Erlang risk models and finite time ruin problems

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Abstract

We consider the joint density of the time of ruin and deficit at ruin in the Erlang(n) risk model. We give a general formula for this joint density and illustrate how the components of this formula can be found in the special case when $n = 2$. We then show how the formula can be implemented numerically for a general value of n. We also discuss how the ideas extend to the generalised Erlang(n) risk model.

1 Introduction

The time of ruin in the Erlang(2) risk model was first studied by Dickson and Hipp (2001). Since then, many of the results in that paper have been generalised, most notably by Li and Garrido (2004) who consider the Erlang(n) risk model, and by Gerber and Shiu (2005) who consider the generalised Erlang(n) risk model. A common feature of these three papers is that the analysis is based on so-called Gerber-Shiu or discounted penalty functions. Special cases of such functions are Laplace transforms, and one approach to finding quantities such as the density of the time of ruin is to invert these transforms. This has been done in the classical risk model by Dickson and Willmot (2005) and Landriault and Willmot (2009). Such an approach is possible in the classical risk model as Lundberg's fundamental equation has a unique positive solution. However, in each of the models studied by Li and Garrido (2004) and by Gerber and Shiu (2005), Lundberg's fundamental equation has n solutions, making the prospect of transform inversion daunting to say the least.

In Dickson and Li (2010) we consider a Laplace transform inversion approach and derive some explicit formulae for the density of the time of ruin and the joint density of the time of ruin and deficit at ruin in the Erlang(2) risk model. This approach is fruitful, particularly when the initial surplus is 0, but generalisation to the Erlang(n) risk model seems difficult, and suggests that transform inversion may not be the best approach to obtain results for the Erlang(n) risk model.

In this paper we consider the joint distribution of the time of ruin and the deficit at ruin in the Erlang(n) risk model. We use probabilistic arguments to obtain a formula for this joint density in Section 3, where we also discuss an important class of distributions We then consider the special case when $n = 2$ in Section 4 and show how formulae can be obtained for the components of our formula for the joint density function. Finally, in Section 5 we discuss how our formulae can be applied for a general value of n using standard numerical techniques.

2 Notation

In the Erlang(n) risk model, the distribution of the claim inter-arrival times is Erlang(n, β), where β is the scale parameter. For our analysis, we view the claims as being paid at every nth occurrence of an underlying Poisson process which has parameter β. Let

$$p_m(t) = e^{-\beta t}\frac{(\beta t)^m}{m!}$$

for $m = 0, 1, 2, \ldots$, with $p_m(t) = 0$ for negative m.

We model the individual claim amounts as a sequence of independent and identically distributed random variables with distribution function F and density function f. Let $S(t)$ denote aggregate claims over $(0, t)$. There are r claims in $(0, t)$ if the number of occurrences in the underlying Poisson process is one of $rn, rn + 1, \ldots, rn + n - 1$. Thus

$$\Pr[S(t) \leq x] = \sum_{r=0}^{\infty}\left(\sum_{j=0}^{n-1} p_{rn+j}(t)\right) F^{r*}(x),$$

where F^{r*} is the r-fold convolution of F, with $F^{0*}(x) = 1$ for $x \geq 0$. It is useful to define a function $g_{n,j}$ for $j = 0, 1, \ldots, n-1$ as

$$g_{n,j}(x, t) = \sum_{r=1}^{\infty} p_{rn+j}(t) f^{r*}(x),$$

where $f^{r*}(x) = \frac{d}{dx}F^{r*}(x)$. We can think of this as the density associated with an aggregate claim amount of x in $(0, t)$ where the number of Poisson occurrences is $rn + j$, for $r = 1, 2, 3, \ldots$, so that

$$\frac{\partial}{\partial x}\Pr[S(t) \leq x] = \sum_{j=0}^{n-1} g_{n,j}(x, t).$$

Now define the surplus at time $t \geq 0$ as

$$U(t) = u + ct - S(t),$$

where u is the initial surplus and c is the rate of premium income per unit time. We define the finite time ruin probability as

$$\psi_n(u,t) = \Pr[U(s) < 0 \text{ for some } s,\ 0 < s \leq t],$$

with $\bar{\psi}_n(u,t) = 1-\psi_n(u,t)$. Throughout this paper we will apply ruin related quantities for modified Erlang(n) surplus processes, the modification being that the distribution of the time to the first claim is Erlang(i,β), where i will range from 1 to $n-1$. For each modified surplus process we use the same notation for quantities that are defined for the unmodified process, but use a subscript i. Thus, for example, $\psi_1(u,t)$ is the finite time ruin probability for the modified surplus process under which the time to the first claim is exponentially distributed with mean $1/\beta$.

We define the (defective) density of the time of ruin as

$$w_n(u,t) = \frac{\partial}{\partial t}\psi_n(u,t),$$

and the (defective) joint density of the time (t) of ruin and the deficit (y) at ruin as $w_n(u,y,t)$. Throughout we will use the notation a^* to denote the Laplace transform of a function a.

3 Main formulae

We can adapt Prahbu's (1961) formula for the finite time non-ruin probability to our model as

$$\bar{\psi}_n(u,t) = \Pr[S(t) \leq u+ct] - c\sum_{j=0}^{n-1}\int_0^t g_{n,j}(u+cs,s)\,\bar{\psi}_{n-j}(0,t-s)ds. \quad (3.1)$$

The interpretation of this formula is virtually the same as the interpretation of Prahbu's formula. For non-ruin at time t, we require that $u+ct-S(t) \geq 0$, the probability of which is given by the first term on the right hand side of formula (3.1). The integral terms adjust for realisations of the surplus process that have fallen below 0 before time t. If time s is the last time at which there is an upcrossing of the surplus process through 0, then the number of occurrences of the underlying Poisson process since the previous claim is one of $0, 1, \ldots, n-1$. Suppose the number is j, where $0 \leq j \leq n-1$. Then by the memoryless property of the exponential distribution, the distribution of the time to the next Poisson occurrence is exponential with mean $1/\beta$, and so the distribution of the time to the next claim is Erlang$(n-j,\beta)$. Hence the probability of non-ruin in the time interval from s to t is $\bar{\psi}_{n-j}(0,t-s)$.

As with most recent papers on the time of ruin (e.g. Drekic and Willmot (2003), Dickson and Willmot (2005) and Landriault and Willmot (2009)) we choose to work with the density of the time of ruin. It is straightforward to show by differentiating formula (3.1) that the density of the time of ruin is

$$\begin{aligned} w_n(u,t) &= \beta p_{n-1}(t)\bar{F}(u+ct) \\ &\quad +\beta\sum_{r=1}^{\infty} p_{(r+1)n-1}(t)\int_0^{u+ct} f^{r*}(x)\bar{F}(u+ct-x)dx \\ &\quad -c\sum_{j=0}^{n-1}\int_0^t g_{n,j}(u+cs,s)\,w_{n-j}(0,t-s)ds. \end{aligned} \tag{3.2}$$

Formula (3.2) can also be written down by considering $w_n(u,t)dt$ as the probability that ruin occurs in the time interval $(t,t+dt)$. For ruin to occur in this time interval we require a claim to occur and the surplus to be non-negative up to time t. Thus, up to time t, we require $rn-1$ occurrences of the Poisson process (for $r=1,2,\ldots$) with a further occurrence in $(t,t+dt)$. If the surplus at time t is $u+ct-x\geq 0$ then we require a claim greater than $u+ct-x$. The first two terms in formula (3.2) give the density associated with these, with the first covering the case $x=0$. The final term adjusts to allow for the surplus having crossed upwards through 0 for the last time at time $s<t$, and the surplus again being below 0 in $(t,t+dt)$.

