

# The perturbed compound Poisson risk model with two-sided jumps

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## Abstract

In this paper, we consider a perturbed compound Poisson risk model with two-sided jumps. The downward jumps represent the claims following an arbitrary distribution, while the upward jumps are also allowed to represent the random gains. Assuming that the density function of the upward jumps has a rational Laplace transform, the Laplace transforms and defective renewal equations for the discounted penalty functions are derived, and the asymptotic estimate for the probability of ruin is also studied for heavy-tailed downward jumps. Finally, some explicit expressions for the discounted penalty functions, as well as numerical examples, are given.

**Keywords:** Compound Poisson risk model; Discounted penalty function; Laplace transform; Defective renewal equation; Asymptotic formula.

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# 1 Introduction

The classical compound Poisson risk model perturbed by a Brownian motion is given by:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma B(t), \quad (1.1)$$

where  $u \geq 0$  is the initial surplus,  $c > 0$  is the constant premium rate. The claim number process  $\{N(t)\}_{t \geq 0}$  is a Poisson process with intensity  $\lambda > 0$ . The individual claims  $X_1, X_2, \dots$  are positive independent and identically distributed (i.i.d.) random variables.  $\sigma > 0$  is the diffusion volatility and  $\{B(t)\}_{t \geq 0}$  is a standard Brownian motion starting from zero. Finally  $\{N(t)\}_{t \geq 0}$ ,  $\{X_i\}_{i=1}^{\infty}$  and  $\{B(t)\}_{t \geq 0}$  are assumed to be mutually independent.

Risk model (1.1) was first proposed by Gerber [5] to extend the classical compound Poisson risk model, in which the Brownian motion can be interpreted as random variability of the premiums income or the claims loss. Since then, many ruin problems associated with model (1.1) have been studied. For example, Dufresne and Gerber [2] studied the ruin probabilities by oscillation or by a claim. Gerber and Landry [6] examined the expected discounted penalty function of the deficit at ruin. Tsai [14] studied some discounted distributions for the surplus process perturbed by diffusion. For the same model, Tsai and Willmot [15] and [16] studied the discounted penalty functions and some (discounted) moments, respectively.

In this paper, we study a modified version of model (1.1). With all others being the same, the only change is to assume that the jumps are two-sided. The upward jumps can be explained to be the random gains of the company, while the downward jumps are interpreted as the random loss of the company. Denote the density of the jump  $X$  by  $f$  which is given by

$$f(x) = pf_+(x)I(x > 0) + qf_-(-x)I(x < 0), \quad (1.2)$$

where  $p, q > 0$ ,  $p + q = 1$ ,  $I(A)$  is an indicator function of the event  $A$ , and  $f_+$ ,  $f_-$  are two density functions on  $[0, \infty)$ . Let  $F_+$ ,  $F_-$  denote the distributions, and  $\mu_+$ ,  $\mu_-$  denote the means of  $f_+$  and  $f_-$ .

The risk model with two-sided jumps has been studied by many authors. Perry et al. [12] studied the joint distribution of the first exit time and overshoot (or undershoot) given that the distributions of the jumps were hyperexponential distribution or a special Coxian distribution. Kou and Wang [8] studied the Laplace transforms of the first passage time and the overshoot for a perturbed compound Poisson risk model where both the upward and downward jumps are

exponentially distributed. Xing et al. [20] extended the results of Kou and Wang [8] to the case when the downward jumps are phase-type distributed and the upward jumps have an arbitrary distribution. Jacobsen [7] considered a more general risk model, the Markov-modulated diffusion risk model with two-sided jumps, and studied the Laplace transform of the time to ruin, the undershoot at ruin, as well as the probability of ruin. More recently, Yang and Zhang [22] studied a compound Poisson risk model with two-sided jumps by analyzing the discounted penalty function.

In this paper, we assume that the downward jumps have an arbitrary density function, while the density of the upward jumps has a rational Laplace transform  $\hat{f}_-(s) = \int_0^\infty e^{-sx} f_-(x) dx$  given by

$$\hat{f}_-(s) = \frac{\nu(s)}{\prod_{i=1}^m (\nu_i + s)^{n_i}}, \quad (1.3)$$

where  $m, n_i \in \mathbb{N}^+$  with  $n_1 + n_2 + \dots + n_m = n$ ,  $\nu_i > 0$ ,  $i = 1, 2, \dots, m$ , with  $\nu_i \neq \nu_j$  for  $i \neq j$ .  $\nu(s)$  is a polynomial function of degree  $n - 1$  or less satisfying  $\nu(0) = \prod_{i=1}^m \nu_i^{n_i}$ . The rational family distributions, widely used in probability applications, include the Erlang, Coxian and phase-type distributions as special case, as well as mixture of them.

In what follows,  $U(t)$  denotes the modified version of (1.1) with two-sided jumps described as above. Let  $T = \inf\{t : U(t) \leq 0\}$  or  $\infty$  otherwise, be the ruin time, and denote the ultimate ruin probability by

$$\psi(u) = Pr(T < \infty | U(0) = u).$$

To guarantee that ruin is possible, we assume the following net profit condition holds, i.e.,

$$c + \lambda q \mu_- > \lambda p \mu_+. \quad (1.4)$$

Since ruin occurs immediately when  $u = 0$ , we have  $\psi(0) = 1$ . By observing the sample paths of  $U(t)$ , we know that ruin can be caused either by the oscillation of the Brownian motion or by a downward jump. Similar to Dufresne and Gerber [2] and Wang [17], we could decompose the probability of ruin as

$$\psi(u) = \psi_s(u) + \psi_d(u),$$

where  $\psi_s(u)$  is the ruin probability when ruin is caused by a downward jump, and  $\psi_d(u)$  is the ruin probability when ruin is caused by oscillation. We have the following initial conditions

$$\psi(0) = \psi_d(0) = 1, \quad \psi_s(0) = 0. \quad (1.5)$$

Now for  $\delta \geq 0$ , define the (Gerber-Shiu) discounted penalty function by

$$\phi(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) | U(0) = u],$$

where  $w(x_1, x_2)$ ,  $x_1, x_2 \geq 0$ , is a nonnegative function of the surplus before ruin  $U(T-)$ , and the deficit at ruin  $|U(T)|$ . Similarly,  $\phi(u)$  can also be decomposed as

$$\phi(u) = \phi_s(u) + \phi_d(u),$$

where

$$\phi_s(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) < 0) | U(0) = u]$$

is the discounted penalty function at ruin that is caused by a downward jump, and

$$\begin{aligned} \phi_d(u) &= E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) = 0) | U(0) = u] \\ &= w(0, 0) E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u] \end{aligned}$$

is the discounted penalty function at ruin that is caused by oscillation. Without loss of generality, we set  $w(0, 0) = 1$ , then the following initial conditions hold

$$\phi(0) = \phi_d(0) = 1, \quad \phi_s(0) = 0. \tag{1.6}$$

In this paper, we study the discounted penalty functions for the perturbed compound Poisson risk model with two-sided jumps. The rest of this paper is organized as follows: In Section 2, we introduce a generalized Lundberg equation and analyze its roots on the right-half complex plane. By using these roots, the Laplace transforms for  $\phi_s(u)$  and  $\phi_d(u)$  are derived in Section 3. The defective renewal equations satisfied by  $\phi_s(u)$  and  $\phi_d(u)$  are derived in Section 4, and accordingly, the analytic expression for the discounted penalty functions are obtained. In Section 5, the asymptotic behavior for the probability of ruin is studied when the downward jumps are heavy-tailed. Finally, assuming that the density  $f_+$  also has a rational Laplace transform, the explicit expressions for the discounted penalty functions as well as numerical examples are given.

