# On a discrete-time Sparre Andersen model with phase-type claims

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**Abstract**

In this paper a discrete-time Sparre Andersen risk model with general inter-claim times is considered. Assuming that the individual claim amounts follow discrete phase-type distributions, the probability of ruin and the distribution of the deficit at ruin are discussed. Examples are provided thereafter. Using the method proposed in Ren Jiandong (2009), the discounted probability of ruin and discounted distribution of the deficit at ruin are considered in the end.

*Keywords:* Discrete phase-type distribution; Sparre Andersen model; Deficit at ruin

## 1 Introduction

In this paper we consider a discrete-time Sparre Andersen risk process

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad n \in \mathbb{N}^+, \tag{1.1}$$

where $u \in \mathbb{N}$ is the initial surplus, and $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s), denoting the individual claim sizes. $\{X_i\}_{i=1}^{\infty}$ only take positive integer values and follow a common probability function (p.f.) $p(x) = \mathbb{P}\{X_1 = x\}$, for $x = 1, 2, \ldots$. Let $P(x) = 1 - \bar{P}(x) = \mathbb{P}\{X \leq x\}$ be the distribution function (d.f.) of $X_1$, $\mu$ be its mean and $\hat{p}(z) = \sum_{x=1}^{\infty} z^x p(x), z \in \mathbb{C}$, be the probability generating function (p.g.f.).

The counting process $\{N(n); n \in \mathbb{N}\}$ denotes the number of claims up to time $n$ and is defined as $N(n) = \max\{k : W_1 + W_2 + \cdots + W_k \leq n\}$, where the inter-claim times $W_i$'s are also assumed to be i.i.d. positive integer-valued r.v.'s with the common

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p.f. $k(t) = \mathbb{P}\{W_1 = t\}, t = 1, 2, \ldots$, the d.f. $K(t)$, the mean $\mathbb{E}[W_1] < \infty$, and the p.g.f. $\hat{k}(z) = \sum_{t=1}^{\infty} z^t k(t), z \in \mathbb{C}$.

Further, we assume that $\{W_i; i \in \mathbb{N}^+\}$ and $\{X_i; i \in \mathbb{N}^+\}$ are mutually independent, and $\mathbb{E}[W_1] = (1+\theta)\mathbb{E}[X_1] = (1+\theta)\mu, \theta > 0$, in order to have a positive safety loading.

For risk model (1.1), let the r.v. $T = \min\{n \in \mathbb{N}^+ : U(n) < 0\}$ denote the time of ruin and $T = \infty$ if $U(n) \geq 0$ for all $n \in \mathbb{N}^+$. As usual, $\psi(u) = \mathbb{P}\{T < \infty | U(0) = u\}, u \in \mathbb{N}$, is the ultimate ruin probability.

The Sparre Andersen risk model is a well recognized risk model. As it was commented by Gerber and Shiu (2005), "although the model was proposed almost half a century ago, it remains an important area of research in actuarial science". A large number of researchers have studied this model on a variety of topics, for instance, the probability of ruin and the Gerber-Shiu functions, after assuming that the inter-claim time distribution is known. The type of distributions considered includes Erlang(2), Erlang($n$), generalised Erlang($n$), $K_n$ distributions, and also some discrete distributions for discrete-time Sparre Andersen models. Some very recent papers on the Sparre Andersen model include Alfa and Drekic (2007), Albrecher et al (2007), Borovkov and Dickson (2008) and Yang and Zhang (2008).

On the contrary, some researchers assumed arbitrary inter-claim time distributions in the Sparre Andersen model. Having specified the individual claim size distributions, interesting results have been obtained for some of the problems. Relevant references see Wang and Liu (2002), Willmot (2007), Landriault and Willmot (2008), Wu and Li (2009), etc.

In this paper we shall assume an arbitrary inter-claim time distribution $k(t)$ and phase-type distributed individual claim sizes. After giving a brief review of the properties for discrete phase-type distributions in Section 2, we shall consider the ruin probability and the distribution of the deficit at ruin within the rest of this paper. The discounted probability of ruin and discounted distribution of the deficit at ruin are also discussed in this paper.

