(i,I)$ is an unbiased estimator of $E[R(i,I)|\Delta]$ for i = 1,2,.., I.

Proof. See Mack (1993).

□

It is also convenient to re-write (4.2) in the form:

$$E[R(i,j+1)/R(i,j)|\Delta] = \nu(j). \tag{4.4}$$

4.2 Independent increments

Replace Assumptions 1 and 2 by 1a, 2a and 3 as follows.

Assumption 1a. $E[R(i,j+1)]/E[R(i,j)] = \eta(j).$ (4.5)

Assumption 2a. $X(i_1,j_1)$ and $X(i_2,j_2)$ are stochastically independent for $(i_1,j_1) \neq (i_2,j_2)$.

Define the set

$$D_i = \{(g,h): g \le I-k-1, h \le k+1, \ \ k = I-i,\dots,I-1\}. \tag{4.6}$$

Assumption 3. $T(g,h) > 0$ for $(g,h) \in D_i$.

Remark 3. It is implicit in Assumption 1a that $E[R(i,j)] \neq 0$. By the assumed non-negativity of the $X(i,j)$, $E[R(i,j)] > 0$ for each i,j.

Remark 4. A comparison of (4.4) and (4.5) indicates that $\nu(j)$ and $\eta(j)$ are different quantities (for fixed j) since

$$E[R(i,j+1)/R(i,j)] \neq E[R(i,j+1)]/E[R(i,j)]. \tag{4.7}$$

This fact was pointed out by Mack (1994).

By Assumption 3, applied to (3.1), all $\hat{\nu}(k)$ appearing in (3.2) are defined and strictly positive.

The conditions of Theorem 1 no longer hold, and so the CLF is not necessarily unbiased.

5. Earlier results

A definitive result under Assumptions 1a and 2a was obtained by Hachemeister and Stanard (1975). This result was not well known for some time and was subsequently re-discovered (Renshaw and Verrall, 1994; Schmidt and Wünsche, 1998).

Theorem 2 (Hachemeister and Stanard). Suppose that

$$X(i,j) \sim \text{Poisson}\left[a(i)b(j)\right] \tag{5.1}$$

for constants $a(i), b(j) > 0,\ i = 0,1,...,I;\ b = 0,1,...,I$. Suppose also that Assumption 2a holds. Then $\hat{\nu}(j)$ is the MLE of $\sum_{h=0}^{j+1} b(h) / \sum_{h=0}^{j} b(h)$ and $\hat{R}(i,I)$ the MLE of $R(i,I)$.

□

Remark 5. The quantity $\sum_{h=0}^{j+1} b(h) / \sum_{h=0}^{j} b(h)$ is equal to $\eta(j)$.

In fact, the results of Schmidt and Wünsche establish equivalence between CLF and MLE for more general distributional assumptions than (5.1). However, these more general cases do not fall within the current area of interest in which Assumption 2a holds.

Note that, because of (5.1), Theorem 2 deals with one of the log-linear cross-classification structure mentioned in Section 1. The theorem provides a case in which the CLF is ML.

Bias in models of this type was investigated by Verrall (1991) and Doray (1996). Suppose that

$$\log X(i,j) = \log a(i) + \log b(j) + \varepsilon(i,j) \tag{5.2}$$

where

$$\varepsilon(i,j) \sim N\left(0, \sigma^2\right) \tag{5.3}$$

and the $\varepsilon(i,j)$ are stochastically independent.

In this structure, Assumptions 1a and 2a hold with $\eta(j)$ as in Remark 5.

Suppose the data triangle Δ is available and let $\hat{\theta}$ denote an estimate of IBNR claims:

$$\hat{\theta} = \sum_{i=1}^{I} \sum_{j=I-i+1}^{I} \hat{\theta}(i,j) \tag{5.4}$$

where $\hat{\theta}(i,j)$ is an estimator of $X(i,j)$.

