The joint distribution of the time to ruin and the number of claims until ruin in the classical risk model

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Abstract

We use probabilistic arguments to derive an expression for the joint density of the time of ruin and the number of claims until ruin in the classical risk model. From this we obtain a general expression for the probability function of the number of claims until ruin. We also consider the moments of the number of claims until ruin and illustrate our results in the case of exponentially distributed individual claims. We find a very strong correlation between the number of claims until ruin and the time of ruin in this case. Finally, we briefly discuss joint distributions involving the surplus prior to ruin and deficit at ruin.

1 Introduction

The distribution of the number of claims until ruin has been studied by a number of authors over the years. One of the earliest references is Beard (1971). A more fruitful approach was that of Stanford and Stroiński (1994) who derive some recursive procedures to calculate the probability of ruin at the nth claim in the classical risk model. Egídio dos Reis (2002) derives the Laplace transform of the probability function of the number of claims until ruin in the classical risk model. He inverts this for certain claim size distributions, and, using a duality argument, finds moments of the number of claims until ruin when the initial surplus is 0.

In a recent paper, Landriault et al (2011) consider a Sparre Andersen risk model with exponential claims. Using Gerber-Shiu type analysis (see Gerber and Shiu (1998)) they derive a number of results including an expression for the probability function of the number of claims until ruin. They also consider the classical risk model and apply the approach adopted by Dickson and Willmot (2005) to find the density of the time to ruin in order to derive an expression for the joint density of the time to ruin and the number of claims until ruin. (For convenience, we use the term joint density throughout when referring to two or more variables, even if one of the variables is discrete.) An important idea in Landriault et al (2011) is that some known results relating to the time to ruin can be interpreted in terms of the number of claims until ruin, and we show that this idea has applications to other results in ruin theory.

Dickson (2007) uses probabilistic arguments to find an expression for the density of the time to ruin in the classical risk model, and we now adapt this approach to obtain an expression for the joint density of the time to ruin and the number of claims until ruin, from which we can easily extract the marginal distributions. We extend the results of Egídio dos Reis (2002) by considering the moments of the number of claims until ruin when the initial surplus is greater than 0. Central to our analysis is the case when the initial surplus is 0. We can obtain all results of interest for this case using Gerber-Shiu type analysis, with the function introduced by Landriault et al (2011), and Lagrange's implicit function theorem, as in Dickson and Willmot (2005).

2 Notation

Throughout this paper we consider the classical risk model. Let $\{U(t)\}_{t\geq 0}$ denote the surplus process of an insurer with

$$U(t) = u + ct - S(t),$$

where $u \geq 0$ is the initial surplus, c is the rate of premium income per unit time, and $\{S(t)\}_{t\geq 0}$ is the aggregate claims process. We have $S(t) = \sum_{i=1}^{M(t)} X_i$, where $\{M(t)\}_{t\geq 0}$ is a Poisson process with Poisson parameter λ, X_i denotes the amount of the ith claim, and $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, each with distribution function F, such that $F(0) = 0$, and probability density function f. We denote $E[X_1^k]$ by m_k and we assume that m_1 and m_2 exist and that $c > \lambda m_1$.

For this process, let T_u denote the time to ruin from initial surplus u, so that

$$T_u = \inf\{t\colon\ U(t) < 0\}$$

and define $\psi(u) = \Pr(T_u < \infty) = 1 - \chi(u)$ and $\psi(u,t) = \Pr(T_u \leq t)$, and let

$$w(u,t) = \frac{d}{dt}\psi(u,t)$$

denote the defective density of the time to ruin. Let N_{T_u} denote the number of claims until ruin (including the claim causing ruin), and let $w_n(u,t)$ denote the joint density of T_u and N_{T_u}, given initial surplus u, defined for $n = 1, 2, 3, \ldots$ and $t > 0$. Further let $p_n(u)$ denote the probability of ruin at the nth claim given initial surplus u, so that

$$p_n(u) = \int_0^\infty w_n(u,t)dt.$$

As in Landriault et al (2011) we define

$$\phi_{r,\delta}(u) = E\left[r^{N_{T_u}}\, e^{-\delta T_u}\, I(T_u < \infty)\right]$$

for $0 < r \leq 1$ and $\delta \geq 0$, where I is the indicator function. Thus

$$\phi_{r,\delta}(u) = \sum_{n=1}^{\infty} r^n \int_0^\infty e^{-\delta t} w_n(u,t)dt. \tag{1}$$

This function allows us to find the joint density of T_u and N_{T_u} when $u = 0$, as shown in the next section.