This interpretation leads to a formula for the joint density of the time of ruin and deficit at ruin. We have

$$\begin{aligned} w_n(u,y,t) &= \beta p_{n-1}(t)f(u+ct+y) \\ &\quad +\beta\sum_{r=1}^{\infty} p_{(r+1)n-1}(t)\int_0^{u+ct} f^{r*}(x)f(u+ct-x+y)dx \\ &\quad -c\sum_{j=0}^{n-1}\int_0^t g_{n,j}(u+cs,s)\,w_{n-j}(0,y,t-s)ds. \end{aligned} \tag{3.3}$$

Formula (3.3) generalises formula (9) of Dickson (2007) for the joint density of the time of ruin and the deficit at ruin in the classical risk model. Indeed, that formula can be obtained by setting $n=1$ in formula (3.3). To obtain formula (3.3), the $\bar{F}$ terms in formula (3.2) are replaced by densities as we now require a claim of size $u + ct - x + y$ rather than a claim exceeding $u + ct - x$. Similarly the final term is an adjustment term to allow for the surplus having been below 0 before time t, with the surplus being in $(-y, -y - dy)$ in $(t, t + dt)$. For example, if s is the last time at which there was an upcrossing of the surplus process through 0, there will have been r claims if the number of Poisson occurrences is $rn+j$ (for $j = 0, 1, \ldots, n-1$). If the number of Poisson occurrences is $rn + j$, then a further $n - j$ must occur until the next claim. The density associated with ruin with a deficit of y at time t is thus $w_{n-j}(0, y, t - s)$.

For a given claim size density f, we may well be able to write down a formula for the first two terms in formulae (3.2) and (3.3). To evaluate the integral terms in these formulae requires expressions for $w_{n-j}(0, t)$ and $w_{n-j}(0, y, t)$ for $j = 0, 1, \ldots, n - 1$. In Section 4 we illustrate how such formulae can be obtained for particular claim size distributions in the special case when $n = 2$. However, we shall not try to find formulae for either $w_n(u, t)$ or $w_n(u, y, t)$. Rather, once we know the required functions when $u = 0$ we have sufficient information to find either $w_n(u, t)$ or $w_n(u, y, t)$ by numerical integration. Similarly, for the general case $n \geq 2$, our approach will be to use numerical integration to compute either $w_n(u, t)$ or $w_n(u, y, t)$. In Section 5 we explain how $w_{n-j}(0, t)$ and $w_{n-j}(0, y, t)$ can be computed numerically to provide inputs for numerical integration in formula (3.2) or (3.3).

Formula (3.3) has a convenient form when the individual claim amount density satisfies a particular factorisation, and we discuss this next. We then conclude this section by discussing how the above ideas extend to a generalised Erlang(n) risk model.

3.1 An important special case

We now consider the important special case when the individual claim amount density is such that

$$f(x + y) = \sum_{i=1}^{m} \eta_i(x)\tau_i(y) \tag{3.4}$$

where $\{\eta_i\}_{i=1}^m$ are functions and $\{\tau_i\}_{i=1}^m$ are density functions. This factorisation was introduced by Willmot (2007) who shows that many well known density functions have this form. When this factorisation applies we find that

$$w_j(0, y, t) = \sum_{i=1}^{m} h_{j,i}(0, t)\tau_i(y) \tag{3.5}$$

for $j = 1, 2, \dots, n$, and as an immediate consequence of formula (3.5), equation (3.3) can be written in the very useful form

$$w_n(u, y, t) = \sum_{i=1}^{m} h_{n,i}(u, t)\tau_i(y)$$

where

$$\begin{aligned} h_{n,i}(u,t) &= \beta\, p_{n-1}(t)\, \eta_i(u+ct) \\ &\quad +\beta \sum_{r=1}^{\infty} p_{(r+1)n-1}(t) \int_0^{u+ct} f^{r*}(x)\eta_i(u+ct-x)dx \\ &\quad -c \sum_{j=0}^{n-1} \int_0^t g_{n,j}(u+cs, s)\, h_{n-j,i}(0, t-s)ds. \end{aligned} \tag{3.6}$$

In order to show formula (3.5) we use Laplace transforms. Let

$$\phi_n(u) = E\left[e^{-\delta T - \kappa Y} I(T < \infty)\right]$$

denote the bivariate Laplace transform of the time of ruin, T, and the deficit at ruin, $Y = |U(T)|$, where I is the indicator function. For $j = 1, 2, \dots, n-1$, let $\phi_j(u)$ denote the corresponding bivariate Laplace transform of the time of ruin and the deficit at ruin for the modified Erlang(n) surplus process in which the distribution of the time to the first claim is Erlang(j, β).

It follows from Li and Garrido (2004) that

$$\phi_n(0) = \int_0^\infty \int_0^\infty e^{-\delta t - \kappa y} w_n(0, y, t)dydt = \frac{\beta^n}{c^n} T_{r_n} T_{r_{n-1}} \dots T_{r_1}\omega(0), \tag{3.7}$$

where $r_1, r_2, \dots, r_n$ are the n roots of Lundberg's fundamental equation for the Erlang(n) risk model,

$$a_n(s) := \left(\frac{\beta+\delta}{\beta} - \frac{c}{\beta}s\right)^n = f^*(s), \tag{3.8}$$

for a function A,

$$T_r\, A(x) = \int_x^\infty e^{-r(u-x)} A(u)du$$

as in Dickson and Hipp (2001), and

$$\omega(u) = \int_0^\infty e^{-\kappa y} f(u+y)dy.$$

With the decomposition for f given by formula (3.4) we see that

$$\omega(u) = \int_0^\infty e^{-\kappa y} f(u+y)dy = \sum_{i=1}^m \eta_i(u)\tau_i^*(\kappa)$$

and by (3.7)

$$\begin{aligned} \phi_n(0) &= \frac{\beta^n}{c^n} T_{r_n} T_{r_{n-1}} \dots T_{r_1} \omega(0) \\ &= \frac{\beta^n}{c^n} \sum_{i=1}^m \tau_i^*(\kappa) \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^n r_i x_i} \eta_i \left(\sum_{i=1}^n x_i \right) dx_1 dx_2 \dots dx_n. \end{aligned} \tag{3.9}$$

Inversion of (3.9) gives

$$w_n(0,y,t) = \sum_{i=1}^m h_{n,i}(0,t)\tau_i(y),$$

where

$$\begin{aligned} &\int_0^\infty e^{-\delta t} h_{n,i}(0,t)dt \\ = &\frac{\beta^n}{c^n} \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^n r_i x_i} \eta_i \left(\sum_{i=1}^n x_i \right) dx_1 dx_2 \dots dx_n. \end{aligned}$$

To find $w_j(0,y,t)$ we can adapt equations (5.5) and (5.8) of Gerber and Shiu (2005) to our model and write

$$\beta\phi_{j+1}(u) = (\beta+\delta)\phi_j(u) - c\phi_j'(u), \tag{3.10}$$

for $j = 0, 1, 2, \dots, n-2$, with $\phi_0(u) = \phi_n(u)$. It then follows that

$$\phi_j(u) = \left[(1+\delta/\beta)\mathcal{I} - (c/\beta)\mathcal{D}\right]^j \phi_n(u),$$

for $j = 0, 1, 2, \dots, n-2$, where $\mathcal{I}$ is the identity operator and $\mathcal{D}$ is the differentiation operator.