The MLE's of the parameters in (5.2) and (5.3) are found by regression, because of the normal error structure. The MLE of σ^2 takes the form

$$\hat{\sigma}^2 = \text{RSS}/n \tag{5.5}$$

where RSS is the residual sum of squares and n the number of data points $\left(=(I+1)(I+2)/2\right)$.

This last estimator is biased, and its unbiased form is:

$$\tilde{\sigma}^2 = \text{RSS}/(n-p) \tag{5.6}$$

where p is the number of parameters (other than σ^2) in (5.2).

Doray considers a number of estimators $\hat{\theta}$, including:

- $\hat{\theta}_U$, the unique uniformly minimum variance unbiased estimator;
- $\hat{\theta}_V$, a modification of the MLE in which $\hat{\sigma}^2$ is replaced by $\tilde{\sigma}^2$.

Theorem 3 (Doray). Suppose that $X(i,j)$ are log normally distributed according to (5.2) and (5.3) with the $\varepsilon(i,j)$ stochastically independent (so that Assumptions 1a and 2a hold). Then

$$\hat{\theta}_U < \hat{\theta}_V \tag{5.7}$$

$$E\left[\hat{\theta}_U\right] < E\left[\hat{\theta}_V\right]. \tag{5.8}$$

□

This shows that, for the log normal case, $\hat{\theta}_V$, an approximation to MLE is biased upward.

There is no particular connection between Theorems 2 and 3, one relating to Poisson variates and the other to log normal. Nonetheless, the results obtained are that:

- the CLF is MLE in one case;
- an MLE approximation is biased upward in the other case.

This raises a question as to whether the CLF is generally biased upward. Section 6 considers this question.

6. Chain ladder bias

Theorem 4. Define

$$Y=\prod_{k=I-i}^{I-1}\hat{v}(k). \tag{6.1}$$

Then

$$\frac{\partial^2 Y}{\partial X^2(g,h)}=0 \text{ for } (g,h)\notin D_i; \tag{6.2}$$

for $(g,h)\in D_i$ and $h\leq I-i$,

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)}=2\sum_{k=I-i}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\text{x}\left[\frac{1}{T(i-1,I-i)}+\sum_{l=I-i+1}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}\right] \tag{6.3}$$

for $(g,h)\in D_i$ and $h>I-i$,

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)}=2\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\sum_{l=h}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}. \tag{6.4}$$

These results do **not** depend on the Assumptions 1a, 2a and 3.

Proof. See appendix. □

Theorem 5. Under Assumptions 1a, 2a and 3, and if $X(g,h)$ is not degenerate for at least one $(g,h)\in D_i$, the CLF $\hat{R}(i,I)$ is biased upward as an estimate of $E\left[R(i,I)\right]$ in the sense that

$$E\left[\hat{R}(i,I)\right]>E\left[R(i,I)\right]. \tag{6.5}$$

Proof. See appendix. □

Remark 6. An alternative form of (6.5) is:

$$E_{R(i,I-i)}E\left[\hat{R}(i,I)\,|\,R(i,I-i)\right] > E\left[R(i,I)\right]. \tag{6.6}$$

Note that Theorem 5 does **not** state that

$$E\left[\hat{R}(i,I)|\,R(i,I-i)\right] > E\left[R(i,I)\right].$$

Indeed, it follows from (3.2), (6.1) and the fact that Y is independent of $R(i, I-i)$ that

$$E\left[\hat{R}(i,I)|\,R(i,I-i)\right] = R(i,I-i)E\left[Y\right] > E\left[R(i,I)\right]\frac{R(i,I-i)}{E\left[R(i,I-i)\right]}, \tag{6.7}$$

the last step following from (A.23).

It is evident that (6.6) (equivalently (6.5)) follows from (6.7).

7. Conclusion

Theorem 5 shows that, under very general distribution free conditions, **the CLF is biased upward**. A simulation test of prediction bias in the chain ladder and other models was carried out by Stanard (1985). One of his experiments dealt with the case in which the total number of claims in an accident year is a Poisson variate and is multinomially distributed over development years. It may be shown that distinct cells in a row of the claim count triangle are then stochastically independent, and so Theorem 5 applies.