Finally, we use the notation $\tilde{a}$ for the Laplace transform of a function a.

3 The case $u = 0$

In this section we consider the case $u = 0$. Although formulae (4) to (7) below are obtained by Landriault et al (2011) we include them for completeness as results for the case $u = 0$ are required for the more general case when $u > 0$. Also, the approach below is more direct than theirs if we are interested in using transform inversion techniques only for the case $u = 0$, and not for the case $u > 0$ as in Landriault et al (2011). We apply a different approach to the case $u > 0$ in the next section.

By the standard argument of conditioning on the time and the amount of the first claim we obtain

$$\frac{d}{du}\phi_{r,\delta}(u) = \frac{\lambda+\delta}{c}\phi_{r,\delta}(u) - \frac{\lambda r}{c}\bar{F}(u) - \frac{\lambda r}{c}\int_0^u \phi_{r,\delta}(u-x)f(x)dx, \tag{2}$$

and taking the Laplace transform of this equation with respect to u we obtain

$$\tilde{\phi}_{r,\delta}(s) = \frac{c\,\phi_{r,\delta}(0) - \lambda r\int_0^\infty e^{-su}\bar{F}(u)du}{cs - (\lambda+\delta) + \lambda r\tilde{f}(s)}. \tag{3}$$

Now let ρ be the unique positive solution of

$$cs - (\lambda+\delta) + \lambda r\tilde{f}(s) = 0.$$

(See Landriault et al (2011).) Then, as ρ is a zero of the denominator of equation (3), it must also be a zero of the numerator, giving

$$\phi_{r,\delta}(0) = \frac{\lambda r}{c} \int_0^\infty e^{-\rho u} \bar{F}(u) du.$$

Landriault et al (2011) adapt the technique in Dickson and Willmot (2005) of employing Lagrange's implicit function theorem to express a Laplace transform with transform parameter ρ as a Laplace transform with transform parameter δ. Formula (44) of Landriault et al (2011) allows us to invert $\phi_{r,\delta}(0)$ with respect to δ as

$$\lambda r e^{-\lambda t} \bar{F}(ct) + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda r)^{n+1} t^{n-1}}{n! c} \int_0^{ct} y \, f^{n*}(ct - y) \bar{F}(y) dy,$$

and hence by equation (1), we have

$$w_1(0,t) = \lambda e^{-\lambda t} \bar{F}(ct) \tag{4}$$

and for $m = 2, 3, 4, \ldots$,

$$w_m(0,t) = e^{-\lambda t} \frac{\lambda^m t^{m-1}}{(m-1)!} \int_0^{ct} \frac{y}{ct} \, f^{(m-1)^*}(ct - y) \bar{F}(y) dy. \tag{5}$$

Summing $w_m(0,t)$ over m yields the known formula (see Dickson and Willmot (2005))

$$w(0,t) = \lambda e^{-\lambda t} \bar{F}(ct) + \sum_{m=2}^{\infty} e^{-\lambda t} \frac{\lambda^m t^{m-1}}{(m-1)!} \int_0^{ct} \frac{y}{ct} \, f^{(m-1)^*}(ct - y) \bar{F}(y) dy.$$

Similarly, integrating $w_m(0,t)$ over t yields

$$p_1(0) = \int_0^\infty \lambda e^{-\lambda t} \bar{F}(ct) dt \tag{6}$$

and for $m = 2, 3, 4, \ldots$,

$$p_m(0) = \int_0^\infty e^{-\lambda t} \frac{\lambda^m t^{m-1}}{(m-1)!} \int_0^{ct} \frac{y}{ct} \, f^{(m-1)^*}(ct - y) \bar{F}(y) dy dt. \tag{7}$$

Formula (5) is required to find $w_m(u,t)$, and formula (7) is required to find $p_m(u)$, both for $m = 2, 3, 4, \ldots$, as shown in the next section.