## 2 Analysis of a generalized Lundberg equation

In this section, we introduce a generalized version of the Lundberg equation for the perturbed compound Poisson model with two-sided jumps, and then analyze

the number of its roots on the right-half complex plane. Hereafter, we denote the Laplace transform of a function by adding a hat on the corresponding letter.

We seek for a number  $s$  such that the process  $\{e^{-\delta t + sU(t)}\}_{t \geq 0}$  is a martingale. Here the martingale condition is

$$Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) = 0, \quad (2.1)$$

where  $D = \sigma^2/2$ ,  $\hat{f}(s) = E[e^{-sX_1}] = p\hat{f}_+(s) + q\hat{f}_-(-s)$ . Equation (2.1) is called the generalized Lundberg equation for the perturbed compound Poisson risk model with two-sided jumps. When  $p = 1, q = 0$ , (2.1) simplifies to the Lundberg equation (5) of Gerber and Landry [6], and it is shown that the Lundberg equation has exactly one positive root  $\rho > 0$ .

**Lemma 1** *When  $0 < p, q < 1$  and the Laplace transform  $\hat{f}_-(s)$  is given by (1.3), the generalized Lundberg equation (2.1) has exactly  $n+1$  roots,  $\rho_1(\delta), \rho_2(\delta), \dots, \rho_{n+1}(\delta)$ , on the right-half complex plane.*

**Proof.** It suffices to show that the following equation

$$\prod_{i=1}^m (\nu_i - s)^{n_i} \left[ Ds^2 + cs - (\lambda + \delta) + \lambda p \hat{f}_+(s) \right] + \lambda q \nu(-s) = 0 \quad (2.2)$$

has exactly  $n + 1$  roots with positive real part. Let  $r > 0$  be a sufficiently large number, and denote by  $C_r$  the contour containing the imaginary axis running from  $-ir$  to  $ir$  and a semicircle with radius  $r$  running clockwise from  $ir$  to  $-ir$ . For  $s$  on the semicircle, we have  $|Ds^2 + cs - (\lambda + \delta)| \rightarrow \infty$  as  $r \rightarrow \infty$ , while

$$|\lambda \hat{f}(s)| \leq \lambda p + \lambda q \frac{|\nu(-s)|}{\left| \prod_{i=1}^m (\nu_i - s)^{n_i} \right|} \rightarrow \lambda p, \quad |s| \rightarrow \infty$$

because  $\nu(s)$  is a polynomial of degree  $n - 1$  or less. For  $s$  on the imaginary axis, we have

$$|Ds^2 + cs - (\lambda + \delta)| \geq \lambda + \delta > \lambda \geq |\lambda \hat{f}(s)|.$$

Thus, when  $r$  is sufficiently large, we have for  $s$  on  $C_r$

$$\left| \prod_{i=1}^m (\nu_i - s)^{n_i} [Ds^2 + cs - (\lambda + \delta)] \right| > \left| \prod_{i=1}^m (\nu_i - s)^{n_i} \lambda \hat{f}(s) \right|.$$

By Rouché's theorem, we know that equation (2.1) has the same number of roots as the following equation

$$\prod_{i=1}^m (\nu_i - s)^{n_i} [Ds^2 + cs - (\lambda + \delta)] = 0$$

inside  $C_r$ . Since the latter has  $n + 1$  roots inside  $C_r$ , so does the former. Finally, we complete the proof by letting  $r \rightarrow \infty$ .  $\square$

If we denote the root with the smallest real part by  $\rho_{n+1}(\delta)$ , then it is easy to see that  $\rho_{n+1}(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . The roots of the generalized Lundberg equation play an important role in the rest of this paper. In what follows, we denote them by  $\rho_1, \rho_2, \dots, \rho_{n+1}$  for simplicity, and only consider the case when they are all distinct since the analysis of the other case is more tedious.

### 3 Laplace transforms for $\phi_s(u)$ and $\phi_d(u)$

In this section, we derive the Laplace transforms for the discounted penalty functions. To this end, we first derive integro-differential equations for  $\phi_s(u)$  and  $\phi_d(u)$ . Furthermore, we assume that  $\phi_s(u)$  and  $\phi_d(u)$  are twice continuously differentiable in  $u$  over  $(0, \infty)$ . The sufficient conditions for the continuous differentiability can be found in [10, 18].

Now we consider  $\phi_s(u)$ . Let  $h$  be a small positive number. By a heuristic argument as in [15], we have

$$\begin{aligned} \phi_s(u) &= (1 - \lambda h)e^{-\delta h} E[\phi_s(u + ch + \sigma B(h))] \\ &\quad + \lambda h e^{-\delta h} E \left[ p \int_0^{u+ch+\sigma B(h)} \phi_s(u + ch + \sigma B(h) - x) f_+(x) dx \right. \\ &\quad + p \int_{u+ch+\sigma B(h)}^{\infty} w(u + ch + \sigma B(h), x - u - ch - \sigma B(h)) f_+(x) dx \\ &\quad \left. + q \int_0^{\infty} \phi_s(u + ch + \sigma B(h) + x) f_-(x) dx \right] + o(h). \end{aligned} \quad (3.1)$$

By Taylor's expansion, we have

$$E[\phi_s(u + ch + \sigma B(h))] = \phi_s(u) + c\phi_s'(u)h + D\phi_s''(u)h + o(h).$$

Plugging above formula into (3.1), applying some rearrangements, dividing both sides by  $h$  and letting  $h \rightarrow 0$ , we obtain

$$\begin{aligned} D\phi_s''(u) + c\phi_s'(u) &= (\lambda + \delta)\phi_s(u) - \lambda p \int_0^u \phi_s(u - x) f_+(x) dx - \lambda p \omega(u) \\ &\quad - \lambda q \int_0^{\infty} \phi_s(u + x) f_-(x) dx. \end{aligned} \quad (3.2)$$

where  $\omega(u) = \int_u^\infty w(u, x - u) f_+(x) dx$ .