# 2 Discrete phase-type distributions

In this section, we shall review some of the key properties of the discrete phase-type distributions. Formalised introductions for the discrete phase-type distributions date back to mid 1970's, see Neuts (1975). However, more researchers have been focusing on the studies of the continuous phase-type distributions. Detailed discussions of continuous phase-type distributions can be found in Neuts (1981) and Latouche and Ramaswami (1999). Brief overviews of either discrete or continuous phase-type distributions and their properties can be found in Asmussen (1992, 2000), Stanford and Stroiński (1994), Drekic et al (2004), Ng and Yang (2005), Eisele (2006), Hipp (2006) and the references therein.

A discrete phase-type distribution is defined by considering a Markov chain with $m$ transient states and one absorbing state, say 0. It has an associated transition matrix

$$\tilde{\mathbf{T}} = \left[ \begin{array}{cc} 1 & \vec{\mathbf{0}} \\ \vec{\mathbf{t}}^\top & \mathbf{T} \end{array} \right],$$

where $\vec{\mathbf{0}} = (0, 0, \ldots, 0)_{1\times m}, \vec{\mathbf{t}} = (t_{10}, t_{20}, \ldots, t_{m0})$, and $\mathbf{T} = (t_{ij})_{m\times m}$. Matrix $\mathbf{T}$ is a substochastic matrix, holding the transition rates among the $m$ transient states, and $\vec{\mathbf{t}}$ contains the absorption rates into state 0 from the transient states. Given a starting distribution $\vec{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \ldots, \alpha_m)$, we denote by $X$ the minimum time for the Markov chain to get to state 0, then $X$ has a discrete phase-type distribution with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$. Although the Markov chain may start from the absorption state 0, as in the sequel we are going to apply the discrete phase-type distributions for the positive individual claim sizes $X_i, i = 1, 2, \ldots$, we shall assume $\alpha_0 = 0$, or equivalently $\sum_{i=1}^m \alpha_i = 1$. Let $X$ represent an arbitrary $X_i$. Further, we know $\vec{\mathbf{t}}^\top = \vec{\mathbf{1}}^\top - \mathbf{T}\vec{\mathbf{1}}^\top$, where $\vec{\mathbf{1}} = (1, \ldots, 1)_{1\times m}$.

The probability function of $X$ is then given by

$$p(k) = \vec{\boldsymbol{\alpha}}\mathbf{T}^{k-1}\vec{\mathbf{t}}^\top, \qquad k \in \mathbb{N}^+. \tag{2.1}$$

The distribution function of $X$ is $P(k) = 1 - \vec{\boldsymbol{\alpha}}\mathbf{T}^k\vec{\mathbf{1}}^\top$ and its p.g.f. is $\hat{p}(z) = \sum_{k=1}^\infty p(k)z^k = z\vec{\boldsymbol{\alpha}}(\mathbf{I} - z\mathbf{T})^{-1}\vec{\mathbf{t}}^\top$ where $\mathbf{I}$ is an $m \times m$ identity matrix. The mean of $X$ is $\mu = \vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-2}\vec{\mathbf{t}}^\top = \vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-1}\vec{\mathbf{1}}^\top$. Similar to the interpretation in Drekic et al (2004) for a continuous phase-type distribution, the $j$th component of $\vec{\boldsymbol{\alpha}}\mathbf{T}^{k-1}$ can be interpreted as the probability that absorption has not occurred by time $k-1$, and the Markov chain is in transient state $j$ at time $k-1$. It then explains why the distribution function of $X$ has the above expression.

It has been remarked by a number of authors that a discrete phase-type distribution has a rational p.g.f., see for example Asmussen (2000) and Eisele (2006), and it can be expressed as follows:

$$\hat{p}(z) = \frac{A(z)}{B(z)} = \frac{a_1 z + a_2 z^2 + \ldots + a_m z^m}{1 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

and $\hat{p}(1) = 1$ gives $A(1)/B(1) = 1$.

