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On the expected discounted penalty function at ruin of a surplus process with interest

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Abstract

In this paper, we study the expected value of a discounted penalty function at ruin of the classical surplus process modified by the inclusion of interest on the surplus. The 'penalty' is simply a function of the surplus immediately prior to ruin and the deficit at ruin. An integral equation for the expected value is derived, while the exact solution is given when the initial surplus is zero. Dickson's (1992) formulae for the distribution of the surplus immediately prior to ruin in the classical surplus process are generalised to our modified surplus process.

Keywords: ruin penalty function, surplus prior to ruin, deficit at ruin, Laplace transform, Volterra equation, compound Poisson process, force of interest.

1 Introduction

Consider a compound Poisson risk model. Assume that $T_n = \sum_{k=1}^{n} Y_k$ is the time of the n-th claim and X_n is the amount of the n-th claim. Suppose that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are two independent sequences of i.i.d. positive random variables, where $\{X_n, n \geq 1\}$ have common distribution $F(x) = \Pr\{X_1 \leq x\}$ with mean $\mu > 0$, and $\{Y_n, n \geq 1\}$ have common exponential distribution $\Pr\{Y_1 \leq x\} = 1 - \exp\{-\lambda x\}$, $x \geq 0$, where $\lambda > 0$.

The number of claims up to time t is denoted by $N(t) = \sup\{n : T_n \leq t\}$. The claim number process is a Poisson process with rate λ, and the aggregate claim amount up to time t is

$$Z(t) = \sum_{n=1}^{N(t)} X_n .$$

Assume that the insurer receives interest on its surplus at a constant force δ per unit time. Let $U_\delta(t)$ denote the surplus at time t. Then

$$U_\delta(t) = ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)} - \int_0^t e^{\delta(t-x)} dZ(x) \tag{1.1}$$

where u is the initial surplus and $c = (1+\theta)\lambda\mu$ is the rate of premium income per unit time, where $\theta > 0$ is the premium loading factor.

Let the time of ruin be

$$T_\delta = \begin{cases} \inf\{t : U_\delta(t) < 0\} \\ \infty \text{ if } U_\delta(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

Denote by $\psi_\delta(u)$ the ruin probability for the surplus process given by equation (1.1). Then

$$\psi_\delta(u) = \Pr\{T_\delta < \infty\} = \Pr\{\cup_{t\geq 0}(U_\delta(t) < 0)\}.$$

The following notation applies throughout this paper:

$$\begin{aligned}
f(x) &= \frac{d}{dx}F(x); \\
F_1(x) &= 1 - \bar{F}_1(x) = \frac{1}{\mu}\int_0^x \bar{F}(t)dt; \\
\bar{\psi}_\delta(u) &= 1 - \psi_\delta(u); \quad \psi(u) = \psi_{\delta=0}(u); \quad \psi(0) = \frac{\lambda\mu}{c} = \frac{1}{1+\theta}; \\
U(T_\delta^-) &= \text{the surplus immediately prior to ruin;} \\
|U(T_\delta)| &= \text{the deficit at ruin;} \\
F_\delta(u,x) &= \Pr\{U(T_\delta^-) \leq x,\, T_\delta < \infty\}; \quad f_\delta(u,x) = \frac{d}{dx}F_\delta(u,x); \\
H_\delta(u,x,y) &= \Pr\{U(T_\delta^-) \leq x,\, |U(T_\delta)| \leq y,\, T_\delta < \infty\}; \\
h_\delta(u,x,y) &= \frac{\partial^2}{\partial x \partial y}H_\delta(u,x,y).
\end{aligned}$$

We consider the expected value of a discounted function of the surplus immediately prior to ruin and the deficit at ruin when ruin occurs as a function of the initial surplus u, namely,

$$\Phi_{\delta,\alpha}(u) = E(w(U(T_\delta^-),\, |U(T_\delta)|)\, e^{-\alpha T_\delta}\, I(T_\delta < \infty))$$

where $I(A)$ is the indicator function of a set A, w is a non-negative function, and α is a non-negative valued parameter. We can interpret $\exp\{-\alpha T_\delta\}$ as the 'discounting factor'.

The function $\Phi_{\delta,\alpha}(u)$ provides a unified means of studying the joint distribution of the surplus immediately prior to ruin and the deficit at ruin. The distributions of these quantities, both joint and marginal, have been studied by many authors including Dickson (1992), Dufresne and Gerber (1988), Gerber *et al* (1987), Gerber and Shiu (1997, 1998) and Lin and Willmot (1999). In particular, Gerber and Shiu (1998) studied the function $\Phi_{\delta=0,\alpha}(u)$ in detail, but they did not consider the case when $\delta > 0$.

