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THE UNIVERSITY OF MELBOURNE

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RESEARCH PAPER NUMBER 2

APRIL 1993

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Abstract

In the classical compound Poisson risk model, Lundberg's inequality provides both an upper bound for, and an approximation to, the probability of ultimate ruin. The result can be applied only when the moment generating function of the individual claim amount distribution exists. In this paper we will derive an exponential upper bound for the probability of ultimate ruin when the moment generating function of the individual claim amount distribution does not exist.

1. Notation

We will consider the classical continuous time risk model under which the aggregate claims process is a compound Poisson process with Poisson parameter λ. Individual claim amounts are identically distributed with distribution function $P(x)$, and are independent of each other and of the number of claims. We will assume throughout that $P(0)=0$. Premiums are received continuously at a constant rate c. We will write $c=(1+\theta)\lambda p_1$ where θ is the premium loading factor and p_1 is the mean individual claim amount. The stochastic process $\{Z(t)\}_{t\geq 0}$ denotes the insurer's surplus at time t, given an initial surplus of $Z(0)$, which we denote u. The time until ruin is denoted T, and defined by

$$\begin{aligned} T &= \inf\{t:\ Z(t)<0,\ t>0\} \\ &= \infty \text{ if } Z(t)\geq 0 \text{ for all } t>0 \end{aligned}$$

The probability of ultimate ruin given an initial surplus of u is denoted $\psi(u)$ and defined by

$$\psi(u) = \Pr(T<\infty)$$

Lundberg's inequality states that

$$\psi(u)\leq \exp\{-Ru\} \tag{1.1}$$

where R is the unique positive number satisfying

$$\lambda + cR = \lambda\int_0^\infty \exp\{Rx\}dP(x)$$

Since we are assuming that $P(0)=0$, an alternative definition of R is that it is the unique positive number satisfying

$$\int_0^\infty \exp\{Rx\}\frac{1-P(x)}{p_1}dx = M_F(\text{R}) = 1+\theta \tag{1.2}$$

where $M_F(r)$ is the moment generating function, evaluated at r, of the distribution

$$F(u) = \frac{1}{p_1}\int_0^u (1-P(x))dx$$

2. A known upper bound for $\psi(u)$

Panjer [1986] derived an upper bound for $\psi(u)$ as follows. The ultimate ruin probability can be expressed as

$$\psi(u) \;=\; 1 \;-\; \sum_{n=0}^{\infty}(1-\psi(0))\psi(0)^n F^{n*}(u) \;\text{ for }\; u>0$$

where *F(u)* is as defined in Section 1, $F^{n*}(u)$ denotes the n-fold convolution of the distribution *F(u)* with itself, and $\psi(0)=1/(1+\theta)$.

If we define a discrete distribution *L(x)*, with probability function *l(x)* defined by

$$l(x) = F(x) - F(x-1) \;\text{ for }\; x = 1,2,3,\ldots$$

then $L(x) \le F(x)$ for all $x \ge 0$, and consequently

$$\psi(u) \;\le\; 1-\sum_{n=0}^{\infty}(1-\psi(0))\psi(0)^n L^{n*}(u) \;=\; \phi(\mathrm{u})$$

Since $\phi(u)$ gives the tail probability for a compound geometric distribution, we can calculate $\phi(u)$ recursively from the formulae

$$\phi(0) = 1/(1+\theta)$$

and

$$\phi(u) = (1+\theta)^{-1}\Big(\sum_{j=1}^{u} l(j)\phi(u-j) + \sum_{j=u+1}^{\infty} l(j)\Big) \;\text{ for }\; u = 1,2,3,\ldots \qquad (2.1)$$

In the next section we will derive an upper bound for $\phi(u)$ for integer values of u which also provides an upper bound for $\psi(u)$. If u is not an integer, we can still bound $\psi(u)$ since $\phi(u) = \phi([u])$ where $[u]$ is the greatest integer less than u.