Another property of interest for the phase-type random variables is that their equilibrium distributions are also (defective) phase-type. In the following, we shall consider one of the two existing definitions for the discrete equilibrium distributions, which was employed in Fagiuoli and Pellerey (1994) and Pavlova et al (2006).

For a distribution $P(x)$, its equilibrium distribution is defined as

$$P_1(x) = \begin{cases} 0, & x = 0, \\ \mu^{-1} \sum_{k=0}^{x-1} \bar{P}(k) & x = 1, 2, \ldots. \end{cases}$$

We have the following result for $P_1(x)$.

**Proposition 1** *Given that distribution $P(x)$ is phase-type with representation ($\vec{\boldsymbol{\alpha}}$, $\mathbf{T}$), its equilibrium distribution is also phase-type with representation ($\vec{\boldsymbol{\pi}}$, $\mathbf{T}$), where $\vec{\boldsymbol{\pi}} = \mu^{-1}\vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-1}$ satisfying $\vec{\boldsymbol{\pi}}\vec{\mathbf{1}}^\top = 1$.*

**Proof.** Using the above definition for equilibrium distributions, we have for $x \in \mathbb{N}^+$,

$$\begin{aligned} P_1(x) &= \frac{1}{\mu}\sum_{k=0}^{x-1} \bar{P}(k) = \frac{1}{\mu}\sum_{k=0}^{x-1} \vec{\boldsymbol{\alpha}}\mathbf{T}^k\vec{\mathbf{1}}^\top \\ &= \frac{1}{\mu}\vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-1}(\mathbf{I} - \mathbf{T}^x)\vec{\mathbf{1}}^\top = 1 - \frac{1}{\mu}\vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-1}\mathbf{T}^x\vec{\mathbf{1}}^\top, \end{aligned}$$

which is a phase-type distribution function with representation ($\vec{\boldsymbol{\pi}}$, $\mathbf{T}$), where $\vec{\boldsymbol{\pi}} = \mu^{-1}\vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-1}$. Further the p.f. of $P_1$ has the form $p_1(x) = \vec{\boldsymbol{\pi}}\mathbf{T}^{x-1}\vec{\mathbf{t}}^\top$, for $x \in \mathbb{N}^+$.

# 3 The ruin probability

To study the probability of ruin for risk model (1.1), there are two tools of often use, which are the ladder heights and the maximal aggregate loss. There has been much discussion in literature on the applications of them to a variety of risk models including the classical risk model and the continuous-time Sparre Andersen models. Useful references include Asmussen (1992, 2000), Neuts (1981), Latouche and Ramaswami (1999), Asmussen and Rolski (1991) and the references therein.

In the following we shall prove several results of the ladder height distributions, the distribution of the maximal aggregate loss and the probability of ruin for model (1.1), given phase-type individual claim amounts. One will see that all of these results are the counterpart of those obtained in a continuous-time Sparre Andersen model with continuous phase-type claim amounts (details see for example Asmussen (2000)).

First of all, we denote the time of ruin when $u = 0$ by $T_0$, and we denote $U_{T_0(i)}$ the $i$th ladder height r.v.. It is well known that the ladder heights in a continuous-time Sparre Andersen model are i.i.d. r.v.'s. In the discrete-time case $U_{T_0(i)}, i = 1, 2, \ldots$ are also i.i.d. with the common distribution function $G$.

**Proposition 2** *Given that individual claim amounts follow the phase-type distribution with representation ($\vec{\boldsymbol{\alpha}}, \mathbf{T}$), the ladder height distribution $G$ is of defective phase-type with representation ($\vec{\boldsymbol{\alpha}}_+, \mathbf{T}$), in which $\vec{\boldsymbol{\alpha}}_+$ satisfies $\vec{\boldsymbol{\alpha}}_+ = \varphi(\vec{\boldsymbol{\alpha}}_+)$, where*

$$\varphi(\vec{\boldsymbol{\alpha}}_+) = \vec{\boldsymbol{\alpha}}\hat{k}(\mathbf{T} + \vec{\mathbf{t}}^\top\vec{\boldsymbol{\alpha}}_+) = \vec{\boldsymbol{\alpha}}\sum_{t=1}^{\infty}(\mathbf{T} + \vec{\mathbf{t}}^\top\vec{\boldsymbol{\alpha}}_+)^t k(t). \tag{3.1}$$