In particular, for $j = 1, 2, \dots, n-1$,

$$\phi_j(0) = \sum_{i=0}^j b_{j,i}(\delta)\phi_n^{(i)}(0) \tag{3.11}$$

where for $i = 0, 1, 2, \ldots, j$,

$$b_{j,i}(\delta) = \binom{j}{i}(-1)^i \left(\frac{c}{\beta}\right)^i \left(\frac{\beta+\delta}{\beta}\right)^{j-i}.$$

Now we aim to find $\phi_n^{(i)}(0)$ for $i = 0, 1, \ldots, j$. It follows from equation (10.9) of Gerber and Shiu (2005) that for $k = 1, 2, \ldots, n$,

$$(-1)^k a_n[r_1, r_2, \ldots, r_k, \mathcal{D}]\phi_n(u) = \phi_n * (T_{r_k} \cdots T_{r_1} f)(u) + T_{r_k} \cdots T_{r_1}\omega(u), \tag{3.12}$$

where $a_n[r_1, r_2, \ldots, r_k, s]$ is the kth divided difference of $a_n(s)$ (where a_n is given by formula (3.8)), with the following recursive definition (Gerber and Shiu, 2005, p.53):

$$\begin{aligned} a_n[r_1, s] &= \frac{a_n(s) - a_n(r_1)}{s - r_1}, \\ a_n[r_1, r_2, s] &= \frac{a_n[r_1, s] - a_n[r_1, r_2]}{s - r_2}, \\ &\vdots \\ a_n[r_1, r_2, \ldots, r_k, s] &= \frac{a_n[r_1, r_2, \ldots, r_{k-1}, s] - a_n[r_1, r_2, \ldots, r_{k-1}, r_k]}{s - r_k}. \end{aligned}$$

As $a_n(s)$ is a polynomial in s of degree n, the arguments in Section 10 of Gerber and Shiu (2005) tell us that $a_n[r_1, r_2, \ldots, r_k, s]$ is a polynomial of degree $n - k$ and so we can write

$$(-1)^k a_n[r_1, r_2, \ldots, r_k, s] = \sum_{i=0}^{n-k} d_{i,n-k}(\delta, r_1, r_2, \ldots, r_k)s^i.$$

Setting $u = 0$ and $k = n, n-1, \ldots, n-j$ in equation (3.12) yields the following $j+1$ linear equations:

$$\begin{aligned} d_{0,0}(\delta, r_1, r_2, \ldots, r_n)\phi_n(0) &= T_{r_n} \cdots T_{r_1}\omega(0), \\ d_{0,1}(\delta, r_1, r_2, \ldots, r_{n-1})\phi_n(0) + d_{1,1}(\delta, r_1, r_2, \ldots, r_{n-1})\phi_n'(0) &= T_{r_{n-1}} \cdots T_{r_1}\omega(0), \\ &\vdots \\ \sum_{i=0}^{j} d_{i,j}(\delta, r_1, r_2, \ldots, r_{n-j})\phi_n^{(i)}(0) &= T_{r_{n-j}} \cdots T_{r_1}\omega(0). \end{aligned}$$

The solutions to this system of equations above can be expressed as

$$\begin{aligned} \phi_n(0) &= \frac{\beta^n}{c^n} T_{r_n} \cdots T_{r_1}\omega(0), \\ \phi_n^{(i)}(0) &= \sum_{k=n-i}^{n} g_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) T_{r_k} \cdots T_{r_1}\omega(0), \end{aligned} \tag{3.13}$$

for $i = 1, 2, \ldots, j$.

Substituting (3.13) into (3.11) and changing the order of double summation gives

$$\phi_j(0) = \sum_{k=n-j}^{n} v_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) T_{r_k} \cdots T_{r_1} \omega(0),$$

where $v_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) = g_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) \sum_{i=n-k}^{j} b_{j,i}(\delta)$.

Finally under the assumption that

$$\omega(u) = \sum_{i=1}^{m} \eta_i(u) \tau_i^*(\kappa)$$

we have for $j = 1, 2, \ldots, n-1$,

$$\begin{aligned} \phi_j(0) &= \int_0^\infty \int_0^\infty e^{-\delta t - \kappa y} w_j(0, y, t) dy dt \\ &= \sum_{i=1}^{m} \tau_i^*(\kappa) \sum_{k=n-j}^{n} v_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) T_{r_k} \cdots T_{r_1} \eta_i(0). \end{aligned}$$

Inversion gives the result we set out to show, namely

$$w_j(0, y, t) = \sum_{i=1}^{m} h_{j,i}(0, t) \tau_i(y) \tag{3.14}$$

for $j = 1, 2, \ldots, n-1$, where $h_{j,i}(0, t)$ is such that

$$\int_0^\infty e^{-\delta t} h_{j,i}(0, t) dt = \sum_{k=n-j}^{n} v_{j,k}(\delta, r_1, r_2, \ldots, r_{n-1}) T_{r_k} \cdots T_{r_1} \eta_i(0).$$

In Section 4 we illustrate how these transforms can be inverted when $n = 2$. For the case $n > 2$, we explain in Section 5 how to apply formula (3.6) using numerical techniques.

3.2 Extension to the generalised Erlang(n) model

In the Erlang(n) risk model, we model the times between claims as the sum of n independent and identically distributed exponential random variables. In the generalised Erlang(n) risk model, we model the time between claims as the sum of n independent exponential random variables with different means. The arguments used to obtain formulae (3.2) and (3.3) also hold for the generalised Erlang(n) risk model as we are still dealing with a sum of exponentially distributed random variables.

We will not give full details here, but simply illustrate via an example how results for the Erlang(n) risk model can be adapted. Consider formula (3.2) when $n = 2$:

$$\begin{aligned} w_2(u,t) &= \beta p_1(t)\bar{F}(u+ct) + \beta \sum_{r=1}^{\infty} p_{2r+1}(t) \int_0^{u+ct} f^{r*}(x)\bar{F}(u+ct-x)dx \\ &\quad -c\sum_{j=0}^{1}\int_0^t \sum_{r=1}^{\infty} p_{2r+j}(s) f^{r*}(u+cs) w_{2-j}(0,t-s)ds. \end{aligned} \tag{3.15}$$

How does this formula change if we model the claim inter-arrival times as $V = V_1+V_2$ where V_1 and V_2 are independent random variables with parameters λ_1 and λ_2 respectively? If we think of the claim inter-arrival times as comprising of two phases, in the first line of (3.15) we require an odd number of phases up to time t, followed by a claim in $(t, t+dt)$. Thus, $\beta p_1(t)$ is replaced by

$$\lambda_2 \int_0^t \lambda_1 e^{-\lambda_1 s} e^{-\lambda_2(t-s)} ds = \frac{\lambda_1\lambda_2\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)}{\lambda_1 - \lambda_2}.$$