3. An upper bound for $\phi(u)$

We can rewrite (2.1) as

$$\phi(u) = (1+\theta)^{-1}\Big(1-\sum_{j=1}^{u} l(j) + \sum_{j=1}^{u} l(j)\phi(u-j)\Big) \;\text{ for }\; u = 1,2,3,\ldots$$

Now fix an integer value t and define a new function $l^*(x)$ as follows:

$$l^*(x) = l(x) \quad \text{for} \quad x = 1,2,\ldots,t$$
$$= 0 \quad \text{for} \quad x = t+1, t+2,\ldots$$

Then, provided that $u \le t$ we can write $\phi(u)$ in terms of the function $l^*(x)$ as

$$\phi(u) = (1+\theta)^{-1}\left(1-\sum_{j=1}^{u} l^*(j) + \sum_{j=1}^{u} l^*(j)\phi(u-j)\right) \quad \text{for} \quad u = 1,2,3,\ldots$$

If we define

$$P(\alpha) = \sum_{j=1}^{t} \exp\{\alpha j\} l^*(j)$$

then it is easy to show that there exists a unique positive number K such that

$$P(K) = 1+\theta \tag{3.1}$$

We can now derive our bound for $\phi(u)$.

Result: For $0 \le u \le t$,

$$\phi(u) \le \exp\{-Ku\} + \beta \tag{3.2}$$

where $\beta = (1-L(t))/(1+\theta-L(t))$.

Proof: We prove the result by induction on u. Note that the result is true when $u = 0$ since

$$\phi(0) = (1+\theta)^{-1} < 1+\beta$$

(We note at this stage that the upper bound is greater than one when $u = 0$. We will discuss this point later.)
Now assume that the result is true for $u = 0,1,2,\ldots,r-1$, and show that it is then true for $u = r$, where $r < t$. We have

$$\begin{aligned}
\phi(r) &= (1+\theta)^{-1}(1-\sum_{j=1}^{r} l^*(j) + \sum_{j=1}^{r} l^*(j)\phi(r-j)) \\
&= (1+\theta)^{-1}(1-\sum_{j=1}^{t} l^*(j) + \sum_{j=r+1}^{t} l^*(j) + \sum_{j=1}^{r} l^*(j)\phi(r-j)) \\
&\le (1+\theta)^{-1}(1-L(t) + \sum_{j=1}^{t} l^*(j)[\exp\{-K(r-j)\} + \beta]) \\
&= (1+\theta)^{-1}(1-(1-\beta)L(t) + P(K)\exp\{-Kr\}) \\
&= \exp\{-Kr\} + \beta
\end{aligned}$$

Note that when $r=t$ the proof is virtually identical. We simply omit the second line and replace r by t in all other lines. The above proof is an adaptation of a result stated by Willmot [1992, Section 2].

Although there are analogies between (1.1) and (3.2) and between (1.2) and (3.1), this result differs from the Lundberg bound in two ways. First, although K could be thought of as playing the role of the adjustment coefficient, it is not a constant but is a function of t. We shall see in the examples in the next section how the choice of t affects K. The second difference from the Lundberg bound is the term β, which is also a function of t. This term causes the bound to exceed one for small values of the initial surplus. However, the impact of this term can be reduced by choosing t to be suitably large, since $\beta \to 0$ as $t \to \infty$. We shall see in our examples that, as with the Lundberg bound, this bound is more useful for large values of u.

4. Examples

Example 1: Let the individual claim amount distribution be lognormal with mean 1 and variance 3 so that its parameters are $\mu = -0.69315$ and $\sigma = 1.17741$. Table 1 shows values of K and β for different values of t when $\theta = 0.1$.