4 The case $u > 0$

In this section we consider the case when $u > 0$. We adapt ideas in Dickson (2007) where a formula for $w(u,t)$ is obtained by applying probabilistic arguments from Prabhu (1961). Consider first $w_1(u,t)$. For small dt, we think of $w_1(u,t)\,dt$ as representing the probability that ruin occurs on the first claim and in the interval $(t, t+dt)$. For this to occur, we require no claims up to time t, and a claim exceeding $u + ct$ in the interval $(t, t+dt)$. Hence

$$w_1(u,t) = \lambda e^{-\lambda t} \bar{F}(u+ct),$$

with formula (4) as a special case. Similarly, for $n = 1, 2, 3, \ldots$ we can construct the formula

$$\begin{aligned} w_{n+1}(u,t) &= e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct} f^{n*}(u+ct-x)\lambda \bar{F}(x)dx \\ &\quad -c\sum_{j=1}^{n}\int_0^t e^{-\lambda s}\frac{(\lambda s)^j}{j!} f^{j*}(u+cs)w_{n+1-j}(0,t-s)ds. \qquad (8) \end{aligned}$$

The arguments behind this formula are as follows. We consider $w_{n+1}(u,t)\,dt$ as representing the probability that ruin occurs on the $(n+1)$th claim and in the interval $(t, t+dt)$. The surplus falls below 0 on the $(n+1)$th claim and in the interval $(t, t+dt)$ it there are n claims up to time t of total amount $u+ct-x$, so that the surplus is x at time t, and if a claim exceeding x occurs in $(t, t+dt)$. The first term in formula (8) covers this situation, but makes no allowance for the possibility that the surplus was below 0 prior to time t. The second term in formula (8) allows for this possibility. It is constructed by adapting Prabhu's (1961) argument. Suppose that at time s, $0 < s < t$, there have been j claims, $1 \leq j \leq n$, whose amount is $u + cs$, so that the surplus at time s is 0. Then the joint density associated with the surplus next falling below 0 at time t and on the $(n+1-j)$th claim from time s is $w_{n+1-j}(0, t-s)$, meaning that the surplus falls below 0 on the $(n+1)$th claim. Summing over j and integrating over t gives the adjustment required to the first term in formula (8). Formula (8) is considerably simpler than the formula for $w_n(u,t)$ given by Landriault et al (2011).

It is straightforward to show that

$$\begin{aligned} \sum_{n=0}^{\infty} w_{n+1}(u,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \\ &\quad -c\int_0^t g(u+cs,s)w(0,t-s)ds \end{aligned}$$

which is the formula for $w(u,t)$ in Dickson (2007).

Now consider the probability function $p_n(u)$. We have

$$p_1(u) = \int_0^\infty \lambda e^{-\lambda t} \bar{F}(u+ct)dt \tag{9}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
p_{n+1}(u) &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct} f^{n*}(u+ct-x)\lambda\bar{F}(x)dxdt \\
&\quad -c\sum_{j=1}^n \int_0^\infty \int_0^t e^{-\lambda s}\frac{(\lambda s)^j}{j!} f^{j*}(u+cs)w_{n+1-j}(0,t-s)dsdt \\
&= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct} f^{n*}(u+ct-x)\lambda\bar{F}(x)dxdt \\
&\quad -c\sum_{j=1}^n \int_0^\infty \int_s^\infty e^{-\lambda s}\frac{(\lambda s)^j}{j!} f^{j*}(u+cs)w_{n+1-j}(0,t-s)dtds \\
&= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct} f^{n*}(u+ct-x)\lambda\bar{F}(x)dxdt \\
&\quad -c\sum_{j=1}^n \int_0^\infty e^{-\lambda s}\frac{(\lambda s)^j}{j!} f^{j*}(u+cs)\int_s^\infty w_{n+1-j}(0,t-s)dtds \\
&= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct} f^{n*}(u+ct-x)\lambda\bar{F}(x)dxdt \\
&\quad -c\sum_{j=1}^n p_{n+1-j}(0)\int_0^\infty e^{-\lambda s}\frac{(\lambda s)^j}{j!} f^{j*}(u+cs)ds.
\end{aligned} \tag{10}$$

This expression is quite different to what we obtain when we invert the formula given by Egídio dos Reis (2002) for the Laplace transform of $p_n(u)$. Interestingly, both his formula and ours have the property that we require values of $p_n(0)$ to calculate values of $p_n(u)$.

