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Public Good Menus and Feature Complementarity

by

Christian Roessler

Department of Economics
The University of Melbourne
Melbourne Victoria 3010
Australia.

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Christian Roessler[1]
U Melbourne, Dept. Economics

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[1]Address: Department of Economics, University of Melbourne, Parkville, Victoria 3010, Australia. E-mail: roessler@econ.unimelb.edu.au.

Abstract

The distance metric on the location space for multidimensional public good varieties represents complementarity between the good's features. "Euclidean" feature complementarity has atypical strong properties that lead to a failure of intuition about the optimal-menu design problem. If the population is heterogeneous, increasing the distance between two varieties is welfare-improving in Euclidean space, but not generally. A basic optimal-direction principle always applies: "anticonvex" menu changes increase participation and surplus. A menu replacement is anticonvex if it moves the varieties apart in the common line space. The result extends to some impure public goods with break-even pricing and variety-specific costs. A sufficient condition for menus to be Pareto-optimal is that "personal price" (nominal price plus perceived distance from a variety) is linear in the norm that induces the distance metric.

JEL Codes: H41, D78, D71, R13, R12

Although public goods are nominally free, demand (for most) is limited in practice. Selective participation reflects non-price costs (time and travel requirements) and the diversity of tastes. Many public goods are offered in varieties to appeal to different patrons. For example, library branches and parks are established in several neighborhoods to meet the location preferences of dwellers throughout the city. But physical place is often not the only dimension on which varieties are distinct. Libraries may have a unique genre focus and a quaint or modern atmosphere; parks offer their own wildlife and plant life. These features can also be thought of as locations in an abstract space, so that the typical variety is a point in a multidimensional space.

Suppose the population is very heterogeneous: everyone favors a different mix of features; in fact, every point in a vector space is associated with exactly one person who desires the corresponding design. There seems to be an obvious solution to the variety location problem: a diverse population is best served by a diverse menu. A change is welfare-improving whenever it increases the spatial dispersion of varieties. We demonstrate that this intuition is wrong. Even when there are only two varieties, it is sometimes beneficial to make them more similar (reduce their perceived distance). The reason is that closer varieties, with respect to non-Euclidean distance, may attract participation from a larger subset of the population.

The metric of perceived distance determines feature complementarity in a multidimensional design space. There is no special reason why preferences should be represented by a Euclidean space. Euclidean features are one of many intermediate cases between perfect substitute and perfect complement features. To explain this perspective and derive a fundamental "optimal direction" result that isn't sensitive to feature complementarity (choice of metric) are the main contributions of the paper.

Specifically, with two neighboring varieties and personal consumption restricted to the most preferred, a move to a convex combination of the design loci reduces participation and consumer surplus. An anticonvex move (away from each other in the line space containing the designs) increases both.[1] The surprising aspect is that stronger statements are false when feature complementarity is non-Euclidean. Moving the varieties apart in distance (in an arbitrary direction) may be detrimental.

It may seem that any menu change is Pareto optimal when each variety has a patron (since it benefits the type who ideally prefers the new design). Although this reasoning is accurate for pure public goods, it does not extend to all "impure" public goods, which may have nonzero, non-uniform break-even prices. We obtain a sufficient condition: an arrangement where every variety has nonzero participation is efficient when the "personal price" (nominal price plus distance from the variety) is linear in a p-norm.[2]

Our restrictions on individual demand are comparatively weak; they cover a family of linear and nonlinear functions that are monotonic in personal distance from a variety. In our choice of welfare measures we depart from the social choice approach, where one would derive a class of acceptable decision rules axiomatically. Surveys about potential use, estimates of surplus, and the existence of a lobby are the kinds of data that are potentially available to policymakers. Thus participation, consumer surplus, and Pareto efficiency are important in actual public good provision.

Alternative metrics have been explored axiomatically in bargaining and social choice theory. Lehrer and Nitzan [6] asked when a given choice rule could be "rationalized" by a metric on preference profiles.[3] The class of admissible rules may be further restricted by imposing axioms on the metric. For example, Pfingsten and Wagener [11] derive axioms that induce the

[1] By participation we mean cumulative use frequency, rather than the number of users. If a person visits a festival twice, then participation increases by two. Nevertheless, the result would also apply to the user-count definition of participation.

[2] The restriction is violated by the quadratic distance cost model, which is popular in spatial economics.

[3] Consider the set of consensus profiles that unanimously prefer some allocation. An allocation is rationalized by a metric if it is elected by a consensus profile that is closest to the actual profile in terms of the metric. Metric rationalizability of a choice rule is equivalent to the Paretian property (if an allocation is preferred by all, then it is implemented).

p-metrics; Conley et al. [1] and Voorneveld and van den Nouweland [12] characterize the Euclidean metric ($p = 2$).[4] Our approach provides a different and non-normative interpretation of alternate p-metrics. Here they relate to the complementarity of decision criteria that jointly determine preference.

The paper is organized as follows. In Section 1 we discuss an example of the setting we have in mind, show that it has a geometric representation and a natural role for metrics as models of feature complementarity. It is important that the reader become familiar with the geometry of the perfect substitutes case in this section, as we often cite it later on. Section 2 defines a more general class of problems that is characterized by the same geometry and contains the example from Section 1.

If an isolated variety moves in the design space, participation and surplus are unaffected. This intuitive proposition is verified in Section 3. It provides a simple graphic argument for the non-neutrality of welfare with respect to locations of neighboring varieties. The argument appears in Section 4, together with the optimal direction theorem. The theorem covers pure public goods (that are free) and impure public goods (with positive but non-strategic prices). Personal prices are assumed to be linear. Under this condition, menus with nonzero participation (in all varieties) are found to be Pareto-efficient in Section 5; we show how the proof fails otherwise. Section 6 concludes and connects to strands of social choice theory and industrial economics.

[4] Related to metric rationalizability are research designs that exploit the duality between social choice problems and multicriteria optimization. Conley et al. and Voorneveld / van den Nouweland utilize this framework. The problem is to balance multiple objectives when it is not possible to maximize with respect to each simultaneously. The metric aggregates shortfalls in component objectives, and the least costly solution in this sense is chosen.

1 Feature Complementarity in a Spatial Model of Demand

When patrons must choose among designs with multiple attributes, they will weigh a close fit with their tastes in one respect against disagreement in another. Sometimes there is a natural way to do this. In comparing tax systems, voters are concerned with the total they must pay; they should be willing to trade a higher income tax against a lower property tax dollar-for-dollar. Hence the components of a tax system are perfect substitutes. When it comes to government spending proposals, many voters have ethical priors that result in a perfect complements mentality. Suppose the issue is how to allocate funds between health care and unemployment benefits. Often the policies are ranked by comparing the least acceptable element of each.

Consider two churches, of which one offers a dull sermon and an excellent choir, whereas the other has an inspiring pastor and mediocre singing by the congregation. A music enthusiast may prefer the former if he is willing to sit through (substitute) the sermon for the enjoyment of the choir. But if he tends to drowsiness, he may attend the latter service, as it satisfies the minimal requirement of keeping him awake to experience the music. In this example, the patron's choice reveals whether preaching and music are regarded as substitutes or complements.

Technically, preference is determined by how perceived distance from a variety is constructed from distances in single dimensions - in other words, it depends on the metric. One can appreciate visually that different metrics reflect different degrees of complementarity between features. We assume here and throughout the paper that "types" (personally ideal designs) constitute a normed vector space $(\mathbb{R}^n, \|\cdot\|)$, or equivalently:

(A1) Types are continuously and uniformly distributed on $\mathbb{R}^n$, $n \geq 2$.

(A2) "Perceived distance" between types x and y is $\|x-y\|$, where $\|\cdot\|$ is a norm.

The class of metrics covered by (A2) includes the p-metrics

$$d_p(x,y) \equiv \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p},$$

which are induced by the p-norms

$$\|x\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

The spaces associated with these norms are the L^p spaces. We often refer to the extremes L^1 and L^∞ (as we shall see these are, respectively, the "perfect substitutes" and "perfect complements" cases), and to the familiar Euclidean space L^2. The reader is reminded that $p=\infty$ corresponds to the max norm $\|x\|_\infty \equiv \max_{i\leq n}|x_i|$ and thus induces the max metric $d_\infty(x,y) \equiv \max_{i\leq n}|x_i - y_i|$.

With (A1) and (A2) in place, one can define an r-ball about the point a: $B(a,r) = \{x : \|x-a\| \leq r\}$. This is the set of types who find their preferred bundle of features is no farther than a distance r from a. Figure 1 plots this set in two dimensions for L^1, L^2, L^∞, together with an intermediate case between $p=2$ and $p=\infty$.