To find the replacement term for $\beta p_{2r+1}(t)$, we first note that the convolution of an Erlang(m, λ_1) density and an Erlang(n, λ_2) density can be written as

$$b_{m,n}(t) = \frac{\lambda_1^m \lambda_2^n e^{-\lambda_1 t} t^{m+n-1}}{\Gamma(m+n)}\, {}_1F_1(n, m+n, (\lambda_1-\lambda_2)t)$$

where ${}_1F_1$ is the confluent hypergeometric function. If the $(2r+2)$th claim in the generalised Erlang(2) risk model occurs in $(t, t+dt)$, then we require $2r+1$ phases up to time t, $r+1$ of which have a length that is exponentially distributed with parameter λ_1. Thus, the term $\beta p_{2r+1}(t)$ is replaced by

$$\lambda_2 \int_0^t b_{r+1,r}(s) e^{-\lambda_2(t-s)} ds = \frac{\lambda_2 e^{-\lambda_1 t}(\lambda_1 t)^{r+1}(\lambda_2 t)^r}{\Gamma(2r+2)}\, {}_1F_1(r+1, 2r+2, (\lambda_1-\lambda_2)t).$$

Similarly, the $p_{2r}(s)$ terms in the second line of (3.15) are replaced by

$$\frac{e^{-\lambda_1 s}(\lambda_1 s)^r(\lambda_2 s)^r}{\Gamma(2r+1)}\, {}_1F_1(r, 2r+1, (\lambda_1-\lambda_2)s).$$

Otherwise formula (3.15) is unchanged, except, of course, that the w functions now apply to the generalised Erlang(2) risk model.

These arguments carry through to the case $n > 2$, although the formulae become a little bit more unwieldy.

4 Solutions when $n = 2$

In the special case when $n = 2$, it is sometimes possible to find formulae for the components of formula (3.3). Applying formulae (3.10) we have

$$\phi_1(0) = \frac{\beta+\delta}{\beta}\phi_2(0) - \frac{c}{\beta}\phi_2'(0).$$

To apply this, we note that equation (A.3) of Sun (2005) gives us

$$c^2 r_i \phi_2(0) + c^2\phi_2'(0) - 2(\beta+\delta)c\phi_2(0) + \beta^2\omega^*(r_i) = 0,$$

for $i = 1, 2$, where r_1 and r_2 are the two positive solutions of Lundberg's fundamental equation and

$$\omega^*(r_i) = \int_0^\infty e^{-r_i u}\int_0^\infty e^{-\kappa x} f(x+u)dxdu. \tag{4.1}$$

Thus

$$\phi_1(0) = \frac{\phi_2(0)}{\beta}(cr_i - \beta - \delta) + \frac{\beta}{c}\omega^*(r_i)$$

and from equations (3) and (4) of Dickson and Li (2010) we know that

$$cr_i - \beta - \delta = (-1)^i \beta q^*(r_i)$$

where $q^*(r_i)^2 = f^*(r_i)$. Thus

$$\begin{aligned}\phi_1(0) &= \frac{\beta}{c}\omega^*(r_1) - \phi_2(0)q^*(r_1) &(4.2)\\ &= \frac{\beta}{c}\omega^*(r_2) + \phi_2(0)q^*(r_2) &(4.3)\\ &= \frac{\beta}{2c}\left(\omega^*(r_2)+\omega^*(r_1)\right) + \frac{\phi_2(0)}{2}\left(q^*(r_2) - q^*(r_1)\right). &(4.4)\end{aligned}$$

Note that $\omega^*(r_1)$, $\omega^*(r_2)$ and $\phi_2(0)$ are all bivariate transforms, and these can be inverted for certain forms for f. Depending on the form of q^*, we may be able to apply either (4.2) or (4.3) with ease, as illustrated in Section 4.1. When this is not the case we can apply (4.4), as illustrated in Section 4.2.

4.1 Erlang(2) claims

In this section we consider the situation when the individual claim amount distribution is Erlang(2,α) so that $f(x) = \alpha^2 x e^{-\alpha x}$. From Dickson and Li

(2010) we know that in this case we have

$$\begin{aligned}\phi_2(0) &= \frac{\beta^2}{c^2}\left(\frac{\alpha}{(\alpha+r_1)^2(\alpha+r_2)}+\frac{\alpha}{(\alpha+r_1)(\alpha+r_2)^2}\right)\frac{\alpha}{\alpha+\kappa}\\ &\quad +\frac{\beta^2}{c^2}\frac{1}{(\alpha+r_1)(\alpha+r_2)}\left(\frac{\alpha}{\alpha+\kappa}\right)^2.\end{aligned}$$

Also, as $f^*(r_1) = (\alpha/(\alpha+r_1))^2$, we know that $q^*(r_1) = \alpha/(\alpha+r_1)$. To find $\omega^*(r_1)$ we apply formula (4.1) noting that f satisfies the factorisation given by equation (3.4), so that

$$\omega^*(r_1) = \frac{\alpha^2}{(\alpha+r_1)^2(\alpha+\kappa)}+\frac{\alpha^2}{(\alpha+r_1)(\alpha+\kappa)^2}.$$

Thus,

$$\begin{aligned}\phi_1(0) &= \frac{\beta}{c}\left(\frac{\alpha^2}{(\alpha+r_1)^2(\alpha+\kappa)}+\frac{\alpha^2}{(\alpha+r_1)(\alpha+\kappa)^2}\right)\\ &\quad -\frac{\beta^2}{c^2}\left(\frac{\alpha^2}{(\alpha+r_1)^3(\alpha+r_2)}+\frac{\alpha^2}{(\alpha+r_1)^2(\alpha+r_2)^2}\right)\frac{\alpha}{\alpha+\kappa}\\ &\quad +\frac{\beta^2}{c^2}\frac{\alpha}{(\alpha+r_1)^2(\alpha+r_2)}\left(\frac{\alpha}{\alpha+\kappa}\right)^2\\ &= \frac{\alpha}{\alpha+\kappa}\left(\frac{\beta}{c}\frac{\alpha}{(\alpha+r_1)^2}-\frac{\beta^2}{c^2}\frac{\alpha^2}{(\alpha+r_1)^3(\alpha+r_2)}-\frac{\beta^2}{c^2}\frac{\alpha^2}{(\alpha+r_1)^2(\alpha+r_2)^2}\right)\\ &\quad +\left(\frac{\alpha}{\alpha+\kappa}\right)^2\left(\frac{\beta}{c}\frac{1}{\alpha+r_1}+\frac{\beta^2}{c^2}\frac{\alpha}{(\alpha+r_1)^2(\alpha+r_2)}\right).\end{aligned}$$

Let $C_{n,m}(t)$ be the inverse of $1/(\alpha+r_1)^{n+1}(\alpha+r_2)^{m+1}$. Then the above transforms tell us that

$$w_2(0,y,t) = h_{2,1}(0,t)\,\alpha^2 y e^{-\alpha y} + h_{2,2}(0,t)\,\alpha e^{-\alpha y}$$

where we already know $h_{2,1}(0,t)$ and $h_{2,2}(0,t)$ from Dickson and Li (2010), and

$$w_1(0,y,t) = h_{1,1}(0,t)\,\alpha^2 y e^{-\alpha y} + h_{1,2}(0,t)\,\alpha e^{-\alpha y}$$

where

$$h_{1,1}(0,t) = \frac{\beta}{c}C_{0,-1}(t) + \frac{\alpha\beta^2}{c^2}C_{1,0}(t)$$

and

$$h_{1,2}(0,t) = \frac{\alpha\beta}{c}C_{1,-1}(t) - \frac{\alpha^2\beta^2}{c^2}C_{2,0}(t) - \frac{\alpha^2\beta^2}{c^2}C_{1,1}(t).$$