5 Moments of N_{T_u}

Let $\delta = 0$ in this section and consider

$$\phi_r(u) = E\left[r^{N_{T_u}} I(T_u < \infty)\right].$$

Then

$$\left.\frac{d^k}{dr^k}\phi_r(u)\right|_{r=1} = E\left[N_{T_u}(N_{T_u}-1)\ldots(N_{T_u}-k+1)I(T_u<\infty)\right].$$

In particular, setting $u = 0$ we can find the moments of N_{T_0} by differentiating

$$\phi_r(0) = \frac{\lambda r}{c} \int_0^\infty e^{-\rho u} \bar{F}(u) du$$

where

$$c\rho - \lambda + \lambda r \tilde{f}(\rho) = 0. \tag{11}$$

For example,

$$\left. \frac{d}{dr} \phi_r(0) \right|_{r=1} = \left. \left(\frac{\lambda}{c} \int_0^\infty e^{-\rho u} \bar{F}(u) du - \frac{\lambda r}{c} \int_0^\infty u \left(\frac{d\rho}{dr} \right) e^{-\rho u} \bar{F}(u) du \right) \right|_{r=1},$$

and from formula (11), $\rho = 0$ when $r = 1$, and differentiating formula (11) with respect to r gives

$$\rho' |_{r=1} = \frac{-\lambda}{c - \lambda m_1}$$

and hence

$$E[N_{T_0} \, I(T_0 < \infty)] = \frac{\lambda m_1}{c} + \frac{\lambda}{c} \frac{\lambda m_2}{2 \, (c - \lambda m_1)}.$$

We remark that this formula can be easily obtained from Egídio dos Reis (2002) who considers the moments of the number of claims from the ruin time to the time of recovery when $u = 0$.

We can now apply ideas in Albrecher and Boxma (2005). From equation (3) we have

$$\tilde{\phi}_r(s) = \frac{c\phi_r(0) - \lambda r \int_0^\infty e^{-su} \bar{F}(u) du}{cs - \lambda + \lambda r \tilde{f}(s)}$$

and so

$$\begin{aligned} \frac{d}{dr} \tilde{\phi}_r(s) &= \frac{c \frac{d}{dr} \phi_r(0) - \lambda \int_0^\infty e^{-su} \bar{F}(u) du}{cs - \lambda + \lambda r \tilde{f}(s)} \\ &\quad - \frac{\left(c\phi_r(0) - \lambda r \int_0^\infty e^{-su} \bar{F}(u) du \right) \lambda \tilde{f}(s)}{\left(cs - \lambda + \lambda r \tilde{f}(s) \right)^2} \\ &= \frac{c \frac{d}{dr} \phi_r(0) - \lambda \int_0^\infty e^{-su} \bar{F}(u) du - \lambda \tilde{f}(s) \tilde{\phi}_r(s)}{cs - \lambda + \lambda r \tilde{f}(s)}. \end{aligned} \tag{12}$$

Then

$$\left. \frac{d}{dr} \tilde{\phi}_r(s) \right|_{r=1} = \frac{cE[N_{T_0} \, I(T_0 < \infty)] - \lambda \int_0^\infty e^{-su} \bar{F}(u) du}{cs - \lambda + \lambda \tilde{f}(s)}$$

$$-\frac{\lambda\tilde{f}(s)\tilde{\psi}(s)}{cs-\lambda+\lambda\tilde{f}(s)}$$

$$= \frac{\tilde{\chi}(s)}{\chi(0)}\left(E[N_{T_0}\,I(T_0<\infty)]-\frac{\lambda}{c}\int_0^\infty e^{-su}\bar{F}(u)du\right)$$

$$-\frac{\tilde{\chi}(s)\lambda\tilde{f}(s)\tilde{\psi}(s)}{c\chi(0)}, \tag{13}$$