Since distance from a is constant along the boundary of $B(a,r)$, $|x_2 - a_2|$ must decrease one-for-one with increases in $|x_1 - a_1|$ when the norm is $\|\cdot\|_1$, or stay fixed while $|x_1 - a_1|$ increases when the norm is $\|\cdot\|_\infty$ and $|x_1 - a_1| < |x_2 - a_2|$. Hence the types treat deviations from the ideal point as perfect substitutes in L^1 and as perfect complements in L^∞ (where only a reduction of the worst deviation makes a difference).

This interpretation of metrics has economic content when a ball in L^p is related to demand. For the moment we use a simple example. The next

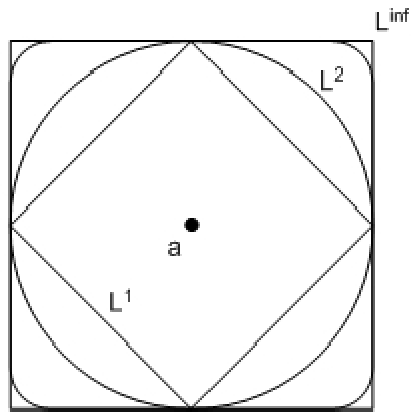

Figure 1: An r-ball with respect to some p-norms

section develops a more general setting for our analysis. Let the optimal quantity (or frequency of use) for type x of variety i be

$$\tilde{q}_{x,i} = \alpha - \beta \left\| x - i \right\| \tag{1}$$

(with α and β constants). We say that x is in the market for design a if $\tilde{q}_{x,a} > 0$. The market is therefore bounded by types $\bar{x}$ such that $\|\bar{x} - a\| = \alpha/\beta$. Letting $r \equiv \alpha/\beta$, the market can be described as the ball $B(a, r)$.

It is often reasonable to assume that individuals consume at most one variety: one cannot visit two parks simultaneously and may always prefer the same one when facing the choice repeatedly over a period of time. To continue with the example, define

$$q_{x,i} = \begin{cases} \tilde{q}_{x,i} & \text{if } \left\| x - i \right\| \leq \min_{j \neq i} \left\| x - j \right\|, \ \left\| x - i \right\| < r \\ 0 & \text{else} \end{cases}, \tag{2}$$

and say that x is in the demand set for design a if $q_{x,a} > 0$. (I.e. if a is strictly the closest variety for x, and $\tilde{q}_{x,a} > 0$.) When varieties are clustered somewhere in the space, a's demand set may be strictly contained in a's

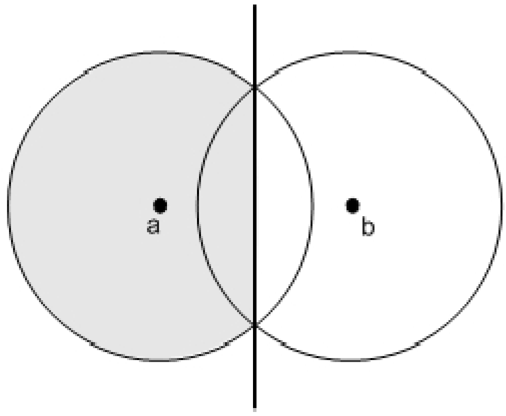

Figure 2: Demand for a in L^2

market. Illustrations in Euclidean and L^1 space follow for two designs a and b. Because these cases reappear throughout the paper, we explain the peculiarities of demand in L^1 with some care.

In Euclidean space (L^2), the set of types who prefer a to b is a halfspace separated by a hyperplane. In Figure 2, a's demand set is then the (shaded) intersection of the market $B(a, r)$ with the halfspace to the left of the line. The hyperplane passes through the intersection of the market boundaries: a shared boundary point $\bar{x}$ satisfies $q_{\bar{x},a} = 0 = q_{\bar{x},b}$, hence $\|\bar{x} - a\| = \alpha/\beta = \|\bar{x} - b\|$, which is indifference in the example. This property of the indifferent set is general (and independent of the metric) when the model is consistent with transitive preferences. If x's opportunity cost of participation (value of not consuming any variety of the good) is c, then if x is indifferent between a and c (in a's market boundary), and between b and c (in b's market boundary), x must be indifferent between a and b.

While the geometry of Euclidean space is rotation-invariant, the same is not true of spaces endowed with other norms. Figure 3 depicts the four cases arising with L^1 and congruent markets.As idiosyncratic as the indifferent sets look, they arise from the same three principles.

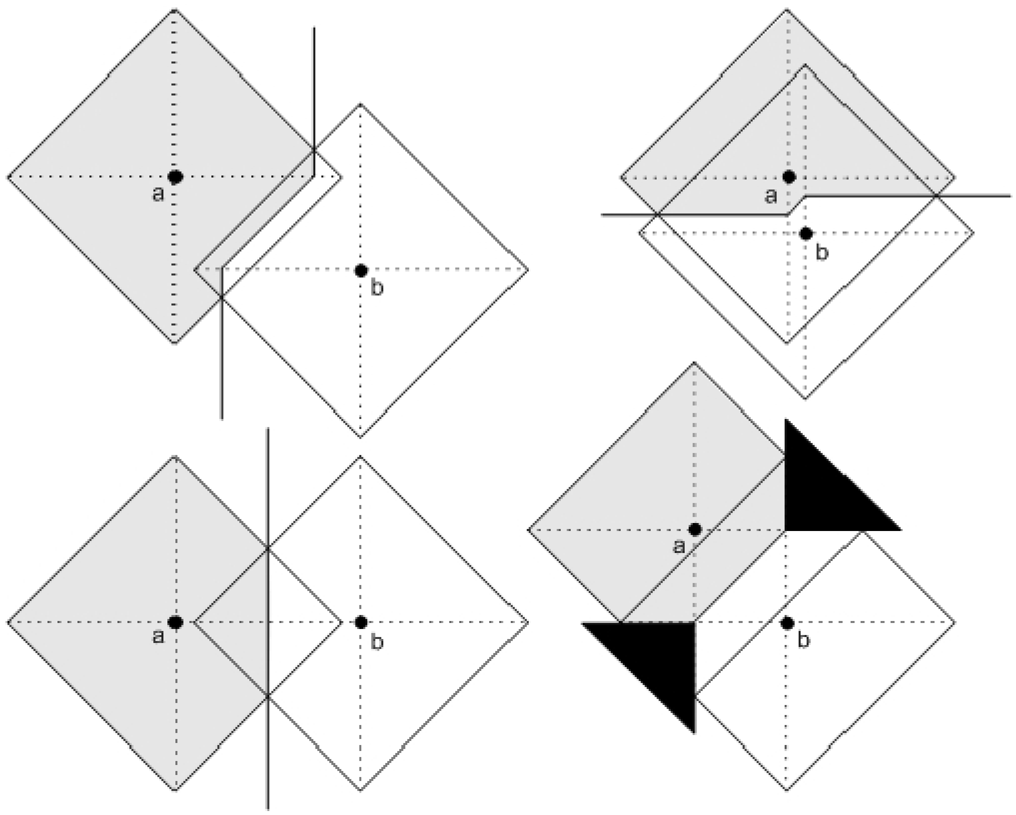

Figure 3: Demand for a in L^1

(1) At points x in the indifferent set such that $\min(a_1, b_1) \leq x_1 \leq \max(a_1, b_1)$ and $x_2 \leq \min(a_2, b_2)$ or $x_2 \geq \max(a_2, b_2)$, a vertical move away from a is also a move away from b, so vertical moves away from center preserve indifference. Analogously, at points x in the indifference set such that $x_1 \leq \min(a_1, b_1)$ or $x_1 \geq \max(a_1, b_1)$ and $\min(a_2, b_2) \leq x_2 \leq \max(a_2, b_2)$, horizontal moves away from center preserve indifference.

(2) At points x in the set such that $\min(a_1, b_1) \leq x_1 \leq \max(a_1, b_1)$ and $\min(a_2, b_2) \leq x_2 \leq \max(a_2, b_2)$, moving away from a in an axis direction entails moving toward b in that direction. To stay indifferent, one must compensate for a step away from a in one dimension with a step toward b in the other. Hence the graph of the indifferent set has slope 1 (is parallel to the edges).[5]

(3) Points such that $x_1 < \min(a_1, b_1)$ and $x_2 < \min(a_2, b_2)$, or $x_1 > \min(a_1, b_1)$ and $x_2 > \min(a_2, b_2)$ all rank a and b identically. Suppose x in this region is indifferent between a and b. Any other point in the region can be reached by a series of steps in the axis directions, where every step is away from a and b or towards both. Such moves cannot change the ordering of a and b; they preserve indifference. Hence if one point in this region is indifferent, *all* of them are. This produces the thick indifferent set in the lower right panel of Figure 3.

Only the lower left panel of Figure 3, where a and b differ in a single dimension, resembles the Euclidean case. At the other extreme, the L^∞ geometry is a 45° rotated version of that in L^1. The p-metrics with $p \in (1, 2)$ and $p > 2$ generate curved indifferent sets that are topological analogues of those in L^1 and L^∞, respectively. The reader will at this point appreciate that the Euclidean geometry is rather special. The paper explores the theme repeatedly in examples from L^1 as we discuss the limits of our results.