From Dickson and Li (2010) we know that

$$C_{n,-1}(t) = \frac{c(ct)^n e^{-(\beta+\alpha c)t}}{n!}\, {}_0F_1(n+2; \alpha\beta c t^2)$$

(where we are using the same notation as in Dickson and Li(2010) for generalised hypergeometric functions), and following the arguments in Dickson and Li (2010) we can show that

$$C_{n,m}(t) = c^2 t (ct)^{n+m} e^{-(\beta+\alpha c)t} \sum_{l=0}^{\infty} \frac{(\alpha\beta c t^2)^l\, \sigma_{l,n,m}}{\Gamma(n+m+2l+2)} \tag{4.5}$$

where

$$\sigma_{l,n,m} \;=\; \sum_{j=0}^{l} (-1)^{l-j} \binom{n+2j+1}{j} \frac{n+1}{n+2j+1} \binom{m+2(l-j)+1}{l-j} \frac{m+1}{m+2(l-j)+1}. \tag{4.6}$$

As we wish to use Mathematica to perform numerical integration, it is convenient to express the functions $C_{n,m}$ in terms of known functions and we can do this for the combinations of n and m in this example. As discussed in the Appendix, we find that

$$\begin{aligned} C_{1,0}(t) \;&=\; c^3 t^2 e^{-(\beta+\alpha c)t} \left(\frac{1}{2}\, {}_0F_3\left(\frac{3}{2}, \frac{3}{2}, 2; \frac{(\alpha\beta c t^2)^2}{16}\right) \right. \\ &\left. + \frac{2}{\alpha\beta c t^2} \left({}_0F_3\left(\frac{1}{4}, \frac{3}{4}, 2; \frac{(\alpha\beta c t^2)^2}{16}\right) - {}_1F_4\left(\left\{\frac{7}{5}\right\}, \left\{\frac{2}{5}, \frac{1}{2}, \frac{3}{2}, 2\right\}; \frac{(\alpha\beta c t^2)^2}{16}\right)\right)\right), \\ C_{1,1}(t) \;&=\; c^4 t^3 e^{-(\beta+\alpha c)t} \left(\frac{3}{2}\, {}_0F_3\left(\frac{3}{2}, \frac{3}{2}, 3; \frac{(\alpha\beta c t^2)^2}{16}\right) - \frac{4}{3}\, {}_0F_3\left(\frac{5}{4}, \frac{7}{4}, 3; \frac{(\alpha\beta c t^2)^2}{16}\right)\right), \end{aligned}$$

and

$$\begin{aligned} C_{2,0}(t) \;&=\; c^4 t^3 e^{-(\beta+\alpha c)t} \left(\frac{4}{3}\, {}_0F_3\left(\frac{5}{4}, \frac{7}{4}, 3; \frac{(\alpha\beta c t^2)^2}{16}\right) \right. \\ &\left. - \frac{7}{6}\, {}_1F_4\left(\left\{\frac{12}{5}\right\}, \left\{\frac{3}{2}, \frac{7}{5}, \frac{5}{2}, 3\right\}; \frac{(\alpha\beta c t^2)^2}{16}\right)\right) \\ &+ e^{-(\beta+\alpha c)t} \frac{2c^3 t}{\alpha\beta} \left({}_0F_3\left(\frac{1}{2}, \frac{3}{2}, 2; \frac{(\alpha\beta c t^2)^2}{16}\right) - {}_0F_3\left(\frac{3}{4}, \frac{5}{4}, 2; \frac{(\alpha\beta c t^2)^2}{16}\right)\right), \end{aligned}$$

giving us formulae for all the components of $h_{1,1}(t)$ and $h_{1,2}(t)$.

4.2 Mixed exponential claims

We now consider the situation when the individual claim amount distribution is a mixture of two exponential distributions. Let

$$f(x) = p\, ae^{-ax} + (1-p)\, be^{-bx}$$

where $0 < a < b$ and $0 < p < 1$.

Then from formula (10) of Dickson and Li (2010) it is straightforward to show that

$$\phi_2(0) = \frac{\beta^2}{c^2}\left(\frac{a}{a+\kappa}\frac{p}{(a+r_1)\,(a+r_2)} + \frac{b}{b+\kappa}\frac{1-p}{(b+r_1)\,(b+r_2)}\right)$$

so that

$$w_2(0,y,t) = h_{2,1}(0,t)\, ae^{-ay} + h_{2,2}(0,t)\, be^{-by}$$

where

$$\int_0^\infty e^{-\delta t} h_{2,1}(0,t)dt = \frac{\beta^2}{c^2}\frac{p}{(a+r_1)\,(a+r_2)}$$

and

$$\int_0^\infty e^{-\delta t} h_{2,2}(0,t)dt = \frac{\beta^2}{c^2}\frac{1-p}{(b+r_1)\,(b+r_2)}.$$

We can invert these transforms using the transform relationships (4) and (5) in Dickson and Li (2010). The transform relationship $q^*(s)^2 = f^*(s)$ tells us that $q^{2n*} = f^{n*}$, and we know from Willmot and Woo (2007) that

$$f^{n*}(x) = \sum_{j=0}^{\infty} \gamma_{n,j}\; e(n+j, b; x)$$

where $e(n,b;x)$ denotes the Erlang(n) density with scale parameter b and

$$\gamma_{n,j} = q^n(1-a/b)^j \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(r+j)}{\Gamma(r)\, j!} \left(\frac{ap}{bq}\right)^r.$$

We know that q is a density function as the mixed exponential distribution is infinitely divisible. However we do not need to identify q to obtain $h_{2,1}(0,t)$ and $h_{2,2}(0,t)$ as the following argument shows.

To invert

$$\frac{1}{(z+r_1)\,(z+r_2)}$$

(where we will set z to be a or b) consider first

$$\frac{1}{z+r_1} + \frac{1}{z+r_2}.$$

By the transform relationships (4) and (5) in Dickson and Li (2010) we know that this is the inverse of a function G_z where

$$G_z(t) = 2\left(c e^{-\beta t - zct} + \sum_{n=1}^{\infty} \frac{\beta^{2n} t^{2n-1}}{(2n)!} e^{-\beta t} \int_0^{ct} y e^{-zy} q^{2n*}(ct-y)dy \right).$$

Thus

$$\left(\frac{1}{z+r_1} + \frac{1}{z+r_2} \right)^2$$

inverts to $G_z^{2*}(t)$. Similarly

$$\frac{1}{(z+r_1)^2} + \frac{1}{(z+r_2)^2}$$

inverts to a function J_z where

$$J_z(t) = 2\left(c^2 t e^{-\beta t - zct} + \sum_{n=1}^{\infty} \frac{\beta^{2n} t^{2n-1}}{(2n)!} e^{-\beta t} \int_0^{ct} y^2 e^{-zy} q^{2n*}(ct-y)dy \right).$$