Similarly, for  $\phi_d(u)$ , we could obtain

$$\begin{aligned} D\phi_d''(u) + c\phi_d'(u) &= (\lambda + \delta)\phi_d(u) - \lambda p \int_0^u \phi_d(u - x) f_+(x) dx \\ &\quad - \lambda q \int_0^\infty \phi_d(u + x) f_-(x) dx. \end{aligned} \quad (3.3)$$

Now we consider the integral  $\int_0^\infty \phi_s(u + x) f_-(x) dx$  in (3.2). Note that the Laplace transform  $\hat{f}_-(s)$  has the form of (1.3), then by partial fraction, we can rewrite it as follows

$$\hat{f}_-(s) = \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{\beta_{i,j} \nu_i^j}{(\nu_i + s)^j}, \quad (3.4)$$

where

$$\beta_{i,j} = \frac{1}{\nu_i^j (n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \left\{ \prod_{k=1, k \neq i}^m \frac{\nu(s)}{(\nu_k + s)^{n_k}} \right\} \Big|_{s = -\nu_i}.$$

Inverting (3.4) leads to

$$f_-(x) = \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} \frac{x^{j-1} \nu_i^j e^{-\nu_i x}}{(j-1)!}, \quad (3.5)$$

which is actually a density of a combination of Erlangs. Let

$$k_{i,j}(x) = \frac{x^{j-1} \nu_i^j e^{-\nu_i x}}{(j-1)!}, \quad x > 0, j \in \mathbb{N}^+,$$

denote the density function of Erlang( $j$ ) with parameter  $\nu_i$ . Then

$$\int_0^\infty \phi_s(u+x) f_-(x) dx = \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} \int_0^\infty \phi_s(u+x) k_{i,j}(x) dx = \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} E[\phi_s(u + Y_{i,j})],$$

where  $Y_{i,j}$  is a random variable with density  $k_{i,j}$ . Let  $Z_{i,1}, Z_{i,2}, \dots, Z_{i,j}$  be  $j$  i.i.d. random variables exponentially distributed with mean  $1/\nu_i$ . Then  $Y_{i,j}$  has the same distribution as that of  $Z_{i,1} + Z_{i,2} + \dots + Z_{i,j}$ . Define the following auxiliary functions

$$A_{i,k+1}(u) = E[A_{i,k}(u + Z_{i,k+1})], \quad k = 0, 1, \dots, j-1,$$

with  $A_{i,0}(u) = \phi_s(u)$ ,  $A_{i,j}(u) = E[\phi_s(u + Y_{i,j})]$ . We have

$$A_{i,k+1}(u) = \int_0^\infty A_{i,k}(u+x) \nu_i e^{-\nu_i x} dx.$$

A change of variables  $y = u + x$  brings above equation into

$$A_{i,k+1}(u) = \nu_i \int_u^\infty A_{i,k}(y) e^{-\nu_i(y-u)} dy.$$

Differentiating both sides of above equation with respect to  $u$  leads to

$$A'_{i,k+1}(u) = \nu_i A_{i,k+1}(u) - \nu_i A_{i,k}(u).$$

It follows from above equation and successive substitution that

$$\phi_s(u) = \left( \frac{\nu_i - \mathcal{D}}{\nu_i} \right)^j A_{i,j}(u), \quad (3.6)$$

where  $\mathcal{D}$  denotes the differentiation operator with respect to  $u$ .

Similarly, let  $B_{i,j}(u) = E[\phi_d(u + Y_{i,j})]$ , then we have

$$\int_0^\infty \phi_d(u+x) f_-(x) dx = \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} B_{i,j}(u), \quad (3.7)$$

and

$$\phi_d(u) = \left( \frac{\nu_i - \mathcal{D}}{\nu_i} \right)^j B_{i,j}(u). \quad (3.8)$$

Now let  $\mathbb{C}^+$  denote the set of complex numbers with nonnegative real parts, and let  $\mathbb{C}^1 = \{s : s \in \mathbb{C}^+, \operatorname{Re}(s) < \min_{1 \leq i \leq m} \nu_i\}$ . For  $s \in \mathbb{C}^1$ , we apply Laplace transforms to both sides of (3.2) and (3.6) to obtain

$$\begin{aligned} \left[ Ds^2 + cs - (\lambda + \delta) + \lambda p \hat{f}_+(s) \right] \hat{\phi}_s(s) &= D\phi'_s(0) - \lambda p \hat{\omega}(s) \\ &\quad - \lambda q \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} \hat{A}_{i,j}(s). \end{aligned} \quad (3.9)$$

and

$$\hat{\phi}_s(s) = \left( \frac{\nu_i - s}{\nu_i} \right)^j \hat{A}_{i,j}(s) + l_{s,i,j-1}(s), \quad (3.10)$$

where  $l_{s,i,j-1}(s)$  is a polynomial of degree  $j-1$ .

Let  $\tau(s) = \prod_{i=1}^m (s - \nu_i)^{n_i}$ . Substituting (3.10) into (3.9) and then multiplying both sides by  $\tau(s)$ , we obtain  $s \in \mathbb{C}^1$

$$\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right] \hat{\phi}_s(s) = l_s(s) - \lambda p \tau(s) \hat{\omega}(s), \quad (3.11)$$

where

$$l_s(s) = D\phi'_s(0)\tau(s) + \lambda q\tau(s) \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} \frac{\nu_i^j l_{s,i,j-1}(s)}{(\nu_i - s)^j}.$$

Similarly, by taking Laplace transforms in (3.3), (3.7) and (3.8), we can obtain for  $s \in \mathbb{C}^1$

$$\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right] \hat{\phi}_d(s) = l_d(s) + \tau(s)(Ds + c), \quad (3.12)$$

where

$$l_d(s) = D\phi'_d(0)\tau(s) + \lambda q\tau(s) \sum_{i=1}^m \sum_{j=1}^{n_i} \beta_{i,j} \frac{\nu_i^j l_{d,i,j-1}(s)}{(\nu_i - s)^j},$$

$l_{d,i,j-1}(s)$  is a polynomial of degree  $j - 1$  satisfying

$$\hat{\phi}_d(s) = \left( \frac{\nu_i - s}{\nu_i} \right)^j \hat{B}_{i,j}(s) + l_{d,i,j-1}(s).$$