[5] This assumes $a \not> b$ and $a \not< b$ (that is, a is located southeast or northwest from b), as in the diagrams. Else the slope is -1.

2 Demand

The piecewise linear demand function (2) belongs to a larger family that shares the same aggregative geometry and is within the scope of our analysis. Let's fix some notation. A menu L lists the locations $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ of varieties the types may exclusively participate in.[6] An individual action is a pair of non-negative quantities or use frequencies $\theta \in \mathbb{R}^2_+$ such that $\theta_a > 0 \Rightarrow \theta_b = 0$ and $\theta_b > 0 \Rightarrow \theta_a = 0$. The choice set is therefore:

$$Q \equiv \; Q_a \cup Q_b \text{ where } Q_i \equiv \{(\theta_a, \theta_b) : \theta_i \geq 0, \theta_{j \neq i} = 0\} \text{ for } i = a, b.$$

We refer to the actual demand of x (the most-preferred element of Q) by $q_x = (q_{x,a}, q_{x,b})$. In order to economize on notation, we'll assume that preference is always strict, so that the maximizers are unique.

The example in Section 1 implicitly referred to a zero-price "pure" public good. The setting can be extended to "impure" public goods, which may be excludable and aren't necessarily free or uniformly priced. They are distinguished from private goods only in that the provider's objective is to maximize a social benefit, rather than profit. Examples of impure public goods include community swimming pools, museums, schools and colleges, and many more. Prices are often set to break even at given costs, or targeted at a specified loss or gain.[7]

[6] The setting generalizes to many varieties, where one would reason inductively from the two-variety case. Beyond the simple extension where every pair has at most one close variety (so that one effectively works with two varieties), things get very complicated.

[7] Technically, our analysis covers situations where pricing is disassociated from variety locations (the paper could be titled "Pure Location Effects"). Even some competitive situations satisfy this criterion. For example, Zhang [13] showed that Bertrand duopolists have an incentive to announce price-matching policies. It is conceivable that firms would never find it optimal to adjust their prices if the product location changes, given the anticipated response from competitors.

We define type x's "personal price" as[8]

$$\delta(x,i) \equiv \|x-i\| + \pi_i,$$

where $\pi_i \in \mathbb{R}$ is variety i's nominal price. If personal price is substituted for distance in (2), the market radius depends on price. The optimal demand for variety i is $\tilde{q}_{x,i} = \alpha - \beta\delta(x,i)$, provided that $\delta(x,i) < \delta(x,j)$. The market is bounded by types $\bar{x}$ such that $\|\bar{x}-i\| = \alpha/\beta - \pi_i$.

Let individual demands satisfy two postulates:

(A3) If $\delta(x,i) \leq \delta(y,i)$ and $\delta(x,i) \leq \delta(x,j)$, then $q_{x,i} \geq q_{y,i}$.

(A4) If $\delta(x,i) = \delta(y,j)$ and $\delta(x,j) = \delta(y,i)$, then $q_{x,i} = q_{y,j}$.

To justify these assumptions, we interpret personal price as a measure of preference. (A3) says: if x prefers variety i more than y does, and moreover does not prefer variety j to i, then x demands more of i than y does. Note that the "moreover" qualification must be made: if x preferred j to i, then x could not also choose i. Yet y, who may be more farther from both i and j, but prefer i, could reasonably choose a nonzero quantity of i. Such points are easy to locate in Figures 2 and 3.

(A4) is an anonymity principle: only distances matter for choice, not individual perspective or language. Suppose if y swapped the names of a and b, y would prefer each as strongly as x. Then we should see $q_{x,a} = q_{y,b}$ (which implies $q_{x,b} = q_{y,a} = 0$, hence $q_x = q_y$). In effect, y mimics the choices of x after "renaming" varieties so that y faces the same problem as x. Figure 4 depicts an instance of (A4). If a and b are locations of (otherwise identical) free-to-enter parks, then x and y have parks at the same driving distances from home. Their visiting patterns should be the same.

[8] In the terminology of spatial economics, δ is the "delivered price." The seemingly more general variant $\delta(x,i) \equiv \|x-i\|\tau + \pi_i$, where τ is constant "transport cost," is only a rescaling since norms satisfy $\|x-i\|\tau = \|x\tau - i\tau\|$.

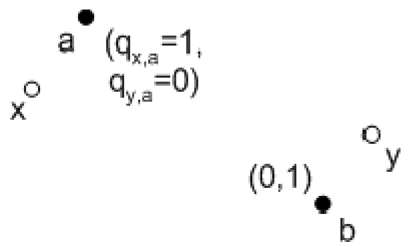

Figure 4: (A4) with equal prices

(A3) and (A4) are usually easy to verify. (A3) holds if $q_{x,i} > 0$ only if $\delta(x,i) \leq \delta(x,j)$ and $q_{x,i}$ is decreasing in $\|x-i\|$ and π_i for $i = a, b$. (A4) requires that $q_{x,i}$ depends on x (only) through $\|x-i\|$ and $\|x-j\|$. The piecewise linear example in Section 1 obviously satisfies these conditions. Many admissible demands could be constructed in a similar fashion, with $q_{x,i}$ nonlinear but monotonic in $\delta(x,i)$.

We demonstrate now that (A1)-(A4) generate demand sets that are consistent with the geometry of Section 1, if a mild nontriviality condition is added:

> (A5) Within a finite distance from i, there exist a type with positive demand for i and a type with zero demand for i and j.

Lemma 1 establishes that x will not choose j over i if x's personal price of i is lower (and a partial converse).

Lemma 1 *(A1)-(A4) imply: if $\delta(x,i) \leq \delta(x,j)$, then $q_{x,i} \geq q_{x,j}$. Conversely, $q_{x,i} > 0$ only if $\delta(x,i) \leq \delta(x,j)$.*

Proof. Let $\delta(x,i) \leq \delta(x,j)$ and consider a "mirror point" y such that $\delta(y,i) = \delta(x,j)$ and $\delta(y,j) = \delta(x,i)$. Such a point exists in a normed vector space; a general proof is found in the appendix (Lemma 7).[9] By

[9] If $\pi_i = \pi_j$ (for example, pure public goods), the argument is simple: let $y = i + j - x$ and confirm the mirror properties by substituting for x in $\|i-x\|$ and $\|j-x\|$.

construction, y satisfies the "if" clause in (A4), hence (i) $q_{x,i} = q_{y,j}$ and (ii) $q_{x,j} = q_{y,i}$. Moreover, since $\delta(y,j) \leq \delta(x,j)$ and $\delta(y,j) \leq \delta(y,i)$, we have (iii) $q_{y,j} \geq q_{x,j}$ by (A3). If $q_{x,j} = 0$, then the lemma holds. Suppose therefore $q_{x,j} > 0$. Then $q_{y,j} > 0$ from (iii), which implies $q_{x,i} > 0$ from (i). But one of $q_{x,i}$ and $q_{x,j}$ must be zero, so there is a contradiction. Infer $q_{x,j} = 0$, hence $q_{x,i} \geq q_{x,j}$.

If $q_{x,i} > 0$, then $q_{x,j} = q_{y,i} = 0$ from (ii), so $q_{x,i} > q_{y,i}$. Suppose $\delta(x,i) > \delta(x,j)$, hence $\delta(x,i) > \delta(y,i)$ and $\delta(y,i) < \delta(y,j)$, by construction of y. (A3) requires $q_{y,i} \geq q_{x,i}$, again a contradiction. Hence $\delta(x,i) \leq \delta(x,j)$.

■

Lemma 1 can be usefully restated if we define the halfspace

$$H_{ij} \equiv \{x : \delta(x,i) \leq \delta(x,j)\}$$

and recall that the demand set for variety i is

$$D_i \equiv \{x : q_{x,i} > 0\}.$$

The converse part of the lemma means $D_i \subseteq H_{ij}$ (since $x \in D_i \Rightarrow x \in H_{ij}$).

Lemma 2 *(A1)-(A5) imply: D_i is nonempty and bounded.*

Proof. Nonemptiness is immediate from (A5). From Lemma 1, $D_i \subseteq H_{ij}$. Suppose z is a point with zero demand for all varieties. If $z \in H_{ji}$ for some $j \neq i$, then a mirror point $z' \in H_{ij}$ with zero demands can be found by Lemma 7 (appendix).[10] Without loss then $z \in H_{ij}$. By (A5), $\|z - i\|$ is finite, by (A3) all $x, z \in H_{ij}$ satisfy $\delta(z,i) \leq \delta(x,i) \Rightarrow q_{z,i} \geq q_{x,i}$. Therefore $q_{x,i} = 0$ for all x beyond some finite distance from i. It follows that $D_i = \{x : q_{x,i} > 0\}$ is bounded.