Table 1

t	K	β
25	0.03892	0.04598
50	0.03458	0.00827
100	0.03259	0.00106
200	0.03074	0.00010

As expected, the value of β decreases with t, as does the value of K. It is mathematically obvious that K should be a decreasing function of t. Intuitively, if we consider K as playing the role of the adjustment coefficient, we would expect K to decrease as t increases. The reason for this is that $l^*(x)$ is replacing the distribution of each increase in the record high of the underlying aggregate loss process. For this new aggregate loss process the probability of ruin should increase with t, and hence we would expect the associated "adjustment coefficient", K, to decrease.

Table 2 shows some values of upper bounds and ruin probabilities for different values of t. The ruin probabilities have been calculated using the algorithm described by Dickson and Waters [1991]. For each value of u, the values of the bound for each value of t are close together and for this reason we have not graphed the different bounds.

Table 2

	t = 25	t = 50	t = 100	t = 200	$\psi(u)$
u = 10	0.7236	0.7159	0.7230	0.7220	0.5344
u = 20	0.5051	0.5091	0.5222	0.5213	0.3467
u = 40	-	0.2591	0.2727	0.2717	0.1538

Although the bounds vary with t there is no pattern to the bounds for a given value of u. This is due to the fact that for two different values of t, if the bounds intersect, then there can only be one point of intersection. For example, with $t=25$, the bound is greater at $u=0$ than when $t=50$, but it also decreases at a greater rate. We note also that for these values of u the bound is not very close to the ruin probability.

Figure 1 shows values of the bound when t is 200, along with the true value for $\psi(u)$ for integer values of u from 100 to 200. From this we can see that the upper bound gives a reasonable approximation to the ruin probability for the larger values of u.

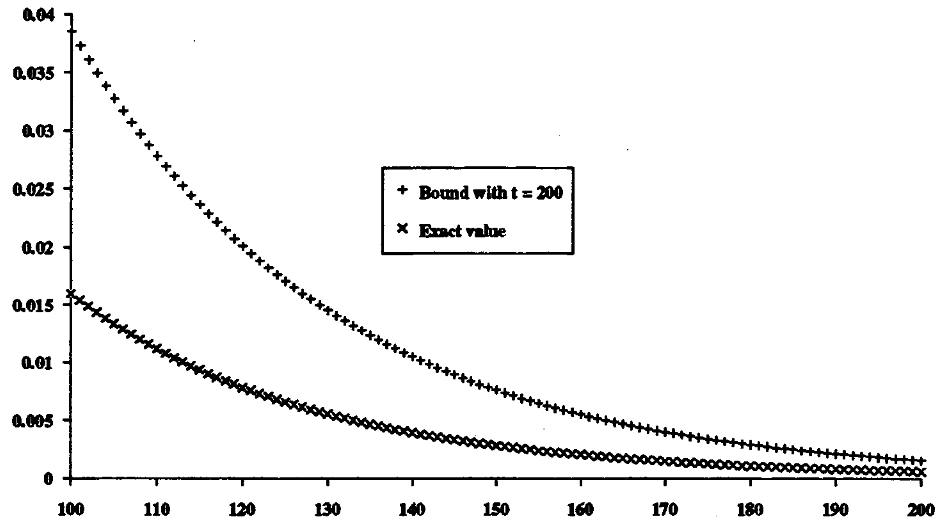

Figure 1: $\psi(u)$ and its upper bound when the individual claim amount distribution is lognormal.

Example 2: In this example we will show how the bound derived in Section 3 can easily be refined. All that is necessary to refine the bound is to define $L(x)$ on $h, 2h, 3h, \ldots$ rather than on the integers. This technique is fully explained by Panjer [1986]. This refinement will alter the value of K for a given value of t but will not alter the value of β as its definition is unchanged by the refinement.
Let the individual claim amount distribution be Pareto with distribution function

$$P(x) = 1-(1+x)^{-2}$$

Table 3 shows values of K, when $h=1$ and $h=0.05$, and β for different values of t when $\theta = 0.1$.