In this paper, we will follow ideas in Sundt and Teugels (1995). In particular, we will consider the function $\Phi_{\delta,\alpha=0}(u) = \Phi_\delta(u)$. We will also derive an integral equation for $\Phi_{\delta,\alpha}(u)$ and find the Laplace transform of an auxiliary function of $\Phi_{\delta,\alpha}(u)$. We then find an exact solution for $\Phi_\delta(0)$ and generalise Dickson's (1992) formulae for the distribution of the surplus prior to ruin when $\delta = 0$ to the situation when $\delta > 0$. Applications of the results will be illustrated by a variety of examples.

2 Integral equations

Using similar arguments to Gerber and Shiu (1998) and Sundt and Teugels (1995), we condition on the time, t, and on the amount, x, of the first claim. We note that if $x \leq ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)}$, then ruin does not occur, but if $x > ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)}$, then ruin occurs. Thus,

$$\begin{aligned}
\Phi_{\delta,\alpha}(u) &= \int_0^\infty \lambda e^{-\lambda t} \int_0^\infty E(w(U(T_\delta^-), |U(T_\delta)|)e^{-\alpha T_\delta} I(T_\delta < \infty) | X_1 = x, Y_1 = t) dF(x)dt \\
&= \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^{ue^{\delta t}+c\bar{s}_{\overline{t}|}^{(\delta)}} \Phi_{\delta,\alpha}(ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)} - x) dF(x)dt \\
&\quad + \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_{ue^{\delta t}+c\bar{s}_{\overline{t}|}^{(\delta)}}^\infty w(ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)},\, x - ue^{\delta t} - c\bar{s}_{\overline{t}|}^{(\delta)}) dF(x)dt.
\end{aligned}$$

Substituting $y = ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)} = ue^{\delta t} + c(e^{\delta t} - 1)/\delta$ in the above equation, we have

$$\begin{aligned}
\Phi_{\delta,\alpha}(u) &= \lambda\,(\delta u + c)^{\frac{\lambda+\alpha}{\delta}} \int_u^\infty (\delta y + c)^{-\frac{\lambda+\alpha}{\delta}-1} \int_0^y \Phi_{\delta,\alpha}(y - x) dF(x)dy \\
&\quad + \lambda\,(\delta u + c)^{\frac{\lambda+\alpha}{\delta}} \int_u^\infty (\delta y + c)^{-\frac{\lambda+\alpha}{\delta}-1} \int_y^\infty w(y,\, x - y) dF(x)dy \\
&= \lambda\,(\delta u + c)^{\frac{\lambda+\alpha}{\delta}} \int_u^\infty (\delta y + c)^{-\frac{\lambda+\alpha}{\delta}-1} \left(\int_0^y \Phi_{\delta,\alpha}(y - x) dF(x) + A(y) \right) dy \quad (2.1)
\end{aligned}$$

where

$$A(t) = \int_t^\infty w(t, s - t) dF(s).$$

Differentiating equation (2.1) with respect to u, we get

$$\frac{d}{du}\Phi_{\delta,\alpha}(u) = \frac{\lambda + \alpha}{c + \delta u}\Phi_{\delta,\alpha}(u) - \frac{\lambda}{c + \delta u}\left(\int_0^u \Phi_{\delta,\alpha}(u - x) dF(x) + A(u) \right). \quad (2.2)$$

Replacing u by t in equation (2.2) and re-arranging, we get for any $t \geq 0$,

$$(\lambda+\alpha)\Phi_{\delta,\alpha}(t) = (c+\delta t)\frac{d}{dt}\Phi_{\delta,\alpha}(t) + \lambda\int_0^t \Phi_{\delta,\alpha}(t-s)dF(s) + \lambda A(t). \tag{2.3}$$

Thus, integrating equation (2.3) from 0 to u, then performing integration by parts, we get,

$$\begin{aligned}
&(\lambda+\alpha)\int_0^u \Phi_{\delta,\alpha}(t)dt \\
=\ & \int_0^u (c+\delta t)d\Phi_{\delta,\alpha}(t) + \lambda\int_0^u\int_0^t \Phi_{\delta,\alpha}(t-s)dF(s)dt + \lambda\int_0^u A(t)dt \\
=\ & (c+\delta u)\Phi_{\delta,\alpha}(u) - c\Phi_{\delta,\alpha}(0) - \delta\int_0^u \Phi_{\delta,\alpha}(t)dt + \lambda\int_0^u\int_s^u \Phi_{\delta,\alpha}(t-s)dtdF(s) + \lambda\int_0^u A(t)dt \\
=\ & (c+\delta u)\Phi_{\delta,\alpha}(u) - c\Phi_{\delta,\alpha}(0) - \delta\int_0^u \Phi_{\delta,\alpha}(t)dt + \lambda\int_0^u\left(\int_0^{u-s} \Phi_{\delta,\alpha}(y)dy\right)dF(s) + \lambda\int_0^u A(t)dt \\
=\ & (c+\delta u)\Phi_{\delta,\alpha}(u) - c\Phi_{\delta,\alpha}(0) - \delta\int_0^u \Phi_{\delta,\alpha}(t)dt + \lambda\int_0^u F(s)\Phi_{\delta,\alpha}(u-s)ds + \lambda\int_0^u A(t)dt,
\end{aligned}$$