Hence, we see that $1/(z+r_1)(z+r_2)$ is the inverse of

$$\frac{1}{2}\left(G_z^{2*}(t) - J_z(t) \right)$$

and so

$$h_{2,1}(0,t) = \frac{p\,\beta^2}{2\,c^2}\left(G_a^{2*}(t) - J_a(t) \right)$$

and

$$h_{2,2}(0,t) = \frac{(1-p)\,\beta^2}{2\,c^2}\left(G_b^{2*}(t) - J_b(t) \right).$$

Next, we find $\phi_1(0)$ from equation (4.4), i.e.

$$\phi_1(0) = \frac{\beta}{2c}\left(\omega^*(r_2) + \omega^*(r_1)\right) + \frac{\phi(0)}{2}\left(q^*(r_2) - q^*(r_1)\right).$$

It is straightforward to show that

$$\omega^*(r_i) = \frac{p\,a}{(a+r_i)(a+\kappa)} + \frac{(1-p)\,b}{(b+r_i)(b+\kappa)}$$

so that

$$\omega^*(r_2)+\omega^*(r_1) = p\left(\frac{1}{a+r_1} + \frac{1}{a+r_2} \right)\frac{a}{a+\kappa} + (1-p)\left(\frac{1}{b+r_1} + \frac{1}{b+r_2} \right)\frac{b}{b+\kappa}.$$

Hence the first term of (4.4) inverts to

$$\frac{\beta}{2c}\left(p\,G_a(t)\,ae^{-ay}+(1-p)\,G_b(t)\,be^{-by}\right).$$

Consider now the second term of (4.4). From the transform relationships (4) and (5) in Dickson and Li (2010) we see that $q^*(r_1)-q^*(r_2)$ is the inverse of of a function $m(t)$ where

$$\begin{aligned} m(t) &= 2\sum_{n=1}^{\infty}\frac{\beta^{2n-1}t^{2n-2}e^{-\beta t}}{(2n-1)!}\int_0^{ct} y\,q^{(2n-1)^*}(ct-y)\,q(y)\,dy \\ &= 2\sum_{n=1}^{\infty}\frac{\beta^{2n-1}t^{2n-2}e^{-\beta t}}{(2n-1)!}\,\frac{ct}{2n}\,q^{2n*}(ct) \\ &= 2c\sum_{n=1}^{\infty}\frac{(\beta t)^{2n-1}e^{-\beta t}}{(2n)!}\,f^{n*}(ct). \end{aligned}$$

(See, e.g., Panjer (1981) for the second step in the above argument.) Hence the second term in (4.4) inverts to

$$\frac{-1}{2}\left(h_{2,1}*m(t)\,ae^{-ay}+h_{2,2}*m(t)\,be^{-by}\right)$$

so that

$$w_1(0,y,t)=\frac{1}{2}\left(\frac{\beta p}{c}\,G_a(t)-h_{2,1}*m(t)\right)\,ae^{-ay}+\frac{1}{2}\left(\frac{\beta\,(1-p)}{c}\,G_b(t)-h_{2,2}*m(t)\right)\,be^{-by}.$$

5 Numerical solution in the general case

In this section we explain how formulae (3.2) and (3.3) can be implemented for a general value of n. Our approach is to use numerical integration. One consequence of this approach is that we have to be able to calculate convolutions f^{n*}. This restriction equally applies in the classical risk model if we want to use formulae to calculate either the density of the time of ruin or the joint density of the time of ruin and the deficit at ruin – see Dickson and Willmot (2005) and Landriault and Willmot (2009). A second feature of our approach is that rather than obtain formulae for the functions $w_j(0,t)$ and $w_j(0,y,t)$ for $j=1,2,\ldots,n$, we solve numerically for these functions. If $w_j(0,y,t)$ decomposes as

$$w_j(0,y,t)=\sum_{i=1}^{n}h_{j,i}(0,t)f_i(y)$$

for density functions $f_i(y)$, $i = 1, 2, \ldots n$, then we can solve numerically for the functions $h_{j,i}(0, t)$.

To illustrate ideas, consider the case $n = 2$, and suppose that we are interested in $w_2(u, t)$. Then setting $u = 0$ in formula (3.2) gives

$$w_2(0,t) = y(t) - c\int_0^t \gamma_1(s)w_2(0,t-s)ds - c\int_0^t \gamma_2(s)w_1(0,t-s)ds \quad (5.1)$$

where

$$\begin{aligned} y(t) &= \beta p_1(t)\bar{F}(ct) + \beta\sum_{r=1}^{\infty} p_{2r+1}(t)\int_0^{ct} f^{r*}(x)\bar{F}(ct-x)dx, \\ \gamma_1(s) &= \sum_{r=1}^{\infty} p_{2r}(s)f^{r*}(cs) \end{aligned}$$

and

$$\gamma_2(s) = \sum_{r=1}^{\infty} p_{2r+1}(s)f^{r*}(cs).$$

The arguments that lead to formula (3.2) lead to a formula for $w_1(u, t)$, and hence for $w_1(0, t)$. We obtain

$$w_1(0,t) = z(t) - c\int_0^t \alpha_1(s)w_1(0,t-s)ds - c\int_0^t \alpha_2(s)w_2(0,t-s)ds \quad (5.2)$$

where

$$z(t) = \beta p_0(t)\bar{F}(ct) + \beta\sum_{r=1}^{\infty} p_{2r}(t)\int_0^{ct} f^{r*}(x)\bar{F}(ct-x)dx,$$

$\alpha_1(s) = \gamma_1(s)$ and

$$\alpha_2(s) = \sum_{r=1}^{\infty} p_{2r-1}(s)f^{r*}(cs).$$

We observe that equations (5.1) and (5.2) are a system of Volterra equations, and we can solve these equations numerically. Following Section 5.6 of Delves and Mohamed (1985), we solve by replacing the integrals in these equations by numerical integrals, using the trapezium rule. Noting that

$$\gamma_1(0) = \gamma_2(0) = \alpha_2(0) = 0,$$

we conclude that $\lim_{t\to 0} w_2(0,t) = 0$ and $\lim_{t\to 0} w_1(0,t) = \beta$. If we let $t = mh$, them equations (5.1) and (5.2) can be written as

$$\begin{aligned} w_2(0,mh) &= y(mh) - ch\sum_{j=1}^{m-1} w_2(0,jh)\gamma_1((m-j)h) \\ &\quad -ch\sum_{j=1}^{m-1} w_1(0,jh)\gamma_2((m-j)h) - \frac{\beta ch}{2}\gamma_2(mh) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} w_1(0,mh) &= z(mh) - ch\sum_{j=1}^{m-1} w_1(0,jh)\alpha_1((m-j)h) \\ &\quad -ch\sum_{j=1}^{m-1} w_2(0,jh)\alpha_2((m-j)h) - \frac{\beta ch}{2}\alpha_1(mh), \end{aligned} \tag{5.4}$$

with the usual convention that $\sum_{j=a}^{b} = 0$ when $b > a$.

Equations (5.3) and (5.4) show that both $w_1(0,mh)$ and $w_2(0,mh)$ can essentially be calculated in a recursive manner. This observation does not actually hinge on choosing the trapezium rule for numerical integration. We chose this method as it is simple to apply, and, as it turns out, is very accurate when h is small. We also note that for general n, calculations will still be performed recursively.