■

[10] The point z' satisfies $\delta(z',i) = \delta(z,j)$ and $\delta(z',j) = \delta(z,i)$ by construction. Therefore $\delta(z,i) \geq \delta(z,j) \Leftrightarrow \delta(z',i) \leq \delta(z',j)$, or: $z \in H_{ji} \Leftrightarrow z' \in H_{ij}$.

The payoff from Lemmas 1 and 2 is the following characterization of demand sets in our model. It provides the geometric basis for our subsequent arguments.

Theorem 3 *(A1)-(A5) imply:* $D_i = B(i, r) \cap H_{ij}$ *for some* r.

Proof. Since D_i is bounded by Lemma 2, $r = \sup\{\|x - i\| : q_{x,i} > 0\}$ exists. Because r is an upper bound, $x \in D_i$ ($\iff q_{x,i} > 0$) only if $\|x - i\| < r$ ($\iff x \in B(i, r)$). Hence $D_i \subseteq B(i, r)$. From the converse of Lemma 1, $D_i \subseteq H_{ij}$; thus (i) $D_i \subseteq B(i, r) \cap H_{ij}$.

If $x \in B(i, r) \cap H_{ij}$, then $\|i - x\| < r$ and $x \in H_{ij}$. Suppose $x \notin D_i$, i.e. $q_{x,i} = 0$. There exists $x' \in H_{ij}$ such that $\|i - x\| < \|i - x'\| < r$, so by (A3) $q_{x',i} = 0$. Now, for all $x'' \in H_{ij}$ with $\|i - x''\| \geq \|i - x'\|$, (A3) implies $q_{x'',i} \leq q_{x',i} = 0$. But then $\|i - x'\|$ is a bound of $\{\|x - i\| : q_{x,i} > 0\}$, and r is not the least upper bound, a contradiction. We conclude $x \in D_i$, i.e. (ii) $B(i, r) \cap H_{ij} \subseteq D_i$, which proves the claim.

■

When (A4) applies, choices are equivalent in that x and y face essentially the same problem and resolve it in the same way. If $s_x : Q \to \mathbb{R}$ and $s_y : Q \to \mathbb{R}$ are surplus measures representing preferences of x and y,[11] it may be reasonable to impose:

(A6) Under the conditions of (A4), $s_x(q_x) = s_y(q_y)$.

Strengthening (A4) in this way is plausible because the personal prices of designs that x and y consume in equal quantities don't differ. Then the choices should be valued the same. A more cautious view is that analogous choices only reflect the same preferences. Types who face similar problems may agree on what is best, but they need not feel the same way about the outcome. Then (A6) is not appropriate; one should only make the ordinal assertion (A4) that choices are *ranked* similarly in similar situations.

[11] Specifically, $s_x(\theta) \geq s_x(\theta') \Rightarrow \theta \succsim_x \theta'$ and $s_x(0) = 0$.

We refer to (A1)-(A5) as the weak system $\mathcal{W}$. In this system we can prove theorems about participation. In order to extend them to theorems about surplus, we need the strong system (A1)-(A6), which we call $\mathcal{S}$.

3 Isolated Varieties and Conservation of Aggregates

We begin by studying the simplest possible situation that is relevant to our problem: an isolated variety. Moving such a variety to a new location in the design space turns out to be welfare-neutral. This should not come as a surprise, since the uniformly populated space has no "special" points, and utilitarian welfare measures like participation and surplus place equal weight on all members of the population.

We are interested in two benefit measures for menus $L = (a, b)$ of pure public good varieties. The first is total participation, defined as the sum of use frequencies by all types who belong to a demand set:

$$P(L) \equiv \int_{x \in \cup_{i \in L} D_i} q_{x,i} dx.$$

In practice, a planner often has this criterion in mind, given that personal enjoyment is typically not observed. One could think of participation as a "revealed" preference indicator. In applied economics, surplus measures that quantify actual value creation are important. Hence we also consider the sum of individual surpluses:

$$V(L) \equiv \int_{x \in \cup_{i \in L} D_i} s_x(q_x) \, dx.$$

If a moves to a' in isolation, there exists a bijective mapping between $D_a = B(a, r)$ and $D_{a'} = B(a', r)$ such that the types paired by the mapping are equidistant from the varieties they consume, and so add equal participation and surplus by (A4) and (A6).

Lemma 4 ("Conservation Lemma") *Suppose a and a' are points in a normed vector space; $B(a,r)$ and $B(a',r)$ are r-balls about these points. There exists a bijective mapping $f : B(a,r) \to B(a',r)$ such that $f(x) = y$ only if $\delta(y,a) = \delta(x,a')$ and $\delta(x,a) = \delta(y,a')$. The restriction $f : B(a,r) \cap H_{aa'} \to B(a',r) \cap H_{a'a}$ has the same property.*

Proof. Our argument is by construction. Let

$$f(x) = a + a' - x. \tag{3}$$

We claim that f is a bijective map on (i) $B(a,r)$ and (ii) $B(a,r) \cap H_{aa'}$, into $B(a',r)$ and $B(a',r) \cap H_{a'a}$ respectively. In addition, we claim that (iii) $f(x) = y$ only if $\|y-a\| = \|x-a'\|$ and $\|y-a'\| = \|x-a\|$. Since $\pi_a = \pi_{a'}$, (iii) leads to $\delta(y,a) = \delta(x,a')$ and $\delta(y,a') = \delta(x,a)$.

Using (3) to substitute for x and $f(x)$, we have

$$\|x-a\| = \|a' - f(x)\|, \quad \|f(x) - a\| = \|a' - x\|. \tag{4}$$

At this point, (iii) follows from the axiomatic property of norms that $\|\alpha x\| = |\alpha| \|x\|$ (let $\alpha = -1$).

To confirm that $f(x)$ is one-to-one, it's enough to show that it belongs to the specified co-domains. Consider the domain (i) $B(a,r)$. It is immediate from $\|x-a\| = \|f(x) - a'\|$ that $\|x-a\| \leq r \Rightarrow \|f(x) - a'\| \leq r$, so $f(x) \in B(a',r)$.

Moreover, since

$$\|x-a\| - \|x-a'\| = \|f(x) - a'\| - \|f(x) - a\|$$

follows from (4) and implies $\|x-a\| \leq \|x-a'\| \Leftrightarrow \|y-a'\| \leq \|y-a\|$, we have $x \in H_{aa'} \Leftrightarrow f(x) \in H_{a'a}$. Therefore $x \in B(a,r) \cap H_{aa'}$ entails $f(x) \in$

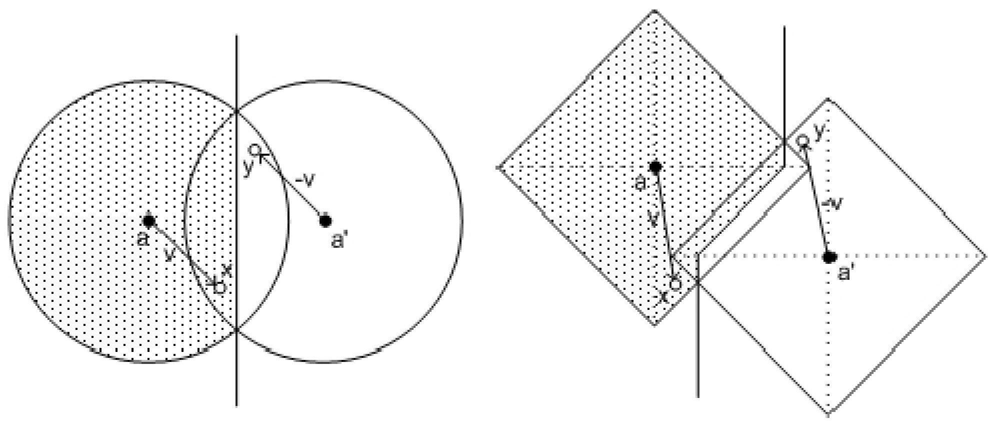

Figure 5: "Conservation Lemma" illustrated for $p = 2$ and $p = 1$

$B(a', r) \cap H_{a'a}$, so that f is also one-to-one on the domain (ii) $B(a, r) \cap H_{aa'}$. Because the inverse of f is one-to-one by analogous reasoning, f is onto and bijective.

■

Figure 5 illustrates the proof for $\|\cdot\|_1$ and $\|\cdot\|_2$. Intuitively, Lemma 4 reflects the translation invariance of norms: the ball $B(a', r)$ can be constructed from $B(a, r)$ by subtracting $a + a'$ from every $x \in B(a, r)$ (and multiplying by -1). Equivalently, observe that any $x \in B(a, r) \cap H_{aa'}$ (the dotted area in Figure 5) is associated with a unique vector v such that $x = a + v$. A point $y \in B(a', r) \cap H_{a'a}$ at the same distance from its preferred design can be constructed as $y = a' - v$. Such a mapping is bijective and can be extended to the whole ball $B(a, r)$. While the result is quite intuitive in Euclidean space, it holds in all normed vector spaces (even if the indifferent set is thick somewhere).