Stanard's simulations did in fact find upward bias in the CLF.

8. Acknowledgements

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The author also acknowledges helpful comment by Dr Thomas Mack.

Appendix
Proof of theorems

Lemma 1. Suppose that

$$y = \prod_{i=1}^{n} f_i(x) \tag{A.1}$$

with $x = (x_1, ..., x_m)^T$ and $f_i(x) > 0$ for each i. Then

$$\frac{1}{y}\frac{\partial^2 y}{\partial x_k^2} = \sum_{i=1}^{n} \frac{1}{f_i}\frac{\partial^2 f_i}{\partial x_k^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{f_i f_j}\frac{\partial f_i}{\partial x_k}\frac{\partial f_j}{\partial x_k}.$$

Proof. Differentiate $\log y$ twice. □

Lemma 2 (multivariate Jensen inequality). Let $X = (X_1, ..., X_m)^T$ where the X_k are pairwise stochastically independent random variables. Let $f : R^m \to R$ be twice differentiable in all its arguments, and suppose that

$$\partial^2 f(X) / \partial X_k^2 \geq 0 \text{ for all } X \text{ and for } k = 1, 2, \ldots, m. \tag{A.2}$$

Then

$$E[f(X)] \geq f(E[X]). \tag{A.3}$$

If strict inequality holds in (A.2) for at least one k, and X_k is not degenerate, then strict inequality holds in (A.3).

Proof. Expand $f(X)$ as the Taylor series:

$$\begin{aligned} f(X) &= f(\mu) + (\partial f(\mu)/\partial X)(X-\mu) \\ &\quad + \tfrac{1}{2}(X-\mu)^T \partial^2 f(\xi)(X-\mu) \end{aligned} \tag{A.4}$$

where $\mu = (\mu_1, ..., \mu_m)^T = E[X]$ and $\xi = \mu + \theta^T(X - \mu)$ for some m-vector θ. Then

$$\begin{aligned} E[f(X)] &= f(\mu) + \tfrac{1}{2}\sum_{k,l=1}^{m} \left[\partial^2 f(\xi)/\partial X_k \partial X_l\right] \mathrm{Cov}[X_k, X_l] \\ &= f(\mu) + \tfrac{1}{2}\sum_{k=1}^{m} \left[\partial^2 f(\xi)/\partial X_k^2\right] \mathrm{Var}[X_k] \end{aligned} \tag{A.5}$$

$$\geq f(\mu), \tag{A.6}$$

where (A.5) follows from the pairwise independence of the X_k, and (A.6) follows from (A.2).

The result (A.6) is the same as (A.3). It is evident that if one of the derivatives in (A.5) is strictly positive and X_k is not degenerate for that k, then the inequality in (A.6) is strict.

□

Proof of Theorem 4. Consider Y defined by (6.1), with $\hat{v}(k)$ defined by (3.1) and (2.3). The observations $X(g,h)$ involved in the $\hat{v}(k)$ constituting Y are just those in D_i. This justifies (6.2).

Now consider $(g,h) \in D_i$. Note that Lemma 1 is applicable to Y because of (6.1). Hence

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)} = \sum_{k=I-i}^{I-1} \frac{1}{\hat{v}(k)}\frac{\partial^2 \hat{v}(k)}{\partial X^2(g,h)} + \sum_{\substack{k,l=I-i \\ k \neq l}}^{I-1} \frac{1}{\hat{v}(k)\hat{v}(l)}\frac{\partial \hat{v}(k)}{\partial X(g,h)}\frac{\partial \hat{v}(l)}{\partial X(g,h)}. \quad \text{(A.7)}$$

There are two cases to be considered in the evaluation of (A.7), according to whether $h \le I-i$ or $h > I-i$.

Case I: $h \le I-i$.