**Proof.** This result is the counterpart of the Proposition 4.1 and 4.3 (Asmussen (2000, pp. 229-230)). Similar to the proofs given by Asmussen (2000), we shall need to construct two terminating Markov processes, $\{m_x\}$ and $\{m_x^*\}$. Here $\{m_x\}$ is obtained by piecing together the ascending ladder heights $U_{T_0(i)}, i = 1, 2, \ldots$, then the lifelength of $\{m_x\}$, denoted by $M$, is the maximal aggregate loss of $\{U(n)\}$. $\{m_x\}$ is a terminating Markov process with a defective initial vector, denoted by $\vec{\boldsymbol{\alpha}}_+$, and a transition matrix $\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+$. By the same arguments as in Asmussen (2000, pp. 227), the ladder height distribution $G$ is of phase-type with representation $(\vec{\boldsymbol{\alpha}}_+, \mathbf{T})$.

To evaluate $\vec{\boldsymbol{\alpha}}_+$, conditioning on $W_1 = t$ we define $\{m_x^*\}$ from $\{U(n+t)-U(t-1)-1\}$ in the same way as $\{m_x\}$ from $\{U(n)\}$. We know $\{m_x^*\}$ has an initial distribution $\vec{\boldsymbol{\alpha}}$ and $m_t^* = m_0$. By knowing that the conditional distribution of $\{m_t^*\}$ given $W_1 = t$ is $\vec{\boldsymbol{\alpha}}(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^t$, we can show by the law of total probability that the distribution $\vec{\boldsymbol{\alpha}}_+$ of $m_0$ satisfies equation (3.1). □

Based on the above discussion, the following result is obtained:

**Theorem 1** *For a discrete-time Sparre Andersen risk model (1.1), if the claim size distribution is of phase-type with representation* $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$*, then*

$$\psi(u) = \vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \vec{\mathbf{1}}^\top, \tag{3.2}$$

*where* $\vec{\boldsymbol{\alpha}}_+$ *satisfies equation (3.1) and can be computed by iteration of (3.1), i.e., by*

$$\vec{\boldsymbol{\alpha}}_+ = \lim_{n\to\infty} \vec{\boldsymbol{\alpha}}_+^{(n)}, \tag{3.3}$$

*where*

$$\vec{\boldsymbol{\alpha}}_+^{(0)} = \mathbf{0}, \qquad \vec{\boldsymbol{\alpha}}_+^{(n)} = \varphi(\vec{\boldsymbol{\alpha}}_+^{(n-1)}), \quad n \geq 1. \tag{3.4}$$

**Proof.** The proof of (3.2) is straightforward. Having known that the maximal aggregate loss $M$ is phase-distributed with representation $(\vec{\boldsymbol{\alpha}}_+, \mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)$ and $\psi(u) = \mathbb{P}\{M > u\}$, the expression of a phase-type distribution function in Section 2 gives the result (3.2) directly.

The convergence of the iteration scheme (3.4) can be proved by applying similar arguments as in Asmussen (2000, pp. 231-232). □