where we have used the well-known formula for the Laplace transform of χ – see, for example, Dickson (2005). Now define

$$b(x)=\int_0^x \psi(y)\chi(x-y)dy.$$

Then by inverting equation (13) we obtain

$$E[N_{T_u}\,I(T_u<\infty)]$$

$$= \frac{\chi(u)}{\chi(0)}E[N_{T_0}\,I(T_0<\infty)]-\frac{\lambda/c}{\chi(0)}\int_0^u \chi(u-x)\bar{F}(x)dx$$

$$-\frac{\lambda}{c\chi(0)}\int_0^u b(u-x)f(x)dx$$

$$= \frac{\chi(u)}{\chi(0)}E[N_{T_0}\,I(T_0<\infty)]-\frac{\chi(u)-\chi(0)}{\chi(0)}$$

$$-\frac{\lambda}{c\chi(0)}\int_0^u b(u-x)f(x)dx$$

$$= 1+\frac{\chi(u)}{\chi(0)}\left(E[N_{T_0}\,I(T_0<\infty)]-1\right)$$

$$-\frac{\lambda}{c\chi(0)}\int_0^u b(u-x)f(x)dx. \tag{14}$$

We can find higher moments of N_{T_u} in a similar fashion. Differentiating formula (12) with respect to r we obtain

$$\frac{d^2}{dr^2}\tilde{\phi}_r(s)=\frac{c\frac{d^2}{dr^2}\phi_r(0)-2\lambda\tilde{f}(s)\frac{d}{dr}\tilde{\phi}_r(s)}{cs-\lambda+\lambda r\tilde{f}(s)}. \tag{15}$$

Then

$$\left.\frac{d^2}{dr^2}\tilde{\phi}_r(s)\right|_{r=1} = \frac{cE[N_{T_0}\,(N_{T_0}-1)\,I(T_0<\infty)]}{cs-\lambda+\lambda\tilde{f}(s)}$$

$$
\begin{aligned}
& -\frac{2\lambda\tilde{f}(s)}{cs-\lambda+\lambda\tilde{f}(s)}\left.\frac{d}{dr}\tilde{\phi}_r(s)\right|_{r=1} \\
= \; & \frac{\tilde{\chi}(s)}{\chi(0)}E[N_{T_0}\,(N_{T_0}-1)\,I(T_0<\infty)] \\
& -\frac{2\lambda\tilde{f}(s)\tilde{\chi}(s)}{c\chi(0)}\left.\frac{d}{dr}\tilde{\phi}_r(s)\right|_{r=1}
\end{aligned}
$$

giving

$$
\begin{aligned}
E[N_{T_u}\,(N_{T_u}-1)\,I(T_u<\infty)] \;=\; & \frac{\chi(u)}{\chi(0)}E[N_{T_0}\,(N_{T_0}-1)\,I(T_0<\infty)] \\
& -\frac{2\lambda}{c\chi(0)}\int_0^u m(u-x)f(x)dx, \qquad (16)
\end{aligned}
$$

where

$$
m(x)=\int_0^x \chi(x-y)E[N_{T_y}\,I(T_y<\infty)]\,dy.
$$

We remark that this argument easily extends to finding higher moments.

Results when $u=0$ can also be found using arguments given by Albrecher and Boxma (2005). From formula (12) we obtain

$$
\left(\frac{d}{dr}\tilde{\phi}_r(s)\right)\left(cs-\lambda+\lambda r\tilde{f}(s)\right)=c\frac{d}{dr}\phi_r(0)-\lambda\int_0^\infty e^{-su}\bar{F}(u)du-\lambda\tilde{f}(s)\tilde{\phi}_r(s).
$$

Setting $r=1$ and $s=0$ we obtain

$$
\left.\frac{d}{dr}\phi_r(0)\right|_{r=1}=\frac{\lambda}{c}\int_0^\infty \bar{F}(u)du+\frac{\lambda}{c}\tilde{\phi}_1(0),
$$

which gives

$$
\begin{aligned}
E[N_{T_0}\,I(T_0<\infty)] \;&=\; \frac{\lambda m_1}{c}+\frac{\lambda}{c}\int_0^\infty \psi(u)du \\
&=\; \frac{\lambda m_1}{c}+\frac{\lambda}{c}\frac{\lambda m_2}{2(c-\lambda m_1)}.
\end{aligned}
$$