Applying (A4), the paired types x and y have equivalent demands: $q_{x,a} = q_{y,b}$. If (A6) is enforced, this implies x and y gain equal satisfaction. Integrating over the demand sets $D_a = B(a, r)$ and $D_{a'} = B(a', r)$, which are bijective images of one another by Lemma 4, must yield the same aggregates:

$$P(a) = \int\limits_{x \in D_a} q_{x,a} dx. = \int\limits_{y \in D_{a'}} q_{y,a'} dy = P(a')$$

and

$$V(a) \equiv \int\limits_{x \in D_a} s_x(q_x)\, dx. = \int\limits_{y \in D_{a'}} s_y(q_y)\, dy = V(b)\,.$$

Location changes of an isolated variety are therefore welfare-neutral with respect to participation and surplus.

The same statement applies to the integrals over $B(a,r) \cap H_{aa'}$ and $B(a',r) \cap H_{a'a}$. Collected in the first of these sets are types who are disadvantaged when a is replaced by a'; the other set contains the beneficiaries. The conservation lemma implies that losses and gains balance.[12] This is the key insight we take to the analysis of neighboring varieties.

4 Participation and Surplus with Neighboring Varieties

The argument that market overlap is wasteful is illustrated in Figure 6. Moving from a toward b, say to a', generates three regions: (1) neutral types who neither gain nor lose because of the move. This is the striped area. Note that anyone who originally consumed b and still does after the move must be neutral. (2) Types who are better off. This includes anyone who switches from b to a' (they wouldn't do so unless they could now do better than b) and anyone now in the demand set of a' who is closer to a' than to a (i.e. to the right of the dotted equistance line). This is the grey area. (3) Types

[12] There is a nice duality with *competition* among two varieties at a and b. The types who end up closer to the design locus when it moves from a to $a' = b$ are the ones who choose b over a when both are available. Hence the benefits created by two competing designs are equal.

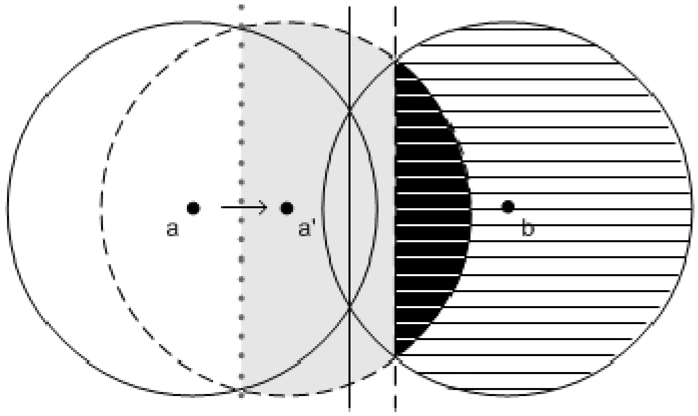

Figure 6: Closer varieties in Euclidean space

who are worse off: anyone who originally consumed a and now drops out of the market or is more distant from a' than from a. This is the solid white area.

The conservation lemma says that the portion of the ball $B(a,r)$ that lies in the halfspace $H_{aa'}$ (to the left of the dotted equidistant line) creates the same benefits as the portion of the ball $B(a',r)$ that lies in the complementary halfspace $H_{a'a}$ (to the right of the equidistance line). But some of $B(a',r) \cap H_{a'a}$ is absorbed into b's demand set (the blackened part of the striped area). In other words, some types who benefit from the move because they are close to a' do not realize the benefit by consuming a'. Instead they continue to prefer b and the benefit is lost. This makes the move toward b welfare-reducing.

These observations suggest it is never a good thing for varieties to move closer together. Consider moves from a that maintain the distance to b. Figure 7 depicts the Euclidean and L^1 cases in the top and bottom panels.

In the Euclidean case, benefit and loss areas correspond.[13] But in L^1

[13] Although the conservation lemma does not say that there exists a bijection from $B(a,r) \cap H_{aa'} \cap H_{ab}$ to $B(b,r) \cap H_{a'a} \cap H_{a'b}$, the claim would be true (only) in Euclidean space if $\|b-a\| = \|b-a'\|$. Then b belongs to the equidistance line between a and a',

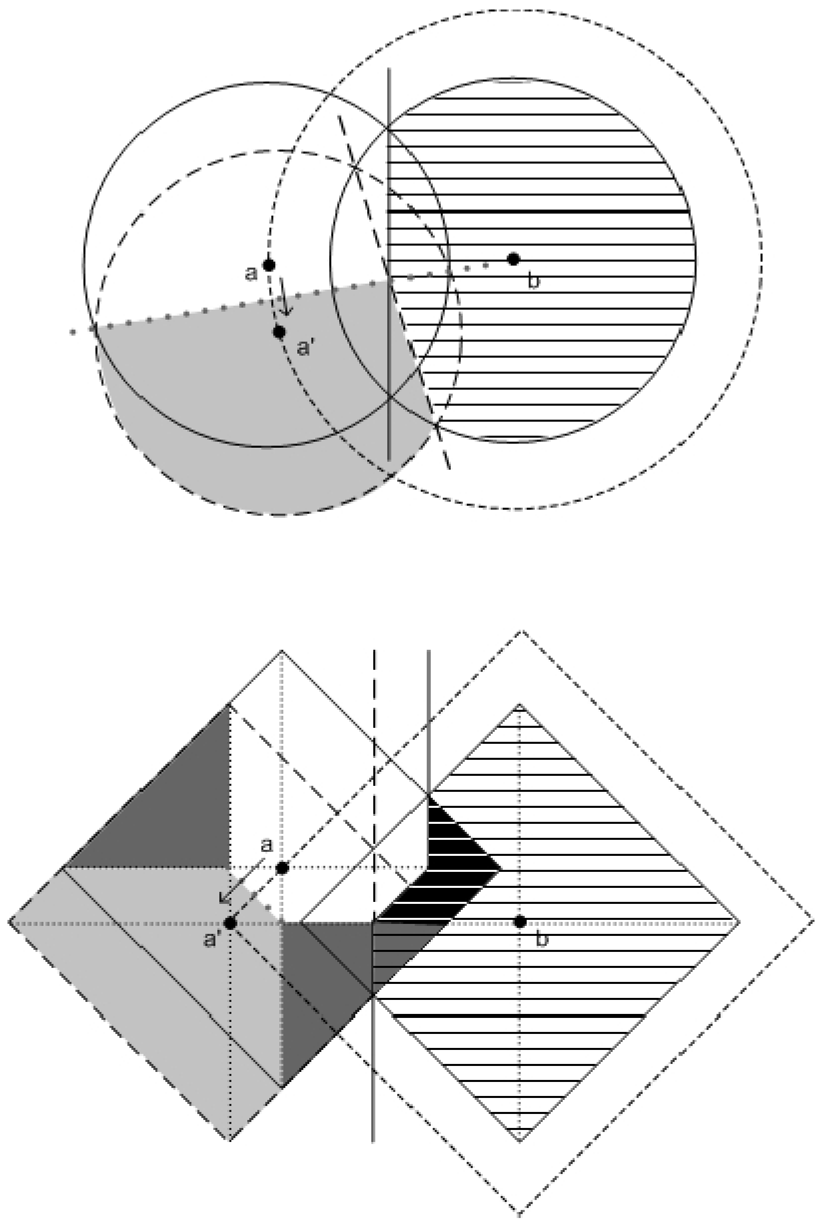

Figure 7: Distance-preserving moves of a in L^2 and L^1

the move to a' causes an increase in the gain area (corresponding to the black striped region), while the distance from b has not changed. According to the conservation lemma, the light grey area and the union of the white and black striped area yield the same participation and surplus. Since only the white area favors the design at a, and it can be mapped to an identical (strict) subset of the grey area, which prefers a', the change is strictly welfare-improving. There are nearby moves such that $\|a' - b\| < \|a - b\|$ that are still beneficial.

So it's possible for closer varieties to be strictly preferred by society. Welfare is not directly linked to menu diversity. Instead we have a weaker theorem which, combined with the special symmetry of Euclidean spaces, implies that welfare *is* distance-monotonic in L^2.