**Theorem 1** For  $s \in \mathbb{C}^+$ , the Laplace transforms  $\hat{\phi}_s(s)$  and  $\hat{\phi}_d(s)$  are given by

$$\hat{\phi}_s(s) = \frac{\lambda p \sum_{i=1}^{n+1} \prod_{j=1, j \neq i}^{n+1} \frac{s - \rho_j}{\rho_i - \rho_j} [\tau(\rho_i)\hat{\omega}(\rho_i) - \tau(s)\hat{\omega}(s)]}{\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right]}, \quad (3.13)$$

$$\hat{\phi}_d(s) = \frac{\sum_{i=1}^{n+1} \prod_{j=1, j \neq i}^{n+1} \frac{s - \rho_j}{\rho_i - \rho_j} [\tau(s)(Ds + c) - \tau(\rho_i)(D\rho_i + c)]}{\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right]}. \quad (3.14)$$

**Proof.** Note that  $\hat{\phi}_s(s)$  and both sides of (3.11) are analytic for  $s \in \mathbb{C}^+$ . By the identity theorem for analytic functions, equation (3.11) holds for all  $s \in \mathbb{C}^+$ . Setting  $s = \rho_i$  for  $i = 1, 2, \dots, n + 1$  makes the left hand side of (3.11) vanish due to Lemma 1, and consequently, we have

$$l_s(\rho_i) = \lambda p \tau(\rho_i) \hat{\omega}(\rho_i).$$

Since  $l_s(s)$  is a polynomial of degree  $n$ , it can be rewritten as

$$l_s(s) = \lambda p \sum_{i=1}^{n+1} \prod_{j=1, j \neq i}^{n+1} \frac{s - \rho_j}{\rho_i - \rho_j} \tau(\rho_i) \hat{\omega}(\rho_i),$$

due to Lagrange interpolation formula. Plugging above formula into (3.11) yields (3.13). (3.14) can also be obtained analogously.  $\square$

## 4 Defective renewal equations for $\phi_s(u)$ and $\phi_d(u)$

In this section, we derive defective renewal equations for  $\phi_s(u)$  and  $\phi_d(u)$ . Firstly, we introduce the Dickson-Hipp operator  $T_s$  defined on a real-valued function  $h$ ,

$$T_s h(x) = \int_x^\infty e^{-s(y-x)} h(y) dy, \quad x \geq 0,$$

where  $s$  is a nonnegative real number (or a complex number with nonnegative real part) such that above integral is convergent. It is easy to see that the Laplace transform of  $h$ ,  $\hat{h}$ , can be expressed as  $T_s h(0)$ . The operator  $T_s$  is commutative, i.e.  $T_s T_r = T_r T_s$ , and furthermore

$$T_s T_r h(x) = \frac{T_s h(x) - T_r h(x)}{r - s}, \quad r \neq s.$$

For more properties of  $T_s$ , we refer to Dickson and Hipp [1] and Li and Garrido [9].

For convenience, let  $\pi(s) = \prod_{i=1}^{n+1} (s - \rho_i)$ ,  $\pi'(\rho_i) = \prod_{j=1, j \neq i}^{n+1} (\rho_i - \rho_j)$ . Now we are ready to derive integral equations for  $\phi_s(u)$  and  $\phi_d(u)$ .

**Theorem 2** *The discounted penalty functions  $\phi_s(u)$  and  $\phi_d(u)$  satisfy the following integral equations*

$$\phi_s(u) = \int_0^u \phi_s(u-x) g(x) dx + H(u), \quad (4.1)$$

$$\phi_d(u) = \int_0^u \phi_d(u-x) g(x) dx + e^{-au}, \quad (4.2)$$

where  $a = \frac{c}{D} + \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \rho_i}{\pi'(\rho_i)}$ ,

$$g(x) = \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} e^{-ax} * T_{\rho_i} f(x), \quad H(u) = \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} e^{-au} * T_{\rho_i} \omega(u),$$

and  $*$  is the convolution operator.

**Proof.** Since  $\tau(s) \left[ s - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} q \hat{f}_-(-s) \right]$  is a polynomial of degree  $n + 1$  with leading coefficient being 1, then  $\tau(s) \left[ s - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} q \hat{f}_-(-s) \right] - \pi(s)$  is a polynomial of degree  $n$  satisfying for  $i = 1, 2, \dots, n + 1$

$$\tau(\rho_i) \left[ \rho_i - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} q \hat{f}_-(-\rho_i) \right] - \pi(\rho_i) = -\tau(\rho_i) \left[ \frac{D}{c} \rho_i^2 + \frac{\lambda}{c} p \hat{f}_+(\rho_i) \right]$$

thanks to Lemma 1. Then by Lagrange interpolation formula, we have

$$\tau(s) \left[ s - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} q \hat{f}_-(-s) \right] - \pi(s) = -\pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \left[ \frac{D}{c} \rho_i^2 + \frac{\lambda}{c} p \hat{f}_+(\rho_i) \right]}{\pi'(\rho_i)(s - \rho_i)}. \quad (4.3)$$

Thus, we have

$$\begin{aligned} & \tau(s) \left[ \frac{D}{c} s^2 + s - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \hat{f}(s) \right] \\ &= \pi(s) + \tau(s) \left[ \frac{D}{c} s^2 + \frac{\lambda}{c} p \hat{f}_+(s) \right] - \pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \left[ \frac{D}{c} \rho_i^2 + \frac{\lambda}{c} p \hat{f}_+(\rho_i) \right]}{\pi'(\rho_i)(s - \rho_i)} \\ &= \pi(s) \left[ 1 - \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \left[ \frac{D}{c} \rho_i^2 + \frac{\lambda}{c} p \hat{f}_+(\rho_i) \right] - \tau(s) \left[ \frac{D}{c} s^2 + \frac{\lambda}{c} p \hat{f}_+(s) \right]}{\pi'(\rho_i)(s - \rho_i)} \right] \\ &= \pi(s) \left[ 1 - \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \left[ \frac{D}{c} \rho_i^2 + \frac{\lambda}{c} p \hat{f}_+(\rho_i) \right] - \tau(\rho_i) \left[ \frac{D}{c} s^2 + \frac{\lambda}{c} p \hat{f}_+(s) \right]}{\pi'(\rho_i)(s - \rho_i)} \right. \\ &\quad \left. + \sum_{i=1}^{n+1} \frac{\tau(s) - \tau(\rho_i)}{\pi'(\rho_i)(s - \rho_i)} \left[ \frac{D}{c} s^2 + \frac{\lambda}{c} p \hat{f}_+(s) \right] \right] \\ &= \pi(s) \left[ 1 + \frac{D}{c} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)(\rho_i + s)}{\pi'(\rho_i)} - \frac{\lambda}{c} p \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} f_+(0) \right. \\ &\quad \left. + \sum_{i=1}^{n+1} \frac{\tau(s) - \tau(\rho_i)}{\pi'(\rho_i)(s - \rho_i)} \left[ \frac{D}{c} s^2 + \frac{\lambda}{c} p \hat{f}_+(s) \right] \right]. \quad (4.4) \end{aligned}$$