Example 5.1 *In this example we illustrate the accuracy of the above numerical approach to obtaining values of functions. We let $f(x) = 4xe^{-2x}$ so that we are able to compare our computed values with exact values obtained from formulae in Section 4.1. The above ideas easily extend to the joint density of the time of ruin and deficit at ruin. Table 5.1 shows exact and approximate values of $h_{1,1}(0,t)$, $h_{1,2}(0,t)$ and $w_1(0,t)$ when $\beta = 2$, $c = 1.1$ and the numerical integration parameter is $h = m/t = 0.01$, while Table 5.2 shows exact and approximate values of $h_{2,1}(0,t)$, $h_{2,2}(0,t)$ and $w_2(0,t)$. We can use the approximate values in Tables 5.1 and 5.2 to calculate values of $h_{2,1}(u,t)$ and $h_{2,2}(u,t)$. In Tables 5.3 and 5.4 we show some exact and approximate values for $u = 5$, 10 and 15. The exact values have been calculated using numerical integration in Mathematica with formulae for the components of the integrals. The approximate values have been calculated using numerical integration with repeated Simpson's rule and numerical values for $h_{i,j}(0,t)$ for $t = 0, 0.01, 0.02, \ldots$. The quality of all these approximations is very good.*

Example 5.2 *As a second example we consider the situation when the individual claim amount distribution is exponential with mean 1. In this case there is an explicit formula for* $\psi_n(u,t)$ *– see Dickson et al. (2005). Table 5.5 shows exact and approximate values for* $\psi_4(0,t)$ *and* $\psi_4(10,t)$ *(i.e. finite time ruin probabilities for the Erlang(4) risk model) when* $\beta = 1/4$ *and* $c = 1.1$. *In order to calculate these we first established a system of 4 Volterra equations for* $\psi_j(0,t)$, $j = 1,2,3,4$, *and solved these numerically by writing these equations in a similar form to equations (5.1) and (5.2). As in the previous example, we calculated these using a numerical integration parameter of* $h = 0.01$, *then used repeated Simpson's rule to calculate* $\psi_4(10,t)$. *The quality of the numerical approximations is again very good. We also calculated approximate values of* $\psi_j(0,t)$ *and* $\psi_j(10,t)$ *for* $j = 1,2,3$ *and these approximations showed the same degree of accuracy, suggesting that the approximation method is not particularly sensitive to the Erlang parameter.*

t	$h_{1,1}(0,t)$		$h_{1,2}(0,t)$		$w_1(0,t)$	
	Exact	Approx.	Exact	Approx.	Exact	Approx.
1	0.5588E–1	0.55877E–1	0.99431E–1	0.99437E–1	0.15531	0.15531
3	0.70531E–2	0.70516E–2	0.16227E–1	0.16228E–1	0.23280E–1	0.23280E–1
5	0.30403E–2	0.30103E–2	0.72904E–2	0.72911E–2	0.10331E–1	0.10331E–1
10	0.10028E–2	0.10025E–2	0.24735E–2	0.24737E–2	0.34763E–2	0.34762E–2
20	0.33077E–3	0.33066E–3	0.82682E–3	0.82692E–3	0.11576E–2	0.11576E–2
40	0.10519E–3	0.10515E–3	0.26465E–3	0.26469E–3	0.36985E–3	0.36984E–3
80	0.30576E–4	0.30566E–4	0.77174E–4	0.77190E–4	0.10775E–3	0.10776E–3

Table 5.1: Values of $h_{1,1}(0,t)$, $h_{1,2}(0,t)$ and $w_1(0,t)$, Erlang(2) claims

t	$h_{2,1}(0,t)$		$h_{2,2}(0,t)$		$w_2(0,t)$	
	Exact	Approx.	Exact	Approx.	Exact	Approx.
1	0.90392E–1	0.90389E–1	0.16860	0.16860	0.25899	0.25899
3	0.14751E–1	0.14750E–1	0.34155E–1	0.34156E–1	0.48906E–1	0.48907E–1
5	0.66277E–2	0.66270E–2	0.15925E–1	0.15926E–1	0.22553E–1	0.22553E–1
10	0.22486E–2	0.22484E–2	0.55490E–2	0.55492E–2	0.77976E–2	0.77976E–2
20	0.75166E–3	0.75158E–3	0.18791E–2	0.18792E–2	0.26308E–2	0.26308E–2
40	0.24060E–3	0.24058E–3	0.60534E–3	0.60537E–3	0.84594E–3	0.84595E–3
80	0.70158E–4	0.70160E–4	0.17708E–3	0.17710E–3	0.24724E–3	0.24724E–3

Table 5.2: Values of $h_{2,1}(0,t)$, $h_{2,2}(0,t)$ and $w_2(0,t)$, Erlang(2) claims

t	$h_{2,1}(5,t)$		$h_{2,1}(10,t)$		$h_{2,1}(15,t)$	
	Exact	Approx.	Exact	Approx.	Exact	Approx.
1	$0.51722E{-}3$	$0.51722E{-}3$	$0.52253E{-}6$	$0.52253E{-}6$	$0.28588E{-}9$	$0.28588E{-}9$
3	$0.19474E{-}2$	$0.19474E{-}2$	$0.16002E{-}4$	$0.16002E{-}4$	$0.50546E{-}7$	$0.50546E{-}7$
5	$0.25887E{-}2$	$0.25888E{-}2$	$0.61850E{-}4$	$0.61851E{-}4$	$0.52366E{-}6$	$0.52367E{-}6$
10	$0.25445E{-}2$	$0.25446E{-}2$	$0.23007E{-}3$	$0.23008E{-}3$	$0.76203E{-}5$	$0.76204E{-}5$
20	$0.16446E{-}2$	$0.16448E{-}2$	$0.41645E{-}3$	$0.41648E{-}3$	$0.45643E{-}4$	$0.45645E{-}4$
40	$0.76594E{-}3$	$0.76595E{-}3$	$0.38349E{-}3$	$0.38357E{-}3$	$0.10169E{-}3$	$0.10170E{-}3$
80	$0.27299E{-}3$	$0.27322E{-}3$	$0.20446E{-}3$	$0.20454E{-}3$	$0.95240E{-}4$	$0.95267E{-}4$

Table 5.3: Values of $h_{2,1}(u,t)$, Erlang(2) claims

t	$h_{2,2}(5,t)$		$h_{2,2}(10,t)$		$h_{2,2}(15,t)$	
	Exact	Approx.	Exact	Approx.	Exact	Approx.
1	$0.16486E{-}2$	$0.16486E{-}2$	$0.20625E{-}5$	$0.20625E{-}5$	$0.13057E{-}8$	$0.13057E{-}8$
3	$0.51534E{-}2$	$0.51534E{-}2$	$0.47207E{-}4$	$0.47207E{-}4$	$0.16310E{-}6$	$0.16310E{-}6$
5	$0.66272E{-}2$	$0.66271E{-}2$	$0.16930E{-}3$	$0.16930E{-}3$	$0.15245E{-}5$	$0.15245E{-}5$
10	$0.64100E{-}2$	$0.64098E{-}2$	$0.59660E{-}3$	$0.59659E{-}3$	$0.20383E{-}4$	$0.20383E{-}4$
20	$0.41364E{-}2$	$0.41363E{-}2$	$0.10583E{-}2$	$0.10583E{-}2$	$0.11747E{-}3$	$0.11747E{-}3$
40	$0.19296E{-}2$	$0.19292E{-}2$	$0.96957E{-}3$	$0.96949E{-}3$	$0.25822E{-}3$	$0.25820E{-}3$
80	$0.68933E{-}3$	$0.68906E{-}3$	$0.51674E{-}3$	$0.51664E{-}3$	$0.24103E{-}3$	$0.24099E{-}3$