Theorem 5 *Suppose $L = (a, b)$ and $L' = (a', b')$, where a' and b' are strict convex combinations of a and b. In $\mathcal{W}$, $P(L) > P(L')$; in $\mathcal{S}$, $V(L) > V(L')$.*

Proof. $D_{a'} \cup D_{b'} \subset D_a \cup D_b$ implies $P(L) > P(L')$ in $\mathcal{W}$ and $\mathcal{S}$, and $V(L) > V(L')$ in $\mathcal{S}$.[14] Therefore it's sufficient to show: if $a' = \lambda_1 a + (1 - \lambda_1) b$ and $b' = \lambda_2 a + (1 - \lambda_2) b$ with $\lambda_1, \lambda_2 \in (0, 1)$, then $D_{a'} \cup D_{b'} \subset D_a \cup D_b$. Since $D_a \cup D_b = B(a, r) \cup B(b, r)$ and $D_{a'} \cup D_{b'} = B(a', r) \cup B(b', r)$, we have

$$\begin{aligned} D_{a'} \cup D_{b'} \subset D_a \cup D_b &\Leftrightarrow B(a, r) \cup B(b, r) \subset B(a', r) \cup B(b', r) \\ &\Leftrightarrow B(a', r) \cap B(b', r) \subset B(a, r) \cap B(b, r). \end{aligned}$$

Given $B(a, r)$ and $B(b, r)$, denote the direction from a toward b by the vector $d \equiv (b - a) / \|b - a\|$. Then every point $x' \in B(b, r)$ can be written as $x' = x + \kappa d$ for some $x \in B(a, r)$ and fixed $\kappa = \|b - a\|$. In particular, $y' = y + \kappa d$ in the boundary of $B(b, r)$ is the image of y in the boundary

which is an axis of symmetry between $B(a, r) \cap H_{aa'} \cap H_{ab}$ and $B(b, r) \cap H_{a'a} \cap H_{a'b}$.

[14] Since all types in $D_a \cup D_b$ have positive demands by definition, $D_{a'} \cup D_{b'} \subset D_a \cup D_b$ entails $P(L) > P(L')$. Moreover, incremental surplus from these demands must be non-negative; else they are not optimal choices. Thus $V(L) > V(L')$.

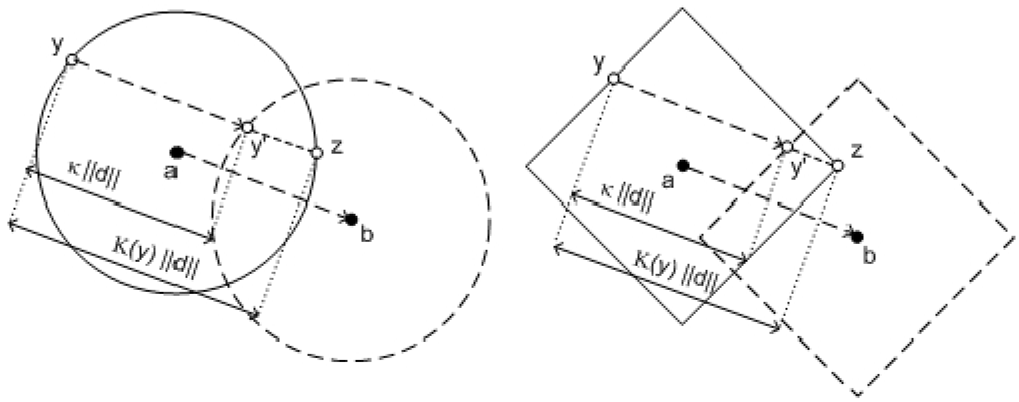

Figure 8: Construction of y' and z in L^2 and L^1 space

of $B(a,r)$. For a boundary point y of $B(a,r)$, let $z = y + K(y)\,d$ be the farthest boundary point of $B(a,r)$ in direction d such that $K(y) \geq 0$. See Figure 8 for examples in L^2 and L^1.

Observe that $B(a,r) \cap B(b,r)$ is the union of convex combinations of y' and z, for all y such that $\kappa \leq K$. So[15]

$$B(a,r) \cap B(b,r) = \left\{ \begin{array}{c} y - (\lambda\kappa + (1-\lambda) K(y))\, d : \\ \lambda \in [0,1], \ \kappa \leq K(y), \ \|a - y\| = r \end{array} \right\}.$$

If κ is larger, fewer y satisfy $\kappa \leq K(y)$; note that κ increases in the distance between a and b. If a' and b' are convex combinations of a and b, then $\|b - a\| \geq \|b' - a'\|$. Hence κ' in the location arrangement L' is smaller than κ in L and enlarges the intersection $B(a', r) \cap B(b', r)$. Therefore $B(a,r) \cap B(b,r) \subseteq B(a', r) \cap B(b, r)$, which is sufficient.

■

The idea of the proof is that $B(a,r) \cap B(b,r)$ increases as a approaches b along a *given* line. Therefore the welfare loss from market overlap increases when a' is a convex combination of a and b (and $b' = b$ or b' is itself a convex

[15] $\lambda y' + (1-\lambda) z = \lambda (y - \kappa d) + (1-\lambda)(y - K(y)\, d)$ reduces to $y - (\lambda\kappa + (1-\lambda) K(y))\, d$.

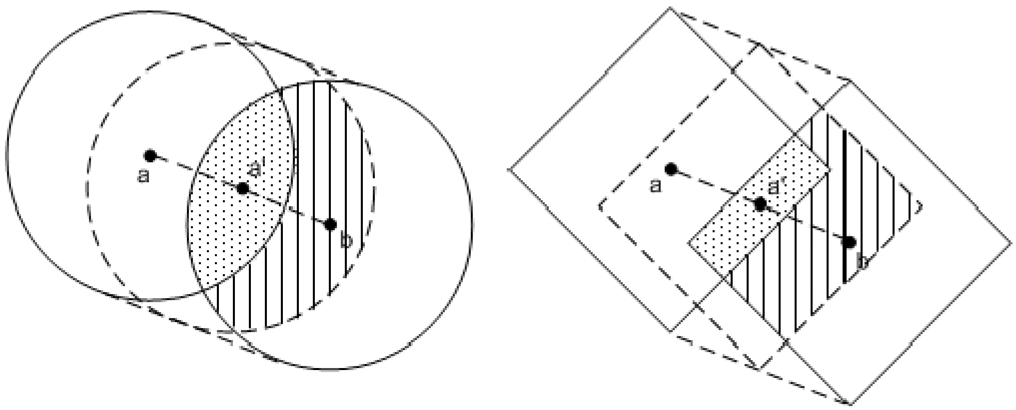

Figure 9: $B(a,r) \cap B(b,r)$ increases if $\|a-b\|$ is reduced along $\overline{ab}$

combination of a and b). In Figure 9, the striped area is the increase in overlap from the move to a'.

The perfect symmetry of a ball in Euclidean space implies that "direction doesn't matter": for an arbitrary a', every $\tilde{a}$ such that $\|\tilde{a}-b\| = \|a'-b\|$ generates identical intersections and distance relationships. So if the distance between a and b is reduced in any manner, there is an equivalent move that satisfies Theorem 5, hence is participation- and surplus-reducing.[16] However, other spaces do not exhibit such symmetry; the same argument does not apply.

In general, a change to a public good menu is welfare-improving if it differentiates all neighboring varieties equally in all features. Does the town folk benefit if, a short walk from the historic center, an old square is renovated in modern style? Not necessarily. If access and flair are substitutes, some patrons who used to come to the square from several blocks away lose interest.

[16] More precisely, in Euclidean space every arrangement $L' = (a', b')$ such that $\|a'-b'\| = \left\|\tilde{a}-\tilde{b}\right\|$ can be obtained from $\tilde{L} = \left(\tilde{a}, \tilde{b}\right)$ by a linear mapping and rotation about $\tilde{b}$. These are isomorphisms in a vector space. In particular, if $\|\tilde{a}-b\| < \|a-b\|$, we can choose a convex combination $L' = (a', b')$ of $L = (a, b)$, so Theorem 5 implies $\tilde{L}$ is welfare-reducing.

A few walk to the center instead; others just stay at home. There is a new clientele who likes the modern design. But some of these people live near the center and continue to go there. The balkers may not be fully replaced, so that there is welfare loss from the move toward menu diversity. A solution is to renovate a different square that is farther from the center, that is develop diversity in a balanced fashion.

5 On the Possibility of Pareto-Dominated Designs

A minimal efficiency requirement for a menu change is that it does not make everyone worse off (and someone strictly so). By (A5) each variety has at least one patron.[17] It is tempting to assert that every menu with a fixed number of varieties is Pareto-optimal. Any alternative arrangement must differ in at least one variety's location, and there is a type who ideally preferred the original variety. However, this type is not necessarily harmed. Consider Figure 10, set in a Euclidean space with $q_{x,i} = \alpha - \beta\delta(x,i)$, $\delta(x,i) = \|i - x\|^2 + \pi_i$, and $\pi_a > \pi_b$.

The move from a to a' *is* inefficient by the Pareto-criterion (the reverse move would be Pareto-improving). Every type who demands b before and after a's location change is neutral (in the striped area). In the drawn scenario, the set of indifferent types (the solid black line) shifts to the left (the broken black line) after the move.[18] No one switches from b to a. Every participant who is closer to a' than to a (to the right of the dotted grey line) previously demanded b. (This set includes a's target a'.) Therefore, the move has no beneficiaries. But every participant in the white region is strictly worse off.

The example depends critically on the restriction that individuals con-

[17] This property is of course preserved by, hence consistent with, anticonvex adjustments.