**Remark.** A special case for $p(x)$ is the zero-truncated geometric distribution with parameter $0 < q < 1$, where $p(x) = (1-q)q^{x-1}, x = 1, 2, \ldots$. It is a discrete phase-type distribution with only one phase, i.e., $m = 1$. By adopting the notation for a phase-type distribution, we have $\vec{\boldsymbol{\alpha}} = (\alpha_1) = (1)$, $\tilde{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ 1-q & q \end{pmatrix}$. Then from Theorem 1 and Proposition 2 we know that the probability of ruin for the discrete-time Sparre Andersen model (1.1), with a general inter-claim time distribution $k(t)$, equals $\psi(u) = \vec{\boldsymbol{\alpha}}_+[q + (1-q)\vec{\boldsymbol{\alpha}}_+]^u$, where $\vec{\boldsymbol{\alpha}}_+$ (a single number) satisfies the equation $\vec{\boldsymbol{\alpha}}_+ = \hat{k}(q + (1-q)\vec{\boldsymbol{\alpha}}_+)$. This ruin probability is the same as the result obtained in Wu and Li (2009) when the discount factor $v = 1$, in which the constant $\xi_1$ is $\vec{\boldsymbol{\alpha}}_+$.

# 4 The distribution of the deficit at ruin

At last we shall consider the conditional distribution of the deficit at ruin in the discrete-time Sparre Andersen model. We will consider the following two functions:

$F(u,y) := \mathbb{P}\{T < \infty, |U(T)| \le y\}$ and $\psi(u,y) := \mathbb{P}\{T < \infty, |U(T)| > y\}$. Immediately we have $\psi(u,y) = \psi(u) - F(u,y)$. The function $F(u,y)$ is the probability that ruin occurs with initial surplus $u$ and a deficit at ruin that is no greater than $y$, which was introduced in the classical risk model by Gerber et al. (1987). The function $\psi(u,y)$ is the probability that ruin occurs and the deficit at ruin exceeds $y$. Further discussions of these functions in the classical risk model can be found in Dufresne and Gerber (1988), Dickson (1989), and Willmot (2000). Drekic et al. (2004) studied these functions in a Sparre Andersen model with phase-type individual claim amounts and derived a simple phase-type representation for the distribution of the deficit. In the sequel, we shall prove the existence of a similar result in the discrete-time Sparre Andersen model.

We further define the following function:

$$F_u(y) := 1 - \bar{F}_u(y) = \frac{F(u,y)}{\psi(u)},$$

which is a non-defective distribution of the deficit at ruin. The p.f.'s associated with $F(u,y)$ and $F_u(y)$ are denoted by $f(u,y)$ and $f_u(y)$, respectively. Let r.v. $Y_u$ denote the deficit at ruin given that ruin occurs, which has the distribution $F_u(y)$. Then we have the following main result:

**Theorem 2** *The deficit $Y_u$ follows a phase-type distribution with representation $(\vec{\boldsymbol{\alpha}}_F, \mathbf{T})$ where the initial distribution $\vec{\boldsymbol{\alpha}}_F$ is*

$$\vec{\boldsymbol{\alpha}}_F = \frac{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u}{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \vec{\mathbf{1}}^\top} = \frac{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u}{\psi(u)}.$$

**Proof.** The following proof is a counterpart of the proof of Proposition 1 in Ng and Yang (2005). Drekic et al. (2004) adopted a different approach to derive the same result as in Ng and Yang (2005).

Recall the terminating Markov process $\{m_x\}$ defined in Section 3, which is obtained by piecing together the ascending ladder heights $U_{T_0(i)}, i = 1, 2, \ldots$, and the lifelength of $\{m_x\}$, $M$, is the maximal aggregate loss of $\{U(n)\}$. Further, $U_{T_0(i)}, i = 1, 2, \ldots$ are i.i.d. phase-distributed with representation $(\vec{\boldsymbol{\alpha}}_+, \mathbf{T})$, and $\{m_x\}$ has an initial vector $\vec{\boldsymbol{\alpha}}_+$ and a transition matrix $\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+$.