Similarly, formula (15) yields

$$
\left.\frac{d^2}{dr^2}\phi_r(0)\right|_{r=1}=\frac{2\lambda}{c}\left.\frac{d}{dr}\tilde{\phi}_r(0)\right|_{r=1},
$$

so that

$$
E[N_{T_0}(N_{T_0}-1)\,I(T_0<\infty)]=\frac{2\lambda}{c}\int_0^\infty E[N_{T_u}\,I(T_u<\infty)]du.
$$

6 Exponential claims

Let $p(x) = \beta e^{-\beta x}$ for $x > 0$. We start with the case $u = 0$ and derive a known formula for $p_n(0)$ – see Landriault et al (2011). First, by formula (6) we have

$$p_1(0) = \int_0^\infty \lambda e^{-\lambda t}\, e^{-\beta ct}\, du = \frac{\lambda}{\lambda + \beta c}.$$

Next, consider the inner integral in formula (7). It is straightforward to show that

$$\int_0^{ct} u\, f^{(n-1)*}(ct-u)\bar{F}(u)du = \frac{\beta^{n-1}(ct)^n e^{-\beta ct}}{n!},$$

and so for $n = 2, 3, 4, \ldots$

$$\begin{aligned} p_n(0) &= \frac{\lambda^n}{c\,(n-1)!}\int_0^\infty e^{-\lambda t} t^{n-2} \frac{\beta^{n-1}(ct)^n e^{-\beta ct}}{n!} dt \\ &= \frac{\lambda^n (\beta c)^{n-1}}{n!\,(n-1)!}\frac{(2n-2)!}{(\lambda+\beta c)^{2n-1}}. \end{aligned}$$

Thus, for $n = 1, 2, 3, \ldots,$

$$p_n(0) = \frac{(2n-2)!}{n!\,(n-1)!}\left(\frac{\lambda}{\lambda+\beta c}\right)^n \left(\frac{\beta c}{\lambda+\beta c}\right)^{n-1}.$$

Summing $p_n(0)$ over n yields $\psi(0) = \lambda/(\beta c)$, so we have

$$\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!\,(n-1)!}\left(\frac{\lambda}{\lambda+\beta c}\right)^n \left(\frac{\beta c}{\lambda+\beta c}\right)^{n-1} = \frac{\lambda}{\beta c},$$

an identity which can also be obtained from the general result (e.g. Graham et al (1994))

$$\sum_{k=0}^{\infty} \binom{2k+1}{k}\frac{z^k}{2k+1} = \frac{1-\sqrt{1-4z}}{2z}.$$

In the case $u > 0$ we obtain the following from formulae (9) and (10):

$$p_1(u) = \frac{\lambda e^{-\beta u}}{\lambda + c\beta}$$

and for $n = 2, 3, 4, \ldots,$

$$p_n(u) = \frac{\lambda^n \beta^{n-1} e^{-\beta u}}{(n-1)!^2}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{c^j\, u^{n-1-j}\,(n+j-1)!}{(\lambda+c\beta)^{n+j}}$$

$$-c\sum_{j=1}^{n-1} p_{n-j}(0)\frac{(\lambda\beta)^j e^{-\beta u}}{j!\,(j-1)!}\sum_{r=0}^{j-1}\binom{j-1}{r}\frac{c^r\,u^{j-1-r}\,(r+j)!}{(\lambda+c\beta)^{r+j+1}}.$$

Moments can be obtained from formulae (14) and (16) as

$$E[N_{T_u}\,I(T_u<\infty)] = \frac{\lambda(c+\lambda u)}{c(c\beta-\lambda)}e^{-(\beta-\lambda/c)u}$$

giving

$$E[N_{T_u}|T_u<\infty] = \frac{\beta(c+\lambda u)}{c\beta-\lambda},$$

and

$$E[N_{T_u}\,(N_{T_u}-1)\,I(T_u<\infty)] = \left(\frac{2\beta c\lambda^2(1+\beta u)}{(c\beta-\lambda)^3}+\frac{\beta u^2\lambda^3}{c(c\beta-\lambda)^2}\right)e^{-(\beta-\lambda/c)u}$$

giving

$$E[N_{T_u}(N_{T_u}-1)|T_u<\infty] = \frac{2\,(\beta c\lambda)^2\,(1+\beta u)}{\lambda(c\beta-\lambda)^3}+\frac{(\beta u\lambda)^2}{(c\beta-\lambda)^2}$$

and

$$V[N_{T_u}|T_u<\infty] = \frac{\beta\lambda(c(\beta c+\lambda)+u(\beta^2c^2+\lambda^2))}{(c\beta-\lambda)^3}.$$