By (3.1),

$$\hat{v}(k) = 1 + C(I-k-1,k+1)/T(I-k-1,k), \quad k = I-i,\ldots,I-1. \quad \text{(A.8)}$$

Thus $X(g,h)$ will not appear in the numerator of the fraction here. It will appear as a summand in the dominator, provided that $g \le I-k-1,\ h \le k$. Then, for $g \le I-k-1,\ h \le k$,

$$\partial \hat{v}(k)/\partial X(g,h) = -C(I-k-1,k+1)/T^2(I-k-1,k) \quad \text{(A.9)}$$

$$\partial^2 \hat{v}(k)/\partial X^2(g,h) = 2C(I-k-1,k+1)/T^3(I-k-1,k). \quad \text{(A.10)}$$

Otherwise, $\partial \hat{v}(k)/\partial X(g,h) = \partial^2 \hat{v}(k)/\partial X^2(g,h) = 0$.

Substitution of (3.1), (A.9) and (A.10) into (A.7) yields

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)} = 2\sum_{k=I-i}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T^2(I-k-1,k)}$$
$$+2\sum_{k=I-i}^{I-g-1}\sum_{l=I-i}^{k-1}\frac{C(I-k-1,k+1)C(I-l-1,l+1)}{T(I-k-1,k+1)T(I-k-1,k)T(I-l-1,l+1)T(I-l-1,l)}$$
$$=2\sum_{k=I-i}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}$$
$$\text{x}\left[\frac{1}{T(I-k-1,k)}+\sum_{l=I-i}^{k-1}\frac{C(I-l-1,l+1)}{T(I-l-1,l+1)T(I-l-1,l)}\right] \qquad \text{(A.11)}$$

Note that, by (2.2), the second member within the square bracket may be expanded as follows:

$$\frac{C(I-l-1,l+1)}{T(I-l-1,l+1)T(I-l-1,l)}=\frac{1}{T(I-l-1,l)}-\frac{1}{T(I-l-1,l+1)} \qquad \text{(A.12)}$$

Substitute (A.12) into the square bracket in (A.11) to obtain

$$\frac{1}{T(I-k-1,k)}+\sum_{l=I-i}^{k-1}\frac{1}{T(I-l-1,l)}-\sum_{l=I-i+1}^{k}\frac{1}{T(I-l,l)}$$
$$=\frac{1}{T(i-1,I-i)}+\sum_{l=I-i+1}^{k}\left[\frac{1}{T(I-l-1,l)}-\frac{1}{T(I-l,l)}\right]$$
$$=\frac{1}{T(i-1,I-i)}+\sum_{l=I-i+1}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}, \qquad \text{(A.13)}$$

by (2.1).

Substitute (A.13) for the square bracket in (A.11) to obtain

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)}=2\sum_{k=I-i}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)(I-k-1,k)}\text{x}$$
$$\left[\frac{1}{T(i-1,I-i)}+\sum_{l=I-i+1}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}\right]. \qquad \text{(A.14)}$$

This proves (6.3).

Case II: $h > I - i$.

To evaluate the derivatives in (A.7) note that, according to (3.1), the $\hat{v}(k)$ involve $X(g,h)$ as follows:

- in the numerator, but not the denominator, for $k = h - 1$
- in the denominator, but not the numerator, for $k > h - 1,\ g \le I - k - 1$.

Note that the restriction $g \le I - k - 1$ is not required in the case $k = h - 1$. This is because in this case $I - k - 1 = I - h \ge g$ for $(g,h) \in D_i$, so the restriction is automatically satisfied.