To find the conditional distribution for the deficit $Y_u$, we start with the joint probability $\psi(u,y)$, the probability that ruin occurs and the deficit at ruin exceeds $y$. It represents the joint probability that $M$ exceeds $u$ and the deficit at ruin exceeds $y$. We have

$$\begin{aligned}\psi(u,y) &= \mathbb{P}\{M > u, |U(T)| > y\} \\ &= \sum_{j=1}^{m} \mathbb{P}\{M > u, m_x(u) = j, |U(T)| > y\} \\ &= \sum_{j=1}^{m} \mathbb{P}\{M > u, m_x(u) = j\}\mathbb{P}\{|U(T)| > y | m_x(u) = j\}, \qquad (4.1)\end{aligned}$$

where $m_x(u)$ is the state of the terminating Markov process $\{m_x\}$ at time $u$. Knowing the structure of $\{m_x\}$ and that the ladder height distribution is of phase-type, we can determine the two probabilities in the right-hand side of (4.1) directly as follows:

$$\begin{aligned} \mathbb{P}\{M > u, m_x(u) = j\} &= \vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \vec{\mathbf{1}}_j^\top, \\ \mathbb{P}\{|U(T)| > y | m_x(u) = j\} &= \vec{\mathbf{1}}_j \mathbf{T}^y \vec{\mathbf{1}}^\top, \end{aligned}$$

where $\vec{\mathbf{1}}_j^\top$ is a $m \times 1$ column vector with all zeros except the $j$th element being 1. Substituting these two probabilities into (4.1) gives

$$\psi(u, y) = \vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \mathbf{T}^y \vec{\mathbf{1}}^\top, \tag{4.2}$$

from which we obtain

$$\bar{F}_u(y) = \frac{\psi(u, y)}{\psi(u)} = \frac{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u}{\psi(u)} \mathbf{T}^y \vec{\mathbf{1}}^\top. \tag{4.3}$$

This completes the proof. □

Within the rest of this paper, we shall consider two numerical examples.

**Example 4.1.** In this example we shall continue our discussion for the zero-truncated geometric distribution with parameter $q$. As it was mentioned in Section 3, it is phase-type with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$, where $\vec{\boldsymbol{\alpha}} = (1)$ and $\mathbf{T} = (q)$. From Theorem 2 and its proof we know that the deficit $Y_u$ follows the same distribution as the individual claim amounts, as its initial vector $\vec{\boldsymbol{\alpha}}_F$ satisfies:

$$\vec{\boldsymbol{\alpha}}_F = \frac{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u}{\vec{\boldsymbol{\alpha}}_+(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \vec{\mathbf{1}}^\top} = 1.$$

**Example 4.2.** The second phase-type distribution we shall look at is the zero-truncated negative binomial distribution with parameters 2 and $0 < q < 1$. The p.f. has the form

$$p(x) = \frac{(1-q)^2}{2-q}(x+1)q^{x-1}, x = 1, 2, \ldots$$

It has representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$, where

$$\vec{\boldsymbol{\alpha}} = \left( \frac{q}{2-q}, \frac{2-2q}{2-q} \right), \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} q & 1-q \\ 0 & q \end{pmatrix}.$$

Then we know $\vec{\mathbf{t}}^\top = (0, 1-q)^\top$. The next step is to solve $\vec{\boldsymbol{\alpha}}_+$ using (3.1). Assume $\vec{\boldsymbol{\alpha}}_+ = (\eta_1, \eta_2)$, then we have

$$\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+ = \begin{pmatrix} q & 1-q \\ (1-q)\eta_1 & q + (1-q)\eta_2 \end{pmatrix},$$

and it has two eigenvalues $\lambda_1 = q - (1-q)\gamma_2$ and $\lambda_2 = q - (1-q)\gamma_1$, where

$$\gamma_1 = \frac{-\eta_2 + \sqrt{\eta_2^2 + 4\eta_1}}{2}, \qquad \gamma_2 = \frac{-\eta_2 - \sqrt{\eta_2^2 + 4\eta_1}}{2}.$$

Obviously, $\gamma_1\gamma_2 + \eta_1 = 0$, and $\gamma_1 + \gamma_2 + \eta_2 = 0$. The associated eigenvectors are $(\gamma_1/\eta_1, 1)^\top$ and $(\gamma_2/\eta_1, 1)^\top$, so

$$\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+ = \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix}^{-1}.$$