Note that  $\tau(s)$  is a polynomial of degree  $n$  and the divided difference  $\frac{\tau(s) - \tau(\rho_i)}{s - \rho_i}$  is a polynomial of degree  $n - 1$ . Then by the following formula in interpolation theory

$$\sum_{i=1}^n \frac{(s_i - s)^k}{\prod_{j=1, j \neq i}^n (s_i - s_j)} = \begin{cases} 1, & k = n - 1, \\ 0, & k = 0, 1, 2, \dots, n - 2, \\ -\frac{1}{\prod_{i=1}^n (s - s_i)}, & k = -1, \end{cases} \quad (4.5)$$

Eq. (4.4) simplifies to

$$\pi(s) \left[ \frac{D}{c} (s + a) - \frac{\lambda}{c} p \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} f_+(0) \right]. \quad (4.6)$$

Similarly, the numerator of (3.13) reduces to

$$\lambda p \pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i) \hat{\omega}(\rho_i) - \tau(s) \hat{\omega}(s)}{\pi'(\rho_i)(s - \rho_i)} = \lambda p \pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} \omega(0). \quad (4.7)$$

Substituting (4.7) and (4.8) into (3.13) yields

$$\hat{\phi}_s(s) = \frac{\frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \frac{T_s T_{\rho_i} \omega(0)}{s+a}}{1 - \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \frac{T_s T_{\rho_i} f_+(0)}{s+a}}. \quad (4.8)$$

Rewriting (4.8) as

$$\hat{\phi}_s(s) = \hat{\phi}_s(s) \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \frac{T_s T_{\rho_i} f_+(0)}{s+a} + \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \frac{T_s T_{\rho_i} \omega(0)}{s+a},$$

and then inverting the Laplace transforms in above equation yields (4.1).

The numerator of (3.14) can be rewritten as

$$D\pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} = D\pi(s) \quad (4.9)$$

thanks to (4.5) and the fact that  $\tau(s)$  is a polynomial of degree  $n$ . Substituting (4.7) and (4.9) into (3.14) yields

$$\hat{\phi}_d(s) = \hat{\phi}_d(s) \frac{\lambda p}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \frac{T_s T_{\rho_i} f_+(0)}{s+a} + \frac{1}{s+a}.$$

Finally, inverting the Laplace transforms in above equation gives (4.2).  $\square$

When  $q = 0$  (or equivalently  $n = 0$ ), we have

$$g(x) = \frac{\lambda}{D} e^{-(\frac{c}{D} + \rho)x} * T_\rho f_+(x), \quad H(u) = \frac{\lambda}{D} e^{-(\frac{c}{D} + \rho)u} * T_\rho \omega(u).$$

In this case, equations (4.1) and (4.2) simplify to equation (2.10) of Tsai and Willmot [15] and equation (17) of Gerber and Landry [6], respectively. Furthermore, they show that the corresponding equations are all defective renewal equations. We remark that (4.1) and (4.2) are also defective renewal equations when  $q > 0$  ( $n \geq 1$ ). This can be verified by showing that  $\int_0^\infty g(x) dx < 1$  or equivalently  $\hat{g}(0) < 1$ .

By (4.4) and (4.7), we can obtain

$$\hat{g}(s) = 1 - \frac{\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right]}{D(s+a)\pi(s)}. \quad (4.10)$$

Then for  $\delta > 0$ , we have

$$\hat{g}(0) = 1 - \frac{\delta \prod_{i=1}^m (\nu_i)^{n_i}}{Da \prod_{i=1}^{n+1} \rho_i} < 1. \quad (4.11)$$

Setting  $s = \rho_{n+1}(\delta)$  in (2.1) and then taking derivatives with respect to  $\delta$  yields

$$\rho'_{n+1}(0) = \frac{1}{c + \lambda q \mu_- - \lambda p \mu_+} > 0$$

due to the net profit condition (1.4). Thus, when  $\delta = 0$ , we obtain by L'Hôpital's rule

$$\begin{aligned} \hat{g}(0) &= 1 - \frac{\prod_{i=1}^m \nu_i^{n_i}}{Da \prod_{i=1}^n \rho_i} \times \lim_{\delta \rightarrow 0^+} \frac{\delta}{\rho_{n+1}(\delta)} \\ &= 1 - \frac{(c + \lambda q \mu_- - \lambda p \mu_+) \prod_{i=1}^m \nu_i^{n_i}}{Da \prod_{i=1}^n \rho_i} < 1. \end{aligned} \quad (4.12)$$

Now let  $\frac{1}{1+\beta} = \hat{g}(0) < 1$ , and  $G(x) = (1 + \beta) \int_0^x g(y) dy$ , then  $G(x)$  is a proper distribution. Define the compound geometric distribution function  $A(x) = 1 - \bar{A}(x)$  with

$$\bar{A}(x) = \sum_{k=1}^{\infty} \frac{\beta}{1+\beta} \left( \frac{1}{1+\beta} \right)^k \bar{G}^{*kn}(x),$$

where  $\bar{G}^{*kn}(x) = 1 - G^{*kn}(x)$  is the  $n$ -fold convolution of the d.f.  $G$  with itself. Rewrite (4.1) and (4.2) as

$$\begin{aligned} \phi_s(u) &= \frac{1}{1+\beta} \int_0^u \phi_s(u-x) dG(x) + H(u), \\ \phi_d(u) &= \frac{1}{1+\beta} \int_0^u \phi_d(u-x) dG(x) + e^{-au}. \end{aligned}$$

Then by Theorem 2.1 of Lin and Willmot [11],  $\phi_s(u)$  and  $\phi_d(u)$  can be expressed as

$$\begin{aligned} \phi_s(u) &= \frac{1+\beta}{\beta} \int_0^u H(u-x) dA(x) + H(u), \\ \phi_d(u) &= \frac{1+\beta}{\beta} \int_0^u e^{-a(u-x)} dA(x) + e^{-au}, \quad u \geq 0. \end{aligned}$$

## 5 Asymptotic result for the probability of ruin

The defective renewal equations (4.1) and (4.2) enable us to derive asymptotic formula for the probability of ruin when the downward jumps are heavy-tailed. In the following results of this section, only the case  $\delta = 0$  is considered.