which implies that

$$(c+\delta u)\Phi_{\delta,\alpha}(u) = c\Phi_{\delta,\alpha}(0) - \lambda\int_0^u A(t)dt + \int_0^u (\delta+\alpha+\lambda\bar{F}(u-t))\Phi_{\delta,\alpha}(t)dt,$$

or

$$\begin{aligned}
\Phi_{\delta,\alpha}(u) &= \frac{c\,\Phi_{\delta,\alpha}(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(t)dt + \int_0^u \frac{\delta+\alpha+\lambda\bar{F}(u-t)}{c+\delta u}\Phi_{\delta,\alpha}(t)dt \\
&= \frac{c\,\Phi_{\delta,\alpha}(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(t)dt + \int_0^u k_{\delta,\alpha}(u,t)\Phi_{\delta,\alpha}(t)dt,
\end{aligned} \tag{2.4}$$

where

$$k_{\delta,\alpha}(u,t) = \frac{\delta+\alpha+\lambda\bar{F}(u-t)}{c+\delta u}.$$

In particular, recalling that $\Phi_\delta(u) = \Phi_{\delta,\alpha=0}(u)$, we get

$$(c+\delta u)\Phi_\delta(u) = c\Phi_\delta(0) - \lambda\int_0^u A(t)dt + \int_0^u (\delta+\lambda\bar{F}(u-t))\Phi_\delta(t)dt, \tag{2.5}$$

for any $u \geq 0$, and

$$\Phi_\delta(u) = \frac{c\,\Phi_\delta(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(t)dt + \int_0^u k_\delta(u,t)\Phi_\delta(t)dt, \tag{2.6}$$

where

$$k_\delta(u,t) = k_{\delta,\alpha=0}(u,t) = \frac{\delta+\lambda\bar{F}(u-t)}{c+\delta u}.$$

Both equations (2.4) and (2.6) are types of the following Volterra integral equation

$$\varphi(x) = l(x) + \int_0^x k(x,s)\varphi(s)ds. \tag{2.7}$$

It is well known (see, for example, Mikhlin(1957)) that if l is absolutely integrable and the kernel k is continuous, then for any $x > 0$, the unique solution for $\varphi(x)$ has the following representation

$$\varphi(x) = l(x) + \int_0^x K(x,s)l(s)ds, \tag{2.8}$$

where

$$K(x,s) = \sum_{m=1}^{\infty} k_m(x,s), \quad x > s \geq 0, \tag{2.9}$$

is called the resolvent of equation (2.7), and

$$k_m(x,s) = \int_s^x k(x,t)k_{m-1}(t,s)dt, \quad m = 2,3,\ldots, \quad x > s \geq 0,$$

with $k_1(x,s) = k(x,s)$.

Further, $\varphi(x)$ can be approximated recursively by Picard's sequence (see Mikhlin(1957)) defined by

$$\varphi_n(x) = l(x) + \int_0^x \varphi_{n-1}(s)l(s)ds, \, n = 1,2,\ldots$$

with $\varphi_0(x) = l(x)$.

Therefore, at least in principle, if we can find $\Phi_{\delta,\alpha}(0)$, we can find the form of the solution for $\Phi_{\delta,\alpha}(u)$ and can approximate $\Phi_{\delta,\alpha}(u)$ recursively. Hence it is important to be able to find $\Phi_{\delta,\alpha}(0)$. Gerber and Shiu (1998) have obtained $\Phi_{\delta=0,\alpha}(0)$ using the technique of probability measure transform. However, we will find $\Phi_\delta(0) = \Phi_{\delta,\alpha=0}(0)$ by using Laplace transforms in the next section. In what follows, unless we state otherwise, the term Laplace transform refers to a Stieltjes transform.