Table 5.4: Values of $h_{2,2}(u,t)$, Erlang(2) claims

6 Concluding remarks

In the case when the individual claim amount distribution satisfies the factorisation given by (3.4), we can integrate expression (3.6) for $h_{n,i}(u,t)$ and hence obtain an expression for the joint distribution function of the time of ruin and the deficit at ruin. For individual claim amount distributions for which the factorisation (3.4) does not hold, the numerical approach of the previous section can be applied to find $w(u,y,t)$ for a fixed value of y. In such cases it seems that a tedious numerical procedure would be required to calculate the joint distribution function of the time of ruin and the deficit at ruin.

t	$\psi_4(0,t)$		$\psi_4(10,t)$	
	Exact	Approx.	Exact	Approx.
1	0.292623	0.292625	0.000024	0.000024
3	0.550729	0.550731	0.000404	0.000404
5	0.632257	0.632259	0.001551	0.001551
10	0.714425	0.714427	0.008073	0.008073
30	0.795861	0.795863	0.051934	0.051934
50	0.819086	0.819087	0.088666	0.088667
100	0.839855	0.839856	0.140965	0.140966

Table 5.5: Values of $\psi_4(u,t)$, exponential claims

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7 Appendix

In this appendix we briefly describe the techniques used to obtain the functions $C_{n,m}$ in Section 4.1. We start by considering the sum $\sigma_{l,n,m}$ given by equation (4.6). Let $s_{l,n,m} = \sigma_{l,n-1,m-1}$ so that

$$s_{l,n,m} = \sum_{j=0}^{l} (-1)^{l-j} \binom{n+2j}{j} \frac{n}{n+2j} \binom{m+2(l-j)}{l-j} \frac{m}{m+2(l-j)}.$$

The keys to finding an expression for this sum are formula (5.63) and (5.70) of Graham et al. (1994). From these, we see that if we define $\mathcal{C}$ to be the generating function of the Catalan numbers, given by

$$c_n = \binom{2n+1}{n} \frac{1}{2n+1}$$

for $n = 0, 1, 2, 3, \ldots$, then $s_{l,n,m}$ is the coefficient of z^l in $\mathcal{C}(z)^n \mathcal{C}(-z)^m$.

We define

$$G_{n,m}(z) = \mathcal{C}(z)^n \mathcal{C}(-z)^m,$$

and we note that $G_{n,m}(z) = G_{m,n}(-z)$, so that it suffices to consider only the case $n \geq m$. The idea is to solve recursively, starting with the case $m = n = 1$. Formula (5.68) of Graham et al. (1994) states that

$$\mathcal{C}(z) = \frac{1 - \sqrt{1-4z}}{2z},$$

and from this we obtain

$$-4z^2 \mathcal{C}(z)\mathcal{C}(-z) - 2z\mathcal{C}(z) + 2z\mathcal{C}(-z) + 1 = \sqrt{1-16z^2} = -\sum_{n=0}^{\infty} \binom{2n}{n} \frac{4^n z^{2n}}{2n-1},$$

where the second identity is given on page 531 of Graham et al. (1994). Rearranging, we obtain

$$\begin{aligned} G_{1,1}(z) &= -\frac{1}{2z}\mathcal{C}(z) + \frac{1}{2z}\mathcal{C}(-z) + \frac{1}{4z^2} + \frac{1}{4z^2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{4^n z^{2n}}{2n-1} \\ &= -\frac{1}{2z} \sum_{n=0}^{\infty} \binom{2n+1}{n} \frac{z^n}{2n+1} + \frac{1}{2z} \sum_{n=0}^{\infty} \binom{2n+1}{n} \frac{(-1)^n z^n}{2n+1} \\ &\quad + \frac{1}{4z^2} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{4^n z^{2n}}{2n-1}. \end{aligned}$$

It is clear that the coefficients of the odd powers of z in $G_{1,1}(z)$ are 0. Hence we require to identify the coefficients ν_{2l} of z^{2l} for $l = 0, 1, 2, \ldots$. We get

$$\begin{aligned} \nu_{2l} &= \frac{-1}{2(4l+3)}\binom{4l+3}{2l+1} - \frac{1}{2(4l+3)}\binom{4l+3}{2l+1} + \frac{1}{4}\binom{2l+2}{l+1}\frac{4^{l+1}}{2l+1} \\ &= \binom{2l+2}{l+1}\frac{4^l}{2l+1} - \binom{4l+3}{2l+1}\frac{1}{4l+3}. \end{aligned}$$

This result can be expressed in terms of gamma functions using the fact that

$$\binom{2n}{n} = (-4)^n\binom{-1/2}{n}.$$

After some straightforward manipulation we obtain

$$\nu_{2l} = \frac{4^{2l+1}}{\Gamma(\frac{1}{2})}\left(\frac{\Gamma(l+\frac{1}{2})}{2\Gamma(l+2)} - \frac{\Gamma(2l+\frac{3}{2})}{\Gamma(2l+3)}\right).$$

Now that we have established a formula for the coefficients of powers of z in $G_{1,1}(z)$, we can consider other cases. From formula (5.59) of Graham et al. (1994) we have

$$z\mathcal{C}(z)^2 = \mathcal{C}(z) - 1,$$

which, on multiplying by $\mathcal{C}(z)^r\mathcal{C}(-z)^s$, yields the identity

$$z\mathcal{C}(z)^{r+2}\mathcal{C}(-z)^s = \mathcal{C}(z)^{r+1}\mathcal{C}(-z)^s - \mathcal{C}(z)^r\mathcal{C}(-z)^s,$$

i.e.

$$zG_{r+2,s}(z) = G_{r+1,s}(z) - G_{r,s}(z).$$

For example, setting $r = 0$ and $s = 1$, we obtain

$$zG_{2,1}(z) = G_{1,1}(z) - G_{0,1}(z),$$

and as we know the coefficients of powers of z in both $G_{1,1}(z)$ and $G_{0,1}(z)$ $(= \mathcal{C}(-z))$, we can identify the coefficients of powers of z in $G_{2,1}(z)$.

Similarly, setting $r = 0$ and $s = 2$, we obtain

$$zG_{2,2}(z) = G_{1,2}(z) - G_{0,2}(z).$$

This allows us to find the coefficients of powers of z in $G_{2,2}(z)$ since $G_{1,2}(z) = G_{2,1}(-z)$ and $G_{0,2}(z)$ $(= \mathcal{C}(-z)^2)$ is given by formula (5.70) of Graham et al. (1994).

Having obtained an expression for $\sigma_{l,n,m}$ we can insert this in (4.5), then apply the technique of looking at the ratio of successive terms (and the value of the first term) in the right hand side of (4.5) as described on pages 207-8 of Graham et al. (1994). This allows us to write the $C_{n,m}$ functions in terms of generalised hypergeometric functions.