[18] The indifferent set is linear with $q_{x,i}$ quadratic in distance between x and i; this is a standard case in spatial and industrial economics.

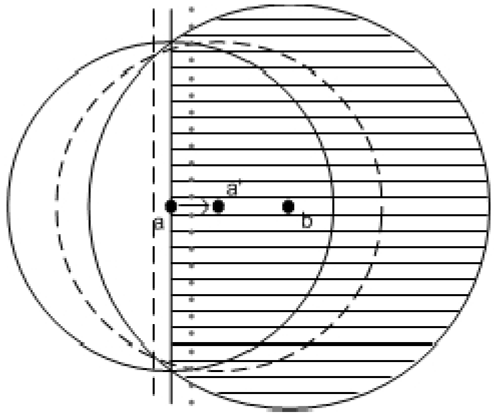

Figure 10: Pareto-domination in L^2 with δ convex in $\|\cdot\|$

sume only one variety, for else a' must benefit from having the ideal design provided. It also requires b to be cheaper. When can we rule out Pareto-domination if prices are possibly non-uniform? It turns out that linearity of personal prices, which we already imposed, is enough.

Theorem 6 *If design i attracts nonzero participation, then it cannot be Pareto-dominated if the personal price δ is linear in the norm: $\delta(x,i) = \|x-i\| + \pi_i$.*

Proof. To synchronize with our discussion, let $i = a$. A move from a to a' is a Pareto-improvement if $\|x-a'\| \le \|x-a\|$ for all $x \in D_a$. If $a \in D_a$, then $\|x-a\| = 0$ at $x = a$ and the inequality holds only if $a' = a$, i.e. there was no move. Hence Pareto improvement is only possible if $a \notin D_a = B(a,r) \cap H_{ab}$. Suppose then that $a \notin H_{ab}$. So $\delta(a,b) < \delta(a,a) \iff \pi_b + \|a-b\| < \pi_a + \|a-a\| = \pi_a$, which leads to[19]

$$\text{(i)} \ \|a-b\| < \pi_a - \pi_b.$$

[19] The inequality can be taken to be strict because else there exists a point y in $B(a,r) \cap H_{ab}$ such that $\|y-a\|$ is arbitrarily small, hence a' must be arbitrarily close to a (or y suffers by the change); in the limit, there is no move.

Now take $x \in B(a,r) \cap H_{ab}$, which is nonempty by nonzero participation. Since x chooses a over b, $\delta(x,b) \geq \delta(x,a) \iff \pi_b + \|x-b\| \geq \pi_a + \|x-a\|$, hence:

$$\text{(ii)}\ \ \|x-b\| - \|x-a\| \geq \pi_a - \pi_b.$$

With (i), (ii) implies

$$\|x-b\| - \|x-a\| > \|a-b\|. \tag{5}$$

Since norms have the triangle property $\|x-b\| \leq \|x-a\| + \|a-b\|$, (**??**) is not possible. Thus nonzero participation implies $a \in B(a,r) \cap H_{ab}$. Every move makes the type at a worse off, so design a is not Pareto-dominated.

■

What happens in Euclidean space with a linear personal price δ? The indifferent set is nonlinear, as in the left panel of Figure 11, and a' is always in $H_{a'b}$. This creates the shaded area of types benefiting from the move to a', hence it is not Pareto-dominated.[20] A corresponding example for L^1 is displayed in the right panel of Figure 11. The reader may check that the indifference sets are drawn in accordance with the principles in Section 1.[21] Again $a' \in H_{a'b}$, so it cannot be dominated.

Some other scenarios in L^1 appear in Figure 12. If $B(a,r)$ is strictly contained in $B(b,r)$, then $D_a = \varnothing$, so Theorem 6 holds trivially in the absence of participation in a.[22]

---

[20] To see where the proof of Theorem 6 fails if δ is nonlinear in the norm, return to the "quadratic model" in Figure 10. The analogues of (i) and (ii), from $\delta(a,b) > \delta(a,a)$ and $\delta(x,b) \geq \delta(x,a)$, are: (i) $\|a-b\|^2 < \pi_a - \pi_b$ and (ii) $\|x-b\|^2 - \|x-a\|^2 \geq \pi_a - \pi_b$. Unlike in the case of a linear personal price, (i) and (ii) do not always conflict. For some x (that satisfy $\|x-b\| > \|x-a\|$), $\|x-b\|^2 - \|x-a\|^2 > \|a-b\|^2$ is consistent with the triangle property of norms $\|x-b\| \leq \|x-a\| + \|a-b\|$.

[21] In particular, the dark grey triangles connected by a dotted line mark the types who are equidistant from a and a'. The solid black line is the indifferent set between a and b; the partially coinciding broken black line that between a' and b.

[22] For example if $\pi_a > \pi_b$ and $a = b$, every type is equidistant from a and b and prefers the cheaper variety (b).

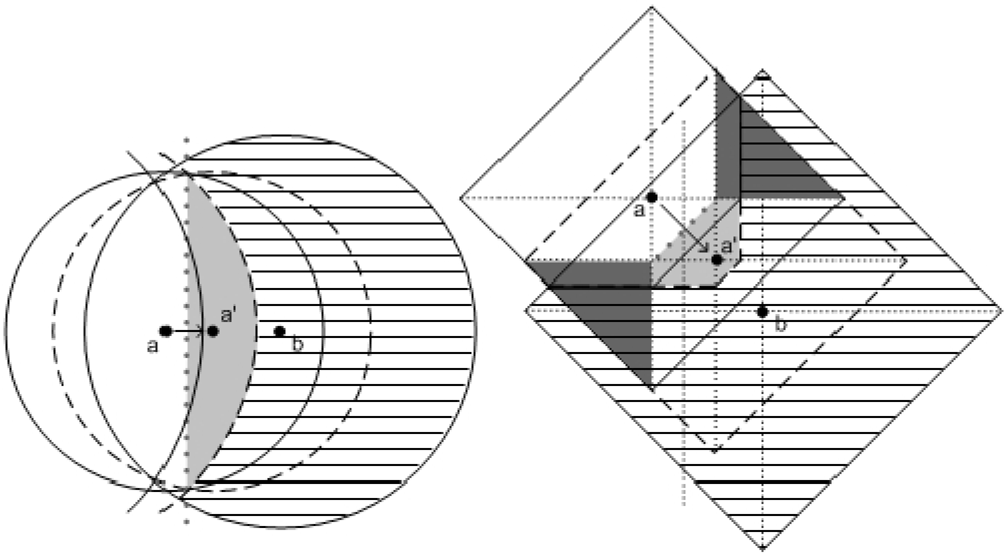

Figure 11: No Pareto-domination with δ linear in $\|\cdot\|$, L^2 and L^1

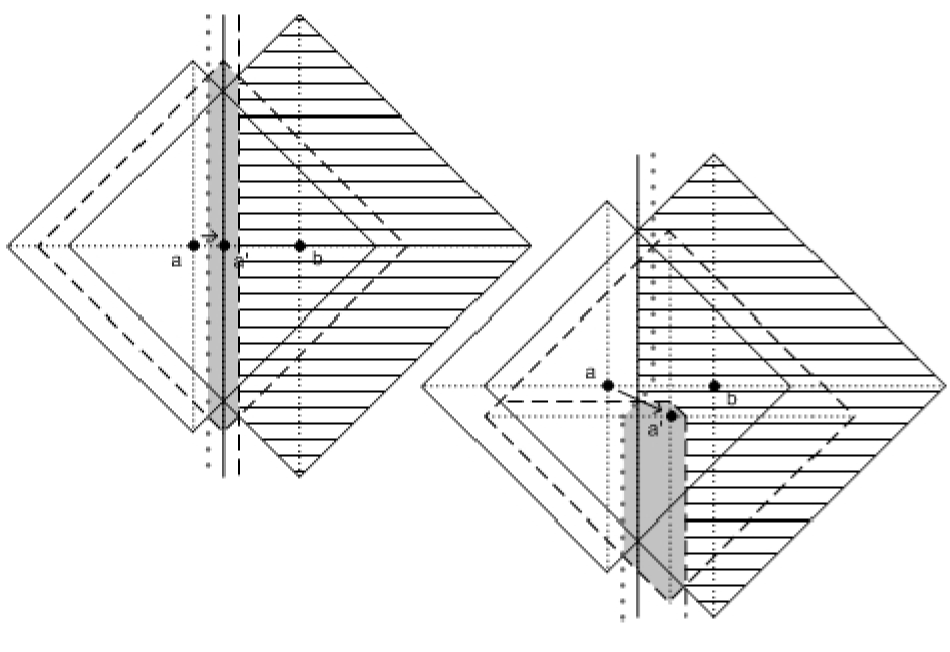

Figure 12: Additional examples of non-domination in L^1

6 Conclusions with Reference to Literature

The population we model represents maximally heterogeneous tastes. Intuitively one expects that a public good menu with a given number of varieties will yield greater benefits (in terms of participation or surplus) if varieties are more dispersed. The conjecture is flawed: menus with neighboring varieties (that "compete" for the same types) cannot be ranked by the distance criterion. Euclidean spaces are a unique exception: if varieties move apart in the Euclidean metric, participation and surplus increase. The paper can be understood to demonstrate the special nature of the Euclidean example, which tends to provide intuition. We argue that Euclidean spaces are no more plausibe than others; they simply reflect an intermediate complementarity between features.