Therefore, equation (3.1) gives

$$\begin{aligned} \vec{\boldsymbol{\alpha}}_+ &= \vec{\boldsymbol{\alpha}} \sum_{t=1}^{\infty} \left[ \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix}^{-1} \right]^t k(t) \\ &= \vec{\boldsymbol{\alpha}} \sum_{t=1}^{\infty} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix}^{-1} k(t) \\ &= \frac{\eta_1}{\gamma_1 - \gamma_2} \vec{\boldsymbol{\alpha}} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{k}(\lambda_1) & 0 \\ 0 & \hat{k}(\lambda_2) \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\gamma_1} \\ -1 & -\frac{1}{\gamma_2} \end{pmatrix}, \end{aligned}$$

from which two equations of $\eta_1$ and $\eta_2$ are obtained:

$$\begin{cases} 2(2-q)\eta_1(\gamma_1 - \gamma_2) = [4(1-q)\eta_1 - q\eta_2][\hat{k}(\lambda_1) - \hat{k}(\lambda_2)] + q(\gamma_1 - \gamma_2)[\hat{k}(\lambda_1) + \hat{k}(\lambda_2)] \\ (2-q)\eta_2(\gamma_1 - \gamma_2) = q[\hat{k}(\lambda_1) - \hat{k}(\lambda_2)] + 2(1-q)\eta_1[\frac{\hat{k}(\lambda_1)}{\gamma_1} - \frac{\hat{k}(\lambda_2)}{\gamma_2}] \end{cases}.$$

Given $q$ and the inter-claim time distribution $k(t)$, $\eta_1$ and $\eta_2$ can be solved using some mathematical software such as *Mathematica*. Having known $\vec{\boldsymbol{\alpha}}_+$, we obtain the following results:

$$\begin{aligned} \psi(u) &= \vec{\boldsymbol{\alpha}}_+ (\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \vec{\mathbf{1}}^\top \\ &= (\eta_1, \eta_2) \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^u & 0 \\ 0 & \lambda_2^u \end{pmatrix} \begin{pmatrix} \frac{\gamma_1}{\eta_1} & \frac{\gamma_2}{\eta_1} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\eta_1}{\gamma_1 - \gamma_2} \left( \gamma_1 \lambda_2^u - \gamma_2 \lambda_1^u, \frac{\gamma_1}{\gamma_2}\lambda_2^u - \frac{\gamma_2}{\gamma_1}\lambda_1^u \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\eta_1}{\gamma_1 - \gamma_2} \left[ (\gamma_1 + \frac{\gamma_1}{\gamma_2})\lambda_2^u - (\gamma_2 + \frac{\gamma_2}{\gamma_1})\lambda_1^u \right], \\ \vec{\boldsymbol{\alpha}}_F &= \frac{1}{\psi(u)} \vec{\boldsymbol{\alpha}}_+ (\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_+)^u \\ &= \frac{1}{(\gamma_1 + \frac{\gamma_1}{\gamma_2})\lambda_2^u - (\gamma_2 + \frac{\gamma_2}{\gamma_1})\lambda_1^u} \left( \gamma_1 \lambda_2^u - \gamma_2 \lambda_1^u, \frac{\gamma_1}{\gamma_2}\lambda_2^u - \frac{\gamma_2}{\gamma_1}\lambda_1^u \right), \\ \bar{F}_u(y) &= \vec{\boldsymbol{\alpha}}_F \mathbf{T}^y \vec{\mathbf{1}}^\top = \vec{\boldsymbol{\alpha}}_F \begin{pmatrix} q^y & (1-q)yq^{y-1} \\ 0 & q^y \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= q^y + (1-q) \frac{\gamma_1 \lambda_2^u - \gamma_2 \lambda_1^u}{(\gamma_1 + \frac{\gamma_1}{\gamma_2})\lambda_2^u - (\gamma_2 + \frac{\gamma_2}{\gamma_1})\lambda_1^u} y q^{y-1}. \end{aligned}$$

# 5 Two discounted functions

To end this paper, we shall consider two generalizations of the probabilities studied in Section 3 and 4, the discounted probability of ruin and the discounted distribution of the deficit at ruin. Both of them are special cases of the well-known Gerber-Shiu expected discounted penalty function that has been receiving much attention for a decade. The probabilities are defined as follows:

$$\begin{aligned} \psi_v(u) &= \mathbb{E}\{v^T I(T < \infty)|U(0) = u\}, \\ \psi_v(u, y) &= \mathbb{E}\{v^T I(T < \infty) I(|U(T)| > y)|U(0) = u\}, \qquad u \in \mathbb{N}, y \in \mathbb{N}^+, \end{aligned}$$

where $0 < v \leq 1$ is a discount factor.