Table 3

t	K ($h = 1$)	K ($h = 0.05$)	β
50	0.02825	0.03077	0.16393
100	0.01962	0.02124	0.09009
200	0.01411	0.01483	0.04739
400	0.00975	0.01002	0.02433

This table shows the same features as Table 1. We can also see that the effect of reducing h is to produce a larger value of K (for a given value of t), which produces a lower value for the upper bound. Unlike in Example 1, the bounds produced using different values of t are not close together. Figure 2a shows bounds calculated with t=200 and 400 and ruin probabilities (calculated as in Example 1) when $\theta = 0.1$ and $h = 0.05$. Figure 2b shows a section of Figure 2a in greater detail. We note the following points about Figures 2a and 2b:

(i) There is one point of intersection of the two bounds shown. For most values of u up to 200 the bound calculated with $t = 200$ is closer to $\psi(u)$.

(ii) For each value of t the bound is closest to the ruin probability at t.

(iii) For the largest value of u shown, the difference between the bound and the ruin probability is about 1%.

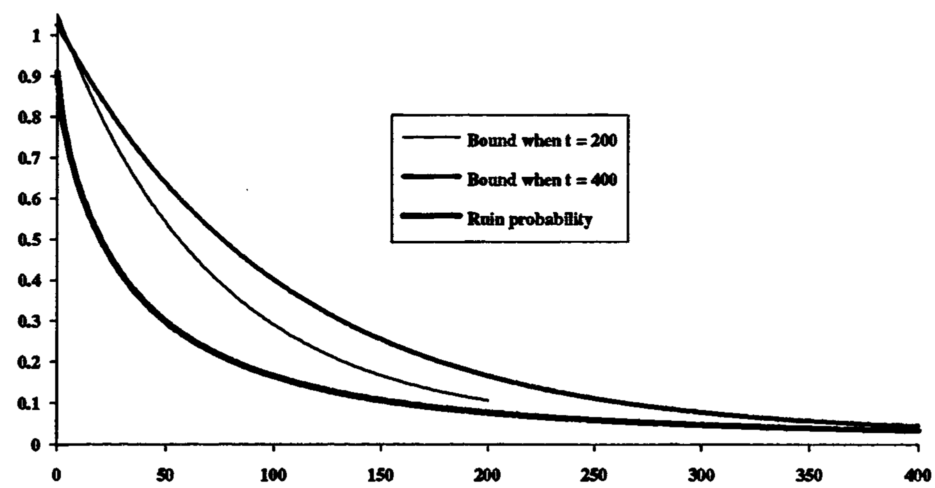

Figure 2a: $\psi(u)$ and upper bounds when the individual claim amount distribution is Pareto.