We remark that

$$E[N_{T_u}|T_u<\infty] = \beta c E[T_u|T_u<\infty].$$

See, for example, Lin and Willmot (2000) or Dickson (2005).

Now consider the covariance between N_{T_u} and T_u given that ruin occurs. We can find $E[T_u\,N_{T_u}\,I(T<\infty)]$ as

$$-\frac{d}{dr}\frac{d}{d\delta}\phi_{r,\delta}(u)\bigg|_{r=1,\delta=0}.$$

For this particular claim size distribution a straightforward approach to obtaining this is to note that equation (2) for $\phi_{r,\delta}(u)$ can be solved as

$$\phi_{r,\delta}(u) = \left(1-\frac{R_{\delta,r}}{\beta}\right)\exp\{-R_{\delta,r}u\} \tag{17}$$

where

$$-cR_{\delta,r}-(\lambda+\delta)+\frac{\lambda r\beta}{\beta-R_{\delta,r}} = 0. \tag{18}$$

If we differentiate equation (17) with respect to both δ and r we obtain

$$\frac{d}{dr}\frac{d}{d\delta}\phi_{r,\delta}(u) = \frac{\left(u(2+u\,(\beta - R_{\delta,r}))R_{\delta,r}^{(1,0)}R_{\delta,r}^{(0,1)} - ((\beta - R_{\delta,r})u - 1)R_{\delta,r}^{(1,1)}\right)\exp\{-R_{\delta,r}u\}}{\beta}$$

where

$$R_{\delta,r}^{(1,0)} = \frac{d}{dr}R_{\delta,r}, \quad R_{\delta,r}^{(0,1)} = \frac{d}{d\delta}R_{\delta,r}, \quad R_{\delta,r}^{(1,1)} = \frac{d}{dr}\frac{d}{d\delta}R_{\delta,r}.$$

If we differentiate equation (18) with respect to δ, and then set $\delta = 0$ and $r = 1$ we obtain

$$R_{\delta,r}^{(0,1)}\Big|_{r=1,\delta=0} = \frac{\lambda}{c(\beta c - \lambda)},$$

if we differentiate equation (18) with respect to r, and then set $\delta = 0$ and $r = 1$ we obtain

$$R_{\delta,r}^{(1,0)}\Big|_{r=1,\delta=0} = \frac{-\beta\lambda}{\beta c - \lambda},$$

and if we differentiate equation (18) with respect to δ, and then r, then set $\delta = 0$ and $r = 1$ we obtain

$$R_{\delta,r}^{(1,1)}\Big|_{r=1,\delta=0} = \frac{\lambda\beta(\lambda + \beta c)}{(\beta c - \lambda)^3}.$$

As $R_{\delta,r} = \beta - \lambda/c$ when $\delta = 0$ and $r = 1$, we obtain

$$E[T_u\, N_{T_u}\,|T_u < \infty] = \frac{\beta\left(c^3\beta + cu(\beta u - 1)\lambda^2 - \lambda^3 u^2 + c^2\lambda(1 + 3u\beta)\right)}{c(\beta c - \lambda)^3}$$

leading to

$$Cov\,[T_u, N_{T_u}|T_u < \infty] = \frac{\beta\lambda(2c + u(\beta c + \lambda))}{(\beta c - \lambda)^3}.$$

A formula for $V[N_{T_u}|T_u < \infty]$ can be found in Lin and Willmot (2000) or Dickson (2005), and so we can calculate the correlation coefficient between T_u and N_{T_u} given that $T_u < \infty$. Table 1 shows some values of this correlation coefficient when $\lambda = \beta = 1$ and $c = 1.1$, 1.2 and 1.3. We observe that the value of the correlation coefficient is very high for each value of c. Whilst we would expect the time to ruin and the number of claims until ruin to show a strong positive correlation, the values in Table 1 are higher than we might expect.