In what follows, we write  $a(x) \sim b(x)$  if  $\lim_{x \rightarrow \infty} a(x)/b(x) = 1$ , where  $a(x)$  and  $b(x)$  are two positive functions. A distribution  $F$  on  $[0, \infty)$  is said to be subexponential, denoted by  $F \in \mathcal{S}$ , if  $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$ . A distribution  $F$  on  $[0, \infty)$  is said to be long-tailed, denoted by  $F \in \mathcal{L}$ , if  $\overline{F}(x+y) \sim \overline{F}(x)$  for any  $y > 0$ . The classes  $\mathcal{S}$  and  $\mathcal{L}$  are two well-known important classes of heavy-tailed distributions with the relation  $\mathcal{S} \subset \mathcal{L}$ . It follows from the remark after Theorem 2.5.1 of Rolski et al. [13] that for all  $\epsilon > 0$ ,  $e^{\epsilon x} \overline{F}(x) \rightarrow \infty$  as  $x \rightarrow \infty$  when  $F$  is a heavy-tailed distribution. For more heavy-tailed distributions, we refer to Embrechts et al. [4] and Rolski et al. [13].

**Lemma 2** *Let  $F_{+e}(x) = \int_0^x \overline{F}_+(y) dy / \mu_+$ , and  $r_+(x) = f_+(x) / \overline{F}_+(x)$  be the equilibrium distribution and failure rate function of  $F_+$ . If  $r_+(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and  $F_{+e} \in \mathcal{S}$ , then  $G \in \mathcal{S}$ .*

Proof. The tail function  $\overline{G}(x) = 1 - G(x)$  is given by

$$\begin{aligned} \overline{G}(x) &= \frac{\lambda p(1+\beta)}{D} \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} \int_x^\infty \int_0^z e^{-a(z-y)} T_{\rho_i} f_+(y) dy dz \\ &= \frac{\lambda p(1+\beta)}{D} \left[ \sum_{i=1}^n \frac{\tau(\rho_i)}{\pi'(\rho_i)} \int_x^\infty \int_0^z e^{-a(z-y)} T_{\rho_i} f_+(y) dy dz \right. \\ &\quad \left. + \frac{\prod_{i=1}^m \nu_i^{n_i}}{\prod_{i=1}^n \rho_i} \int_x^\infty \int_0^z e^{-a(z-y)} \overline{F}_+(y) dy dz \right]. \end{aligned} \quad (5.1)$$

We have

$$\frac{\int_x^\infty \int_0^z e^{-a(z-y)} \overline{F}_+(y) dy dz}{\overline{F}_{+e}(x)} = \frac{\frac{\mu_+}{a} \int_x^\infty \int_0^z a e^{-a(z-y)} dF_{+e}(y) dz}{\overline{F}_{+e}(x)},$$

which tends to  $\mu_+/a$  as  $x \rightarrow \infty$  by Lemma 2.5.2 of Rolski et al. [13].

From the paragraph after Proposition 2 of Embrechts and Villasenor [3], we know that  $r_+(x) \rightarrow 0$  as  $x \rightarrow \infty$  implies that  $F_+ \in \mathcal{L}$ . Note that  $r_+(x)$  is bounded when  $x$  is sufficiently large, and  $\overline{F}_+(y) / \overline{F}_+(x) \leq 1$  for  $y \geq x$ . By the dominated

convergence theorem, we obtain

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{T_{\rho_i} f_+(x)}{\overline{F}_+(x)} &= \lim_{x \rightarrow \infty} \frac{\int_x^\infty e^{-\rho_i(y-x)} f_+(y) dy}{\overline{F}_+(x)} \\ &= \lim_{x \rightarrow \infty} \int_0^\infty e^{-\rho_i s} r_+(s+x) \frac{\overline{F}_+(s+x)}{\overline{F}_+(x)} ds = 0.\end{aligned}$$

By L'Hôpital's rule, we have

$$\lim_{x \rightarrow \infty} \frac{\int_x^\infty \int_0^z e^{-a(z-y)} T_{\rho_i} f_+(y) dy dz}{\overline{F}_{+e}(x)} = \lim_{x \rightarrow \infty} \frac{\mu_+ \int_0^x e^{-ay} T_{\rho_i} f_+(x-y) dy}{\overline{F}_+(x)}. \quad (5.2)$$

Now we show that above limit is zero. By (5.2) and the fact that  $F_+ \in \mathcal{L}$ , we know that for any  $0 < \epsilon < 1$ , there exists a number  $x_0 > 0$  such that when  $x - y > x_0$

$$\left| \frac{T_{\rho_i} f_+(x-y)}{\overline{F}_+(x)} \right| = \left| \frac{T_{\rho_i} f_+(x-y)}{\overline{F}_+(x-y)} \right| \cdot \left| \frac{\overline{F}_+(x-y)}{\overline{F}_+(x)} \right| < \epsilon(1 + \epsilon). \quad (5.3)$$

Note that  $|T_{\rho_i} f_+(x-y)| < 1$ , then

$$\begin{aligned}\left| \frac{\mu_+ \int_0^x e^{-ay} T_{\rho_i} f_+(x-y) dy}{\overline{F}_+(x)} \right| &\leq \left| \frac{\mu_+ \int_0^{x-x_0} e^{-ay} T_{\rho_i} f_+(x-y) dy}{\overline{F}_+(x)} \right| \\ &\quad + \left| \frac{\mu_+ \int_{x-x_0}^x e^{-ay} T_{\rho_i} f_+(x-y) dy}{\overline{F}_+(x)} \right| \\ &< \epsilon(1 + \epsilon) \left| \mu_+ \int_0^{x-x_0} e^{-ay} dy \right| + \left| \frac{\mu_+(e^{-a(x-x_0)} - e^{-ax})}{a\overline{F}_+(x)} \right|.\end{aligned}$$

While the second term in the last line tends to zero as  $x \rightarrow \infty$  because  $F_+$  is heavy-tailed ( $F_+ \in \mathcal{L}$ ). Thus, the limit in (5.3) is zero. Finally, by (5.1), we have

$$\lim_{x \rightarrow \infty} \frac{\overline{G}(x)}{\overline{F}_{+e}(x)} = \frac{\lambda p \mu_+ (1 + \beta) \prod_{i=1}^m \nu_i^{n_i}}{Da \prod_{i=1}^n \rho_i}, \quad (5.4)$$

which further implies  $G \in \mathcal{S}$  because the class  $\mathcal{S}$  is tail-equivalence.  $\square$

We remark that many heavy-tailed distributions widely used in risk theory, such as Pareto, Burr and log-normal, satisfy the conditions in Lemma 2. Now we study the asymptotic behavior of the probability of ruin.