3 The exact solution for $\Phi_\delta(0)$

We define an auxiliary function of $\Phi_\delta(u)$ as

$$Z_\delta(u) = \frac{\Phi_\delta(0) - \Phi_\delta(u)}{\Phi_\delta(0)}. \tag{3.1}$$

Then $Z_\delta(0) = 0$. Also, if the claim size distribution F is sufficiently regular, then $\Phi_\delta(u) \to 0$ as $u \to \infty$. In this case, $\lim_{u\to\infty} Z_\delta(u) = 1$ and we can find the Laplace transform of Z_δ, namely,

$$\gamma_\delta(s) = \int_0^\infty e^{-sx} dZ_\delta(x),$$

with $\gamma_\delta(0) = 1$. Therefore, we assume that $\Phi_\delta(u) \to 0$ as $u \to \infty$. In particular, a sufficient condition for this assumption is that w is bounded. In fact, if $w \leq L$ for some $L > 0$ and F has a finite second moment, then

$$\begin{aligned} (c+\delta u)\Phi_\delta(u) &\leq L(c+\delta u)EI(T_\delta < \infty) \\ &= L\left(c\psi_\delta(u) + \delta u\psi_\delta(u)\right) \leq L\left(c\psi(u) + \delta u\psi(u)\right) \to 0 \end{aligned}$$

as $u \to \infty$. Thus, letting $u \to \infty$ in equation (2.5), we get

$$\Phi_\delta(0) = \frac{\lambda}{c}\int_0^\infty A(t)dt - \frac{\delta}{c}\int_0^\infty \Phi_\delta(t)dt. \tag{3.2}$$

Hence, $\Phi_\delta(0)$ can also be obtained by finding the standard Laplace transform of Φ_δ, namely $\int_0^\infty e^{-st}\Phi_\delta(t)dt$.

Now, equation (3.1) implies that

$$\Phi_\delta(u) = \Phi_\delta(0) - \Phi_\delta(0)Z_\delta(u). \tag{3.3}$$

Inserting equation (3.3) into equation (2.5), we get

$$\begin{aligned} & c\Phi_\delta(0) - c\Phi_\delta(0)Z_\delta(u) + \delta\Phi_\delta(0)u - \delta\Phi_\delta(0)uZ(u) \\ = \; & c\Phi_\delta(0) - \lambda\int_0^u A(t)dt + \delta\int_0^u (\Phi_\delta(0) - \Phi_\delta(0)Z_\delta(t))dt \\ & +\lambda\int_0^u \bar{F}(u-t)(\Phi_\delta(0) - \Phi_\delta(0)Z_\delta(t))dt, \end{aligned}$$

which implies that

$$(c+\delta u)Z_\delta(u) = \delta\int_0^u Z_\delta(t)dt + \frac{\lambda m_A}{\Phi_\delta(0)}A_1(u) - \lambda\mu F_1(u) + \lambda\mu Z_\delta * F_1(u) \tag{3.4}$$

where

$$A_1(u) = \frac{1}{m_A}\int_0^u A(t)dt, \quad m_A = \int_0^\infty A(t)dt,$$

and $Z_\delta * F_1$ is the Stieltjes convolution of Z_δ and F_1.

Thus, differentiating both sides of equation (3.4), we have

$$(c+\delta u)dZ_\delta(u) + \delta Z_\delta(u)\,du = \delta Z_\delta(u)\,du + \frac{\lambda m_A}{\Phi_\delta(0)}\,dA_1(u) - \lambda\mu\,dF_1(u) + \lambda\mu\,dZ_\delta * F_1(u),$$

which gives

$$(c+\delta u)dZ_\delta(u) = \frac{\lambda m_A}{\Phi_\delta(0)}\,dA_1(u) - \lambda\mu\,dF_1(u) + \lambda\mu\,dZ_\delta * F_1(u). \tag{3.5}$$

Taking Laplace transforms of both sides of equation (3.5) yields

$$c\,\gamma_\delta(s) - \delta\,\frac{d}{ds}\gamma_\delta(s) = \frac{\lambda m_A}{\Phi_\delta(0)}\,\beta(s) - \lambda\mu\,\phi(s) + \lambda\mu\,\phi(s)\,\gamma_\delta(s), \tag{3.6}$$

where ϕ and β are the Laplace transforms of F_1 and A_1 respectively, namely,

$$\phi(s) = \int_0^\infty e^{-sx}dF_1(x) = \frac{1}{\mu}\int_0^\infty e^{-sx}\bar{F}(x)dx,$$

or, if F is a continuous distribution,

$$\int_0^\infty e^{-sx}f(x)dx = 1 - \mu s\phi(s), \tag{3.7}$$

and

$$\beta(s) = \int_0^\infty e^{-sx}dA_1(x) = \frac{1}{m_A}\int_0^\infty e^{-sx}A(x)dx. \tag{3.8}$$