u	$c = 1.1$	$c = 1.2$	$c = 1.3$
0	0.998866	0.995859	0.991457
5	0.998867	0.995882	0.991552
10	0.998868	0.995887	0.991573
15	0.998868	0.995889	0.991581
20	0.998868	0.995890	0.991585
25	0.998868	0.995890	0.991588

Table 1: Values of the correlation coefficient

7 Other joint distributions

The arguments in Section 4 together with arguments in Dickson (2007) lead to expressions for joint densities involving the number of claims until ruin, the deficit at ruin and the surplus prior to ruin. Many of the arguments in this section come from Dickson (2007), and consequently we state rather than derive results if no new ideas are involved.

The starting point is a joint density $d_n(u,t,x)$ which is the density associated with n claims in $(0,t)$, non-ruin over $(0,t)$, and a surplus of x at time t. From Gerber (1988) we have that

$$d_n(0,t,x) = \frac{x}{ct}\, e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{n*}(ct-x)$$

for $n = 1,2,3\ldots$ and $0 < x < ct$, with $d_0(0,t,x) = e^{-\lambda t}$ for $x = ct$. Adapting arguments in Dickson and Waters (2006) we have

$$d_0(u,t,x) = e^{-\lambda t}$$

for $x = ct$, and for $n = 1,2,3,\ldots$ and $0 < x < u + ct$,

$$\begin{aligned} d_n(u,t,x) &= e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{n*}(u+ct-x) \\ &\quad - c\, I(t > x/c) \sum_{r=1}^{n-1} \int_0^{t-x/c} e^{-\lambda t} \frac{(\lambda t)^r}{r!} f^{r*}(u+cs)\, d_{n-r}(0,t-s,x)\, ds \\ &\quad - I(t > x/c)\, e^{-\lambda t} \frac{\lambda^n (t-x/c)^n}{n!}\, f^{n*}(u+ct-x). \end{aligned}$$

Now let $w_n(u,y,t)$ denote the joint density of the time to ruin (t), deficit at ruin (y), and number of claims until ruin (n). Then following Dickson (2007), we have

$$w_1(0,y,t) = \lambda e^{-\lambda t} f(ct+y)$$

and for $n = 2, 3, 4, \ldots$,

$$w_n(0, y, t) = \lambda \int_0^{ct} \frac{x}{ct}\, e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} f^{(n-1)*}(ct - x) f(x + y) dx.$$

Similarly,

$$w_1(u, y, t) = \lambda e^{-\lambda t} f(u + ct + y)$$

and for $n = 2, 3, 4, \ldots$,

$$\begin{aligned} w_n(u, y, t) &= \lambda \int_0^{u+ct} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} f^{(n-1)*}(u + ct - x) f(x + y) dx \\ &\quad - c \sum_{r=1}^{n-1} \int_0^t e^{-\lambda s} \frac{(\lambda s)^r}{r!} f^{r*}(u + cs)\, w_{n-r}(0, y, t - s)\, ds. \end{aligned}$$

If we define the joint density $w_n(u, x, y, t)$ where we have now introduced the surplus prior to ruin (x), following Dickson (2007) we have

$$w_1(u, x, y, t) = \lambda e^{-\lambda t} f(u + ct + y)$$

for $x = u + ct$, and for $n = 2, 3, 4, \ldots$ and $0 < x < u + ct$,

$$w_n(u, x, y, t) = \lambda\, d_{n-1}(u, t, x)\, f(x + y).$$

8 Concluding remarks

It was convenient to solve for $\phi_{r,\delta}(u)$ in order to find $E[T_u\, N_{T_u}\, I(T_u < \infty)]$ in Section 6. This approach equally applies to other individual claim amount distributions. Similarly, the arguments from Albrecher and Boxma (2005) can be used to obtain a general expression for $E[T_u\, N_{T_u}\, I(T_u < \infty)]$. As no new techniques are involved, we have omitted the details.

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