**Theorem 3** *Under the conditions in Lemma 2, we have*

$$\psi_s(u) \sim \psi(u) \sim \frac{\lambda p \mu_+}{c + \lambda q \mu_- - \lambda p \mu_+} \overline{F}_{+e}(u). \quad (5.5)$$

**Proof.** Setting  $w \equiv 1$  in (4.1) yields

$$\psi_s(u) = \frac{1}{1+\beta} \int_0^u \psi_s(u-x) dG(x) + H(u), \quad (5.6)$$

where

$$H(u) = \frac{\lambda p}{D} \left[ \sum_{i=1}^n \frac{\tau(\rho_i)}{\pi'(\rho_i)} \int_0^u e^{-a(u-x)} T_{\rho_i} \overline{F}_+(x) dx + \frac{\mu_+ \prod_{i=1}^m \nu_i^{n_i}}{\prod_{i=1}^n \rho_i} \int_0^u e^{-a(u-x)} \overline{F}_{+e}(x) dx \right].$$

Now we show  $H(u)$  is asymptotically proportional to  $\overline{F}_{+e}(u)$ . First, by performing integration by parts, we obtain

$$\int_0^u e^{-a(u-x)} \overline{F}_{+e}(x) dx = \frac{1}{a} \left[ 1 - \int_0^u (1 - e^{-a(u-x)}) dF_{+e}(x) - e^{-au} \right].$$

Then by Lemma 2.5.2 of Rolski et al. [13], we have

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\int_0^u e^{-a(u-x)} \overline{F}_{+e}(x) dx}{\overline{F}_{+e}(u)} &= \lim_{u \rightarrow \infty} \frac{1 - \int_0^u (1 - e^{-a(u-x)}) dF_{+e}(x)}{a \overline{F}_{+e}(u)} \\ &= \lim_{u \rightarrow \infty} \frac{e^{-au}}{a \overline{F}_{+e}(u)} = \frac{1}{a} \end{aligned} \quad (5.7)$$

thanks to  $F_{+e} \in \mathcal{S}$ . While by performing integration by parts again, we have

$$\begin{aligned} \int_0^u e^{-a(u-x)} T_{\rho_i} \overline{F}_+(x) dx &= \int_0^u e^{-a(u-x)} \int_x^\infty e^{-\rho_i(y-x)} \overline{F}_+(y) dy dx \\ &= \int_0^u \int_0^y e^{-a(u-x)} e^{-\rho_i(y-x)} \overline{F}_+(y) dx dy + \int_u^\infty \int_0^u e^{-a(u-x)} e^{-\rho_i(y-x)} \overline{F}_+(y) dx dy \\ &= \frac{1}{a + \rho_i} \left[ \int_0^u e^{-a(u-y)} \overline{F}_+(y) dy + \int_u^\infty e^{-\rho_i(y-u)} \overline{F}_+(y) dy - e^{-au} \int_0^\infty e^{-\rho_i y} \overline{F}_+(y) dy \right]. \end{aligned}$$

Similar to (5.2), we can obtain by L'Hôpital's rule and the dominated convergence theorem,

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\int_u^\infty e^{-\rho_i(y-u)} \overline{F}_+(y) dy}{\overline{F}_{+e}(u)} &= \lim_{u \rightarrow \infty} \int_0^\infty e^{-\rho_i y} \frac{\overline{F}_+(u+y)}{\overline{F}_{+e}(u)} dy \\ &= \lim_{u \rightarrow \infty} \mu_+ \int_0^\infty e^{-\rho_i y} r_+(u+y) \frac{\overline{F}_{+e}(u+y)}{\overline{F}_{+e}(u)} dy = 0. \end{aligned} \quad (5.8)$$

While by Lemma 2.5.2 of Rolski et al. [13] again, we have

$$\begin{aligned} & \lim_{u \rightarrow \infty} \frac{\int_0^u e^{-a(u-y)} \overline{F}_+(y) dy}{\overline{F}_{+e}(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\mu_+ \left[ \left[ 1 - \int_0^u (1 - e^{-a(u-y)}) dF_{+e}(y) \right] - \overline{F}_{+e}(u) \right]}{\overline{F}_{+e}(u)} = 0. \end{aligned} \quad (5.9)$$

Combining (5.8) and (5.9) yields

$$\lim_{u \rightarrow \infty} \frac{\int_0^u e^{-a(u-x)} T_{\rho_i} \overline{F}_+(x) dx}{\overline{F}_{+e}(u)} = 0,$$

which together with (5.7) gives

$$\lim_{u \rightarrow \infty} \frac{H(u)}{\overline{F}_{+e}(u)} = \frac{\lambda p \mu_+ \prod_{i=1}^m \nu_i^{n_i}}{Da \prod_{i=1}^n \rho_i}. \quad (5.10)$$

By Theorem 3.1 of [21], Lemma 2, (4.12) and (5.10), we obtain

$$\psi_s(u) \sim \frac{H(u)}{1 - \frac{1}{1+\beta}} \sim \frac{\lambda p \mu_+}{c + \lambda q \mu_- - \lambda p \mu_+} \overline{F}_{+e}(u).$$

Finally,

$$\psi(u) \sim \frac{H(u) + e^{-au}}{1 - \frac{1}{1+\beta}} \sim \frac{\lambda p \mu_+}{c + \lambda q \mu_- - \lambda p \mu_+} \overline{F}_{+e}(u).$$

This completes the proof.  $\square$

**Remark:** Formula (5.5) shows that the heavy-tailed downward jumps dominate the diffusion perturbation in the sense of ruin. When  $p = 1, q = 0$ , (5.6) recovers the corresponding result for the classical perturbed compound Poisson risk model without upward jumps, see [4, 19].

## 6 Explicit expressions for the discounted penalty functions

In this section, we derive some explicit expressions for the discounted penalty functions. Like the upward jump density, we assume the downward jump density also has a rational Laplace transform, i.e.

$$\hat{f}_+(s) = \frac{f_{l-1}(s)}{f_l(s)}, \quad l \in \mathbb{N}^+, \quad (6.1)$$

where  $f_{l-1}(s)$  is a polynomial of degree  $l - 1$  or less, while  $f_l(s)$  satisfying  $f_l(0) = f_{l-1}(0)$  is a polynomial of degree  $l$  with only negative roots .

Now we start from the Laplace transforms (3.13) and (3.14). Plugging (4.8) into (3.13) gives

$$\begin{aligned}\hat{\phi}_s(s) &= \frac{\lambda p \pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} \omega(0)}{\tau(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda \hat{f}(s) \right]} \\ &= \frac{\lambda p f_l(s) \pi(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} \omega(0)}{\tau(s) f_l(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda q \hat{f}_-(-s) \right] + \lambda p \tau(s) f_{l-1}(s)}.\end{aligned}\quad (6.2)$$

While plugging (4.9) into (3.14) gives

$$\hat{\phi}_d(s) = \frac{D \pi(s) f_l(s)}{\tau(s) f_l(s) \left[ Ds^2 + cs - (\lambda + \delta) + \lambda q \hat{f}_-(-s) \right] + \lambda p \tau(s) f_{l-1}(s)}.\quad (6.3)$$

The common denominator of (6.2) and (6.3), denoted by  $D_{n+l+2}(s)$ , is a polynomial of degree  $n + l + 2$  with leading coefficient  $D$ . The equation  $D_{n+l+2}(s) = 0$  has exactly  $n + l + 2$  roots on the complex plane, with all complex roots being in conjugate pairs. By Lemma 1, we know that  $\rho_1, \rho_2, \dots, \rho_{n+1}$  are  $n + 1$  roots, then  $D_{n+l+2}(s)$  can be expressed as

$$D_{n+l+2}(s) = D \pi(s) \prod_{j=1}^{l+1} (s + R_j).\quad (6.4)$$

We remark that all  $R_j$ 's have a positive real part, since otherwise,  $-R_j$  would also be a root of the generalized Lundberg equation (2.1), which results in a contradiction to Lemma 1.