Equation (3.6) is equivalent to

$$-\delta\,\frac{d}{ds}\gamma_\delta(s) + P_\delta(s)\,\gamma_\delta(s) = Q_\delta(s)$$

where

$$P_\delta(s) = c - \lambda\mu\,\phi(s)$$

and

$$Q_\delta(s) = \frac{\lambda m_A}{\Phi_\delta(0)}\,\beta(s) - \lambda\mu\,\phi(s).$$

When $\delta > 0$, we note that

$$\frac{d}{ds}\left(\gamma_\delta(s)\exp\left(-\frac{1}{\delta}\int_0^s P_\delta(t)dt\right)\right) = -\frac{1}{\delta}Q_\delta(s)\exp\left(-\frac{1}{\delta}\int_0^s P_\delta(t)dt\right)$$

and using the arguments of Sundt and Teugels (1995), we get

$$\gamma_\delta(s)\exp\left(-\frac{1}{\delta}\int_0^s P_\delta(t)dt\right)=\frac{1}{\delta}\int_s^\infty Q_\delta(t)\exp\left(-\frac{1}{\delta}\int_0^t P_\delta(z)dz\right)dt.$$

Hence, $\gamma_\delta(0)=1$ gives

$$\begin{aligned}\delta &= \int_0^\infty Q_\delta(t)\exp\left(-\frac{1}{\delta}\int_0^t P_\delta(z)dz\right)dt\\ &= \frac{\lambda m_A}{\Phi_\delta(0)}\int_0^\infty \beta(t)\exp\left(-\frac{1}{\delta}\int_0^t P_\delta(z)dz\right)dt-\lambda\mu\int_0^\infty \phi(t)\exp\left(-\frac{1}{\delta}\int_0^t P_\delta(z)dz\right)dt,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\Phi_\delta(0) &= \frac{\lambda m_A\int_0^\infty \beta(t)\exp\left(-\frac{1}{\delta}\left(ct-\lambda\mu\int_0^t\phi(s)ds\right)\right)dt}{\delta+\lambda\mu\int_0^\infty\phi(t)\exp\left(-\frac{1}{\delta}\left(ct-\lambda\mu\int_0^t\phi(s)ds\right)\right)dt}\\ &= \frac{\lambda m_A}{\kappa_\delta}\int_0^\infty \beta(\delta z)\exp\left(-cz+\lambda\mu\int_0^z\phi(\delta s)ds\right)dz \qquad (3.9)\end{aligned}$$

where equation (3.9) follows from the substitution $t=\delta z$, and we define

$$\begin{aligned}\kappa_\delta &= 1+\lambda\mu\int_0^\infty\phi(\delta z)\exp\left(-cz+\lambda\mu\int_0^z\phi(\delta s)ds\right)dz\\ &= c\int_0^\infty\exp\left(-cz+\lambda\mu\int_0^z\phi(\delta s)ds\right)dz \qquad (3.10)\end{aligned}$$

using integration by parts.

Note that κ_δ does not depend on the choice of A or w, but β does. We will illustrate the applications of equations (2.6) and (3.9) by examples.

Similarly, if we define an auxiliary function of $\Phi_{\delta,\alpha}(u)$ as

$$Z_{\delta,\alpha}(u)=\frac{\Phi_{\delta,\alpha}(0)-\Phi_{\delta,\alpha}(u)}{\Phi_{\delta,\alpha}(0)}$$

and denote the Laplace transform of $Z_{\delta,\alpha}$ by

$$\gamma_{\delta,\alpha}(s)=\int_0^\infty e^{-sx}dZ_{\delta,\alpha}(x),$$

we get a differential equation for $\gamma_{\delta,\alpha}(s)$, namely

$$c\,\gamma_{\delta,\alpha}(s)-\delta\frac{d}{ds}\gamma_{\delta,\alpha}(s)=\alpha\frac{\gamma_{\delta,\alpha}(s)}{s}+\frac{\lambda m_A}{\Phi_\delta(0)}\beta(s)-\frac{\alpha}{s}-\lambda\mu\,\phi(s)+\lambda\mu\,\phi(s)\,\gamma_{\delta,\alpha}(s).$$

Unfortunately, we are unable to determine $\gamma_{\delta,\alpha}(0)$ using the methods of this section. It seems that the method of Gerber and Shiu (1998) does not apply either.