In Ren (2009), the Gerber-Shiu function in a continuous-time Sparre Andersen model with general inter-claim times was studied. An interesting point made in Proposition 2.1 in the paper is that the discounting effect on the penalty function can be equalized by a particularly defined new inter-claim time distribution. Parallel to Proposition 2.1 in Ren (2009), we have the following result.

**Proposition 3** *The discounted functions $\psi_v(u)$ and $\psi_v(u, y)$ with the inter-claim time distribution $k(t), t \in \mathbb{N}^+$, are the same as the non-discounted functions $\psi(u)$ and $\psi(u, y)$, respectively, with the (defective) inter-claim time distribution $k_v(t) := v^t k(t), t \in \mathbb{N}^+$.*

We remark that the proof of the continuous case in Ren (2009) applies here. Further, we have $\hat{k}_v(z) := \sum_{t=1}^{\infty} z^t k_v(t) = \hat{k}(vz), z \in \mathbb{C}$.

From Proposition 3 we can see that the whole discussion in Section 3 and 4 can be repeated by employing the new (defective) inter-claim time distribution $k_v(t)$ to obtain the discounted functions $\psi_v(u)$ and $\psi_v(u, y)$. We just list some of the corresponding key results in the following without giving proofs.

**Theorem 3** *For a discrete-time Sparre Andersen risk model as defined in (1.1), if the claim size distribution is of phase-type with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$, then:*

*(1) The discounted probability of ruin is*

$$\psi_v(u) = \vec{\boldsymbol{\alpha}}_{+v}(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_{+v})^u \vec{\mathbf{1}}^\top, \tag{5.1}$$

*where $\vec{\boldsymbol{\alpha}}_{+v}$ satisfies equation $\vec{\boldsymbol{\alpha}}_{+v} = \varphi_v(\vec{\boldsymbol{\alpha}}_{+v}) = \vec{\boldsymbol{\alpha}}\hat{k}[v(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_{+v})]$ and can be computed iteratively.*

*(2) The discounted distribution of the deficit $Y_u$, $\psi_v(u, y)$, has the form*

$$\psi_v(u, y) = \vec{\boldsymbol{\alpha}}_{+v}(\mathbf{T} + \vec{\mathbf{t}}^\top \vec{\boldsymbol{\alpha}}_{+v})^u \mathbf{T}^y \vec{\mathbf{1}}^\top.$$

**Remark.** For the positive geometric distribution case, i.e., $p(x) = (1-q)q^{x-1}, x = 1, 2, \ldots$, Theorem 3 gives

$$\psi_v(u) = \vec{\boldsymbol{\alpha}}_{+v}[q + (1-q)\vec{\boldsymbol{\alpha}}_{+v}]^u, \quad \text{and} \quad \psi_v(u, y) = \vec{\boldsymbol{\alpha}}_{+v}[q + (1-q)\vec{\boldsymbol{\alpha}}_{+v}]^u q^y,$$

where $\vec{\boldsymbol{\alpha}}_{+v}$ satisfies equation $\vec{\boldsymbol{\alpha}}_{+v} = \hat{k}[v(q + (1-q)\vec{\boldsymbol{\alpha}}_{+v})]$. The discounted ruin probability $\psi_v(u)$ is the same as the function $\bar{D}_v(u)$ derived in Wu and Li (2009) for the geometric claims, in which the constant $\xi_v$ equals $\vec{\boldsymbol{\alpha}}_{+v}$.

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