Hence, (3.1) yields:

$$\frac{\partial \hat{v}(h-1)}{\partial X(g,h)} = \frac{1}{T(I-h,h-1)} \tag{A.15}$$

$$\frac{\partial^2 \hat{v}(h-1)}{\partial X^2(g,h)} = 0 \tag{A.16}$$

$$\frac{\partial \hat{v}(k)}{\partial X(g,h)} = -\frac{C(I-k-1,k+1)}{T^2(I-k-1,k)}, k \ge h,\ g \le I-k-1 \tag{A.17}$$

$$\frac{\partial^2 \hat{v}(k)}{\partial X^2(g,h)} = \frac{2C(I-k-1,k+1)}{T^3(I-k-1,k)}, k \ge h,\ g \le I-k-1. \tag{A.18}$$

Substitute (A.15) – (A.18) into (A.7) to obtain

$$\begin{aligned}
\frac{1}{Y}\frac{\partial^2 Y}{\partial^2 X(g,h)} &= \frac{1}{\hat{v}(h-1)}\frac{\partial^2 \hat{v}(h-1)}{\partial X^2(g,h)} + \sum_{k=h}^{I-g-1} \frac{1}{\hat{v}(k)}\frac{\partial^2 \hat{v}(k)}{\partial X^2(g,h)} \\
&+ \sum_{\substack{k,l=h \\ k\ne l}}^{I-g-1} \frac{1}{\hat{v}(k)\hat{v}(l)}\frac{\partial \hat{v}(k)}{\partial X(g,h)}\frac{\partial \hat{v}(l)}{\partial X(g,h)} \\
&+ \frac{2}{\hat{v}(h-1)}\frac{\partial \hat{v}(h-1)}{\partial X(g,h)}\sum_{k=h}^{I-g-1}\frac{1}{\hat{v}(k)}\frac{\partial \hat{v}(k)}{\partial X(g,h)} \\
&= 2\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T^2(I-k-1,k)} \\
&+ 2\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\sum_{l=h}^{k-1}\frac{C(I-l-1,l+1)}{T(I-l-1,l+1)T(I-l-1,l)} \\
&- \frac{2}{T(I-h,h)}\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}.
\end{aligned} \tag{A.19}$$

Now apply (A.12) and use the same mode of calculation as led from (A.11) to (A.14). Then (A.19) becomes:

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial X^2(g,h)} = 2\sum_{k=h}^{I-g-1}\frac{C(I-k-1,k+1)}{T(I-k-1,k+1)T(I-k-1,k)}\sum_{l=h}^{k}\frac{R(I-l,l)}{T(I-l-1,l)T(I-l,l)}. \quad \text{(A.20)}$$

This proves (6.4). □

Proof of Theorem 5. Consider Y defined by (6.1). By Theorem 4,

$\partial^2 Y / \partial X^2(g,h) \geq 0$ for all (g,h)

with strict inequality for some (g,h), namely those in D_i. It follows from a multivariate form of Jensen's inequality (see Lemma 2) that

$$E[Y] > \bar{Y} = \prod_{k=I-i}^{I-1} \bar{v}(k) \quad \text{(A.21)}$$

where $\bar{Y}$ is the value obtained by replacing each $X(g,h)$ in Y by its expectation, and $\bar{v}(k)$ is similarly defined.

By (3.1),

$$\begin{aligned}\bar{v}(k) &= E[T(I-k-1,k+1)]/E[T(I-k-1,k)] \\ &= \sum_{g=0}^{I-k-1} E[R(g,k+1)] / \sum_{g=0}^{I-k-1} E[R(g,k)] \qquad [\text{by}(2.3)] \\ &= \eta(k), \text{ by } (4.5).\end{aligned} \quad \text{(A.22)}$$

Substitute (A.22) in (A.21):

$$E[Y] > \prod_{k=I-i}^{I-1} \eta(k) = E[R(i,I)]/E[R(i,I-i)] \quad \text{(A.23)}$$

by (4.5).

Now take expectations on both sides of (3.2):

$$E[\hat{R}(i,I)] = E[R(i,I-i)]E[Y] > E[R(i,I)], \text{ by (A.23)}. \quad \text{(A.24)}$$

The first step leading to (A.24) is justified by the fact that the $X(g,h)$ involved in Y are those in the set D_i (see the start of the proof of Theorem 4), and this excludes row i of the array X (see (4.6)) on which $R(i,I-i)$ depends. Thus, $R(i,I-i)$ and Y are stochastically independent. □

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