By (6.4), formulas (6.2) and (6.3) can be rewritten as

$$\hat{\phi}_s(s) = \frac{\lambda p f_l(s) \sum_{i=1}^{n+1} \frac{\tau(\rho_i)}{\pi'(\rho_i)} T_s T_{\rho_i} \omega(0)}{D \prod_{j=1}^{l+1} (s + R_j)},\quad (6.5)$$

$$\hat{\phi}_d(s) = \frac{f_l(s)}{\prod_{j=1}^{l+1} (s + R_j)}.\quad (6.6)$$

If all  $R_j$ 's are distinct, performing partial fraction leads to

$$\frac{\lambda p f_l(s) \frac{\tau(\rho_i)}{\pi'(\rho_i)}}{D \prod_{j=1}^{l+1} (s + R_j)} = \sum_{j=1}^{l+1} \frac{a_{ij}}{s + R_j}, \quad \frac{f_l(s)}{\prod_{j=1}^{l+1} (s + R_j)} = \sum_{j=1}^{l+1} \frac{b_j}{s + R_j},$$

where

$$a_{ij} = \frac{\lambda p f_l(-R_j) \frac{\tau(\rho_i)}{\pi'(\rho_i)}}{D \prod_{k=1, k \neq j}^{l+1} (R_k - R_j)}, \quad b_j = \frac{f_l(-R_j)}{\prod_{k=1, k \neq j}^{l+1} (R_k - R_j)}. \quad (6.7)$$

Thus, (6.5) and (6.6) becomes

$$\hat{\phi}_s(s) = \sum_{i=1}^{n+1} \sum_{j=1}^{l+1} \frac{a_{ij}}{s + R_j} T_s T_{\rho_i} \omega(0), \quad \hat{\phi}_d(s) = \sum_{j=1}^{l+1} \frac{b_j}{s + R_j}.$$

Finally, inverting the Laplace transforms in above formulas yields the following expressions for  $\phi_s(u)$  and  $\phi_d(u)$ .

**Theorem 4** *If the Laplace transform  $\hat{f}_+(s)$  has the form (6.1) and the roots  $-R_1, -R_2, \dots, -R_{l+1}$  are all distinct, then the discounted penalty functions  $\phi_s(u)$  and  $\phi_d(u)$  can be expressed as*

$$\phi_s(u) = \sum_{i=1}^{n+1} \sum_{j=1}^{l+1} a_{ij} e^{-R_j u} * T_{\rho_i} \omega(u), \quad (6.8)$$

$$\phi_d(u) = \sum_{j=1}^{l+1} b_j e^{-R_j u}, \quad u \geq 0. \quad (6.9)$$

Next, we give a numerical example to illustrate the solution procedure to calculate the discounted penalty functions. Assume in the sequel that

$$\hat{f}_+(s) = \frac{\alpha}{s + \alpha}, \quad \hat{f}_-(s) = \frac{p_1 \nu_1}{s + \nu_1} + \frac{p_2 \nu_2}{s + \nu_2},$$

where  $\alpha, \nu_1, \nu_2 > 0$ ,  $0 < p_1, p_2 < 1$ ,  $p_1 + p_2 = 1$ . Then the downward jumps are exponentially distributed with density  $f_+(x) = \alpha e^{-\alpha x}$ , while the upward jump density  $f_-(x) = p_1 \nu_1 e^{-\nu_1 x} + p_2 \nu_2 e^{-\nu_2 x}$  is a mixture of exponentials. The following generalized Lundberg equation

$$Ds^2 + cs - (\lambda + \delta) + \lambda p \frac{\alpha}{s + \alpha} + \lambda q \left( \frac{p_1 \nu_1}{\nu_1 - s} + \frac{p_2 \nu_2}{\nu_2 - s} \right) = 0 \quad (6.10)$$

has five roots, say  $\rho_1, \rho_2, \rho_3 > 0$ ,  $-R_1, -R_2 < 0$ .

Set  $c = 2$ ,  $D = 1$ ,  $\delta = 0.3$ ,  $\lambda = 1$ ,  $p = 0.6$ ,  $q = 0.4$ ,  $p_1 = 0.2$ ,  $p_2 = 0.8$ ,  $\nu_1 = 0.4$ ,  $\nu_2 = 0.8$ ,  $\alpha = 0.3$ . It is easy to check that the net profit condition (1.4) holds. After solving equation (6.10), we obtain  $\rho_1 = 0.17095$ ,  $\rho_2 = 0.4431$ ,  $\rho_3 = 0.95805$ ,  $R_1 = 0.15783$ ,  $R_2 = 2.51429$ . By (6.7), we obtain  $a_{11} = 0.02435$ ,  $a_{12} = 0.37920$ ,

$a_{21} = 0.00397$ ,  $a_{22} = 0.06190$ ,  $a_{31} = 0.00788$ ,  $a_{32} = 0.12269$ .  $b_1 = 0.06033$ ,  $b_2 = 0.93967$ . By formula (6.9), we obtain

$$\phi_d(u) = 0.06033e^{-0.15783u} + 0.93967e^{-2.51429u}.$$

Now we consider the tail of the distribution of the discounted deficit at ruin when ruin is caused by a downward jump,  $F_s(x|u) := E[e^{-\delta T} I(|U(T)| > x, T < \infty) | U(0) = u]$ , which can be obtained by setting  $w(x_1, x_2) = I(x_2 > x)$ . In this case,  $\omega(u) = e^{-\alpha(u+x)}$ . Accordingly, formula (6.8) becomes

$$F_s(x|u) = \sum_{i=1}^{n+1} \sum_{j=1}^{l+1} \frac{a_{ij} e^{-R_j u - \alpha x}}{(\rho_i + \alpha)(\alpha - R_j)},$$

which gives

$$F_s(x|u) = 0.445298e^{-0.15783u-0.3x} - 0.445298e^{-2.51429u-0.3x}, \quad u, x \geq 0.$$

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