Example 3.1 Let $w(x_1, x_2) = 1$. Then $\Phi_\delta(u) = \psi_\delta(u)$, $A(t) = \bar{F}(t)$, $m_A = \mu$, and $\beta(s) = \phi(s)$, and equation (2.6) gives

$$\psi_\delta(u) = \frac{c\,\psi_\delta(0)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u) + \int_0^u k_\delta(u,t)\psi_\delta(t)dt, \tag{3.11}$$

Equivalently, as $\bar{\psi}_\delta(u) = 1 - \psi_\delta(u)$,

$$\bar{\psi}_\delta(u) = \frac{c\,\bar{\psi}_\delta(0)}{c+\delta u} + \int_0^u k_\delta(u,t)\bar{\psi}_\delta(t)dt, \tag{3.12}$$

which is equation (2) of Sundt and Teugels (1995). In addition, equations (3.9) and (3.10) give

$$\psi_\delta(0) = \frac{\lambda\mu}{\kappa_\delta}\int_0^\infty \phi(\delta z)\exp\left(-cz + \lambda\mu\int_0^z \phi(\delta s)ds\right)dz = \frac{\kappa_\delta - 1}{\kappa_\delta}$$

which is equivalent to equation (14) of Sundt and Teugels (1995). As $1/\kappa_\delta = \bar{\psi}_\delta(0)$, the general expression for $\Phi_\delta(0)$ in (3.9) becomes

$$\Phi_\delta(0) = \lambda m_A \bar{\psi}_\delta(0)\int_0^\infty \beta(\delta z)\exp\left(-cz + \lambda\mu\int_0^z \phi(\delta s)ds\right)dz. \tag{3.13}$$

Example 3.2 Let $w(x_1, x_2) = I(x_1 \le x)I(x_2 \le y)$. Then $\Phi_\delta(u) = H_\delta(u, x, y)$ and

$$\begin{aligned} A(t) &= \int_t^\infty w(t, s-t)dF(s) = \int_t^\infty I(t \le x)I(s-t \le y)dF(s) \\ &= I(t \le x)\int_t^{t+y} dF(s) = I(t \le x)(\bar{F}(t) - \bar{F}(t+y)). \end{aligned}$$

Thus, equation (2.6) gives

$$\begin{aligned} H_\delta(u,x,y) &= \frac{c\,H_\delta(0,x,y)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^{u\wedge x}\left(\bar{F}(t) - \bar{F}(y+t)\right)dt \\ &\quad + \int_0^u k_\delta(u,t)H_\delta(t,x,y)dt \\ &= \frac{c\,H_\delta(0,x,y)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}[F_1(u\wedge x) + F_1(y) - F_1(u\wedge x + y)] \\ &\quad + \int_0^u k_\delta(u,t)H_\delta(t,x,y)dt. \end{aligned} \tag{3.14}$$

In addition,

$$\beta(t) = \frac{1}{m_A}\int_0^\infty e^{-ts}A(s)ds = \frac{1}{m_A}\int_0^x e^{-ts}(\bar{F}(s) - \bar{F}(y+s))ds.$$

Hence, equation (3.9) gives

$$H_\delta(0,x,y) = \frac{\lambda}{\kappa_\delta}\int_0^\infty\int_0^x e^{-\delta sz}(\bar{F}(s) - \bar{F}(y+s))ds\,\exp\left(-cz + \lambda\mu\int_0^z \phi(\delta s)ds\right)dz. \tag{3.15}$$

Thus, equations (3.14) and (3.15) give the main results of Yang and Zhang (2001a). Similarly, letting $x \to \infty$ in these equations we get equations (4) and (12) of Yang and Zhang (2001b).

Example 3.3 For $r \geq 0$, let $w(x_1, x_2) = e^{-rx_2}$. Then

$$\Phi_\delta(u) = E(e^{-r|U(T_\delta)|} I(T_\delta < \infty)) = \tilde{W}_\delta(u, r),$$

the Laplace transform of the deficit at ruin when ruin occurs. Thus, when F is a continuous distribution,

$$A(t) = \int_t^\infty w(t, x-t) dF(x) = \int_t^\infty e^{-r(x-t)} f(x) dx,$$

and by equations (3.8) and (3.7), we have

$$\begin{aligned}
\beta(t) &= \frac{1}{m_A} \int_0^\infty e^{-ty} \int_y^\infty e^{-r(x-y)} f(x) dx dy \\
&= \frac{1}{m_A} \int_0^\infty e^{-rx} f(x) \int_0^x e^{-(t-r)y} dy dx = \frac{1}{m_A} \int_0^\infty f(x) \frac{e^{-rx} - e^{-tx}}{t-r} dx \\
&= \left(\frac{\mu}{m_A}\right) \frac{t\phi(t) - r\phi(r)}{t-r}.
\end{aligned}$$

Thus, using equation (3.13) we get

$$\tilde{W}_\delta(0, r) = \lambda\mu\bar{\psi}_\delta(0) \int_0^\infty \frac{\delta z \phi(\delta z) - r\phi(r)}{\delta z - r} \exp\left(-cz + \lambda\mu \int_0^z \phi(\delta s) ds\right) dz.$$