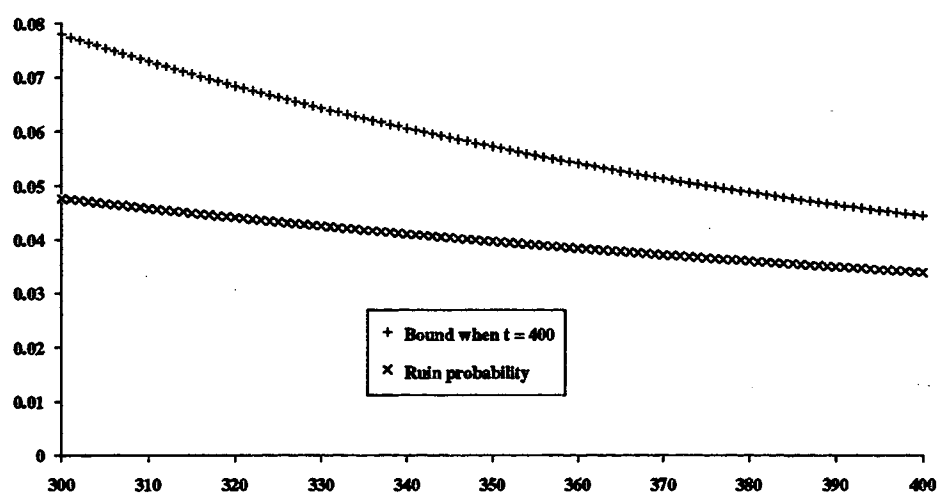

Figure 2b: Detail of Figure 2a

5. Comparison with other bounds

Although our aim has been to produce a bound for $\psi(u)$ in cases when Lundberg's inequality does not apply, it is of interest to compare our new bound with Lundberg's inequality when it applies. When Lundberg's inequality applies we can set $t = \infty$ and so we can calculate a value of K which is independent of t. It is straightforward to show that $K < R$, and so Lundberg's inequality always gives a tighter bound.

When Lundberg's inequality does not apply, we can compare our new bound with that proposed by Broeckx et al [1986]. Their result is

$$\psi(u) \leq e^{-ur(u)} \tag{5.1}$$

where $r(u)$ is a constant for a fixed value of u that satisfies

$$\int_0^u \frac{1-P(x)}{p_1} e^{xr(u)} dx + e^{ur(u)} \int_u^\infty \frac{1-P(x)}{p_1} dx \leq 1+\theta \tag{5.2}$$

Taylor [1986], in his discussion of this bound, shows via an example that it gives lower values than Lundberg's bound for small values of u. Let us now compare this bound with the one developed in Section 3.

In Example 2, the distribution of individual claim amounts is Pareto. Using numerical integration it is straightforward to solve (5.2) for $r(u)$ for a given value of u. Table 4 shows values of $r(u)$ (rounded to five decimal places) for selected values of u , together with Broeckx et al's bound, (5.1), and values of our new bound using values given in Table 3 with $h = 0.05$.

Table 4

u	r(u)	exp{-ur(u)}	exp{-Ku} + β
50	0.01952	0.3767	0.3786
100	0.01473	0.2293	0.2096
200	0.01068	0.1182	0.0989
400	0.00730	0.0539	0.0425

We can see that Broeckx's et al's bound is superior when $u = 50$ but is inferior for the other tabulated values of u. Further calculations confirm that the new bound is superior for larger values of u but inferior for smaller values. The new bound can also be improved by reducing the value of h.

6. Concluding remarks

In the examples in the previous section we have calculated upper bounds and exact values for ruin probabilities. Since algorithms exist to calculate exact values for ruin probabilities (see, for alternative approaches to the earlier reference, Goovaerts and De Vylder[1984] and Panjer [1986]), the reader may well ask why there is a need to calculate an upper bound. The second example in the previous section provides a reason. If we ask the question "What is the initial surplus required such that the ruin probability is at most 1%?" then recursive algorithms provide an answer, but only by trial and error which can require a considerable amount of computer time. By spending a little time calculating upper bounds, we can save a lot of computer time calculating exact values. The methods described in this paper are easy to apply and all calculations relating to the new upper bound were performed on a spreadsheet.

References

Broeckx, F., Goovaerts, M.J. and De Vylder, F. [1986] "Ordering of risks and ruin probabilities". *Insurance: Mathematics and Economics 5, 35-40.*

Dickson, D.C.M. and Waters, H.R. [1991] "Recursive calculation of survival probabilities'. *ASTIN Bulletin 21, 199-221.*

Goovaerts, M.J. and De Vylder, F. [1984] "A stable recursive algorithm for evaluation of ultimate ruin probabilities". *ASTIN Bulletin 14, 53-60.*

Panjer, H.H. [1986] "Direct calculation of ruin probabilities'. *Journal of Risk and Insurance 53, 521-529.*

Taylor, G.C. [1986] "Discussion of 'Ordering of risks and ruin probabilities'". *Insurance: Mathematics and Economics 5, 41-44.*

Willmot, G.E. [1992] "On the tails of compound geometric distributions". University of Waterloo, Institute of Insurance and Pensions Research, Report 93-02.

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