Welfare-monotonicity of anticonvex menu changes is ultimately consistent with the intuition that maximally dispersed varieties are a solution to the menu design problem. But the optimal direction from any *given* arrangement is of practical interest when the planner is budget-constrained and adjustments are costly. One wants to achieve the greatest welfare improvement with a small outlay on a particular variety. Formally the question is: in which direction should it be relocated? The benefit measure may be participation or surplus, and the change should be efficient in the Pareto sense (not everyone is against it).

Any "anticonvex" move (away from the neighboring variety along the shared line) increases participation and surplus (and is Pareto-optimal if personal prices are linear).[23] The most interesting aspect is perhaps that

[23] Menu design with multiple neighboring varieties is a much more difficult problem. The optimal direction is a weighted sum of bilateral anticonvex directions, but the weights cannot be equal unless individual demand functions are binary. With general individual demand, it is not enough to minimize the total intersection with other markets. Patrons who are closer to the locus of the rejected variety have greater potential use frequencies and therefore fail to appropriate larger benefits. They cannot be traded one-for-one against more remote patrons.

stronger claims are false. In practice, features are not equally malleable. A park's physical location can only be adjusted at prohibitive cost, but adding a playground may be possible. (Note that this is not a quality change. Some patrons may want quietude, so the playground is not universally preferred.) Our analysis cautions against the hypothesis that moving away from other varieties (neighboring parks that have no playgrounds) in one feature is welfare-improving. Say the park in question is also unique in offering a rose bed. If quietude and the scent of roses are complements, the playground could deter more visits than it attracts.

Related literatures on public good menus (or monopoly-provided private good menus) almost unanimously support the notion that maximal dispersion is optimal. This is not surprising or inconsistent with our findings, as these models are confined to one-dimensional design spaces, where metrics are equivalent and feature complementarity is not an issue. One-dimensional spaces are meant to be consistent reductions of more realistic multidimensional settings. But we have demonstrated that subtleties are lost in the simplification, and induction from one-dimensional spaces can be quite misleading. It may be worthwhile to reexamine formally similar research under this aspect.

Welfare-maximizing locations of public good varieties have been discussed in the context of social choice theory. Faced with a menu of alternatives (points in an interval), individuals select the preferred one. There exists a "peak" point for every individual such that closer alternatives to the peak are always preferred. The problem is to find menus that satisfy efficiency and consistency criteria. Miyagawa [8] showed that the only solutions that satisfy both Pareto optimality and a fairness restriction for two varieties are the "left-peaks" and "right-peaks" rules. They place the varieties at the two smallest (distinct) peaks or the two largest (distinct) peaks in the population. Ehlers [3], [4] modified Miyagawa's problem and found support for the "extreme-peaks" rule which places varieties at the smallest and largest locations that are peaks in the population. This is also the only admissible rule if Nash's and Arrow's independence axioms are imposed instead of fairness (Ehlers [2]). The "extreme-peaks" rule is similar to offering two remote designs; it formalizes the basic intuition one has about the menu design problem.

In ranking fiscal policies, all individuals prefer a high-quality public good. Yet they are actually offered a bundle of service and personal cost. Valuation of services may vary, so there can be disagreement about the ideal level of taxation. In theory, individuals move to the jurisdiction where the most acceptable tax policy is in force or achievable through voting. This is formally equivalent to participating in the preferred variety of a public good. Perroni and Scharf [10] study this problem with individually preferred tax policies distributed uniformly on the real line. Jurisdictions are elements of a partition of the line, hence everyone participates in one variety of the public good. In equilibrium, jurisdictions are of equal size and select the median policy by majority voting. Hence varieties are evenly spaced along the line; this is surplus-optimal given jurisdiction size.

The same type of result is common in the industrial economics literature on multi-store monopoly. Under typical assumptions, one location arrangement is no more costly to the monopolist than another, for a fixed number of plants. If consumers bear the transport cost and the monopolist can partially appropriate the benefits of reducing it, the monopolist places the plants as a planner would. If consumers are uniformly distributed in a one-dimensional space, as in Katz [5], Pal and Sarkar [9], or Matsumura [7], even spacing of stores occurs in equilibrium. This is another "maximum dispersion" result with many varieties (infinitely many if the line is unbounded). Under cost constraints and with a history, the menu may have a fixed number of varieties with prior locations that can only be changed gradually and slightly. Our analysis shows that moving *toward* even spacing (maximum dispersion) is not necessarily best.

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A Appendix: Existence of a Mirror Point

Lemma 7 *For every x in the normed vector space $(\mathbb{R}^n, \|\cdot\|)$ there exists y such that $\delta(y,i) = \delta(x,j)$ and $\delta(y,j) = \delta(x,i)$.*

Proof. Using the definition of δ, the claim is that there exists y with the properties

$$\|y-i\| = \|x-j\| - \pi_i + \pi_j \quad (6)$$
$$\|y-j\| = \|x-i\| + \pi_i - \pi_j. \quad (7)$$

Equivalently, the intersection of the closed balls

$$Y_i = \{\tilde{y} : \|\tilde{y}-i\| \leq \|x-j\| - \pi_i + \pi_j\}$$

and

$$Y_j = \{\tilde{y} : \|\tilde{y}-j\| \leq \|x-i\| + \pi_i - \pi_j\}$$

is nonempty. Refer to Figure 13 for an illustration with the Euclidean metric.

Consider the line space through i and j, i.e. the affine combinations $\tilde{y}(\lambda) = \lambda i + (1-\lambda) j$, $\lambda \in \mathbb{R}$. Elements in the intersection with Y_i satisfy:

$$\|\tilde{y}(\lambda) - i\| = (1-\lambda)\|i-j\| \leq \|x-j\| - \pi_i + \pi_j, \text{ or } \lambda \leq 1 - \frac{\|x-j\| - \pi_i + \pi_j}{\|i-j\|}.$$

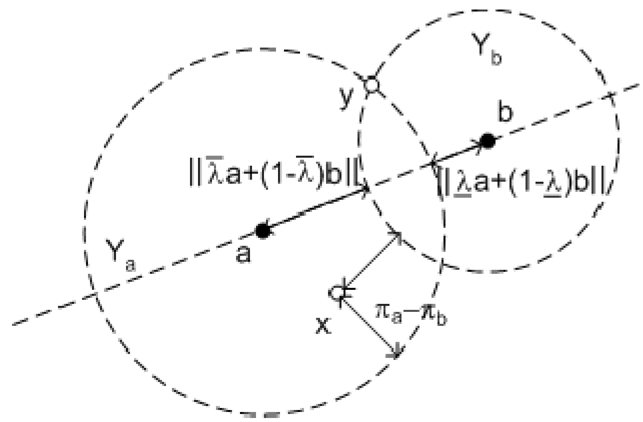

Figure 13: Construction of y with $\mu(y,a)=\mu(x,b)\,,\ \mu(y,b)=\mu(x,a)$

Hence

$$\overline{\lambda}=1-\frac{\|x-j\|-\pi_i+\pi_j}{\|i-j\|}$$

is the greatest λ such that $\tilde{y}(\lambda)\in Y_i$.

Elements in the intersection of the line space with Y_j satisfy:

$$\|\tilde{y}(\lambda)-j\|=\lambda\,\|i-j\|\leq\|x-i\|+\pi_i-\pi_j,$$

or

$$\begin{aligned}\lambda &\leq \frac{\|x-i\|+\pi_i-\pi_j}{\|i-j\|}\\ &= \frac{\|x-i\|+\|x-j\|-\|i-j\|}{\|i-j\|}+1-\frac{\|x-j\|-\pi_i+\pi_j}{\|i-j\|}\\ &= \frac{\|x-i\|+\|x-j\|-\|i-j\|}{\|i-j\|}+\overline{\lambda}.\end{aligned}$$

Letting

$$\underline{\lambda}=\frac{\|x-i\|+\|x-j\|-\|i-j\|}{\|i-j\|}+\overline{\lambda}$$

denote the smallest element such that $\tilde{y}(\lambda)\in Y_j$.

The triangle inequality ensures that $\|x-i\|+\|x-j\|-\|i-j\| \geq 0$; thus $\underline{\lambda} \geq \overline{\lambda}$ and the parameter interval $\left[\overline{\lambda}, \underline{\lambda}\right]$ is nonempty. But the $\tilde{y}(\lambda)$ on this interval belong to $Y_i \cap Y_j$, which implies that the boundary of $Y_i \cap Y_j$ is nonempty and a solution to (6) and (7) exists.

■