Therefore, given that $U_\delta(0) = 0$ and ruin occurs, if the $(n+1)$-th moment of F exists, we get the n-th moment of the deficit at ruin, namely,

$$\begin{aligned}
E(|U(T_\delta)|^n \,|\, T_\delta < \infty) &= \frac{(-1)^n}{\psi_\delta(0)} \left(\frac{d^n}{dr^n} \tilde{W}_\delta(0, r)|_{r=0}\right) \\
= \frac{\lambda\mu\bar{\psi}_\delta(0)}{\psi_\delta(0)} & \int_0^\infty \left(\frac{d^n}{dr^n} \frac{\delta z \phi(\delta z) - r\phi(r)}{\delta z - r}|_{r=0}\right) \exp\left(-cz + \lambda\mu \int_0^z \phi(\delta s) ds\right) dz.
\end{aligned}$$

In particular, when $n = 1$ we have

$$E(|U(T_\delta)| \,|\, T_\delta < \infty) = \frac{\lambda\mu\bar{\psi}_\delta(0)}{\psi_\delta(0)} \int_0^\infty \frac{1 - \phi(\delta z)}{\delta z} \exp\left(-cz + \lambda\mu \int_0^z \phi(\delta s) ds\right) dz.$$

Similarly, we can let $w(x_1, x_2) = e^{-r(x_1 + x_2)}$ to find the Laplace transform of $U(T^-) + |U(T)|$, the amount of the claim causing ruin when ruin occurs, and hence its moments.

4 The distribution of the surplus prior to ruin

Throughout this section we assume that F is a continuous distribution with density f. From equation (3.15), we have

$$h_\delta(0, x, y) = \frac{\partial^2}{\partial y \partial x} H_\delta(0, x, y) = \frac{\lambda}{\kappa_\delta} f(x+y) \int_0^\infty \exp\{-(c + \delta x)z + \lambda\mu \int_0^z \phi(\delta s) ds\} dz, \quad (4.1)$$

and

$$F_\delta(0,x) = \lim_{y\to\infty} H_\delta(0,x,y) = \frac{\lambda}{\kappa_\delta}\int_0^\infty \int_0^x e^{-\delta s z}\bar{F}(s)ds \exp\left(-cz + \lambda\mu\int_0^z \phi(\delta s)ds\right)dz,$$

which gives

$$f_\delta(0,x) = \frac{d}{dx}F_\delta(0,x) = \frac{\lambda}{\kappa_\delta}\bar{F}(x)\int_0^\infty \exp\left(-(c+\delta x)z + \lambda\mu\int_0^z \phi(\delta s)ds\right)dz. \tag{4.2}$$

Thus, equations (4.2) and (4.1) yield

$$h_\delta(0,x,y) = \frac{f(x+y)}{\bar{F}(x)} f_\delta(0,x). \tag{4.3}$$

Equation (4.3) is a special case of a more general result, namely,

$$h_\delta(u,x,y) = \frac{f(x+y)}{\bar{F}(x)} f_\delta(u,x). \tag{4.4}$$

Equation (4.4) is interesting because it shows that the joint distribution of the surplus immediately prior to ruin and the deficit at ruin is determined by the individual claim amount distribution and the distribution of the surplus immediately prior to ruin. The intuition behind this is given in the proof of equation (2.40) of Gerber and Shiu (1998). We also note that the proof of Gerber and Shiu's equation still holds for equation (4.4). However, we will give an alternative analytical proof of equation (4.4) using equations for $h_\delta(u,x,y)$ and $f_\delta(u,x)$.

Due to equation (4.4), the study of the distribution of the surplus immediately prior to ruin is important. Dickson (1992) has found the following formulae for $f_{\delta=0}(u,x)$, which state that when $u \le x$,

$$f_{\delta=0}(u,x) = f_{\delta=0}(0,x)\frac{1-\psi(u)}{1-\psi(0)} \tag{4.5}$$

and when $u > x$,

$$f_{\delta=0}(u,x) = f_{\delta=0}(0,x)\frac{\psi(u-x)-\psi(u)}{1-\psi(0)}. \tag{4.6}$$

Basically, formulae (4.5) and (4.6) show that the distribution of the surplus immediately prior to ruin is a function of the ruin probability ψ. Dickson (1992) also derived the corresponding formulae for $F_{\delta=0}(u,x)$. Gerber and Shiu (1998) have generalised Dickson's formulae under the definition of ruin probability given in their paper. Here we investigate whether Dickson's formulae hold when $\delta > 0$. We will discuss this issue later in this section.

First, differentiating equation (3.14) with respect to x and y successively, we get an integral equation for $h_\delta(u,x,y)$.

For any $u \geq 0$,

$$h_\delta(u,x,y) = \frac{c\,h_\delta(0,x,y)}{c+\delta u} - \frac{\lambda I(u>x)f(x+y)}{c+\delta u} + \int_0^u k_\delta(u,t)h_\delta(t,x,y)dt. \tag{4.7}$$

Letting $y \to \infty$ in equation (3.14), we get an integral equation for $F_\delta(u,x)$. For any $u \geq 0$,

$$F_\delta(u,x) = \frac{c\,F_\delta(0,x)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u \wedge x) + \int_0^u k_\delta(u,t)F_\delta(t,x)dt. \tag{4.8}$$

Differentiating equation (4.8) with respect to x, we get an integral equation for $f_\delta(u,x)$. For any $u \geq 0$,

$$f_\delta(u,x) = \frac{c\,f_\delta(0,x)}{c+\delta u} - \frac{\lambda I(u>x)}{c+\delta u}\bar{F}(x) + \int_0^u k_\delta(u,t)f_\delta(t,x)dt.$$

Now let $K_\delta(x,s)$ be the resolvent of the Volterra equation (2.6), namely, for $x > s \geq 0$,

$$K_\delta(x,s) = \sum_{m=1}^{\infty} k_{m,\delta}(x,s)$$

where

$$k_{m,\delta}(x,s) = \int_s^x k_\delta(x,t)k_{m-1,\delta}(t,s)dt, \quad m = 2,3,\ldots,$$

with

$$k_{1,\delta}(x,s) = k_\delta(x,s) = \frac{\delta + \lambda\bar{F}(x-s)}{c+\delta x}.$$

Then by equations (3.12) and (2.8), we know that

$$\bar{\psi}_\delta(u) = \frac{c\,\bar{\psi}_\delta(0)}{c+\delta u} + c\,\bar{\psi}_\delta(0)\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt,$$

implying that

$$\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt = \frac{\bar{\psi}_\delta(u)}{c\bar{\psi}_\delta(0)} - \frac{1}{c+\delta u}. \tag{4.9}$$

In particular, when $\delta = 0$

$$k_0(x,s) = k_{\delta=0}(x,s) = \frac{\lambda}{c}\bar{F}(x-s).$$

Thus, for $x > s \geq 0$

$$K_0(x,s) = K_{\delta=0}(x,s) = \sum_{n=1}^{\infty}(\lambda/c)^n \bar{F}^{*n}(x-s) = K_0(x-s).$$

In fact, $\theta K_0(u)/(1+\theta)$ is the density function of the compound geometric distribution function

$$\sum_{n=0}^{\infty} \frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^n F_1^{*n}(u),$$

which is the well-known Beekman convolution formula for $\bar{\psi}(u)$.

Hence, equation (4.9) implies that for any $u \geq 0$,

$$\int_0^u K_0(y)dy = \frac{\bar{\psi}(u)}{\bar{\psi}(0)} - 1 = \frac{\psi(0)-\psi(u)}{1-\psi(0)}.$$

Therefore, for any $u > x \geq 0$,

$$\int_0^x K_0(u-t)dt = \int_{u-x}^u K_0(y)dy = \frac{\psi(u-x)-\psi(u)}{1-\psi(0)}. \tag{4.10}$$

In addition, by equations (3.11), (2.8), and (4.9), we know that $\psi_\delta(u)$ has the following representation of solution

$$\begin{aligned}
\psi_\delta(u) &= \frac{c\,\psi_\delta(0)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u) + \int_0^u K_\delta(u,t)\left(\frac{c\,\psi_\delta(0)}{c+\delta t} - \frac{\lambda\mu}{c+\delta t}F_1(t)\right)dt \\
&= \frac{c\,\psi_\delta(0)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u) + c\,\psi_\delta(0)\left[\frac{\bar{\psi}_\delta(u)}{c\bar{\psi}_\delta(0)} - \frac{1}{c+\delta u}\right] - \lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt \\
&= \frac{\psi_\delta(0)\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(u) - \lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt
\end{aligned}$$

and hence

$$\lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt = \frac{\psi_\delta(0)\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \psi_\delta(u) - \frac{\lambda\mu}{c+\delta u}F_1(u). \tag{4.11}$$

We can now give equations satisfied by $F_\delta(u,x)$.

Theorem 4.1 When $u \leq x$,

$$F_\delta(u,x) = \psi_\delta(u) - \frac{1-\psi_\delta(u)}{1-\psi_\delta(0)}\left(\psi_\delta(0) - F_\delta(0,x)\right). \tag{4.12}$$

When $u > x$,

$$F_\delta(u,x) = \left(F_\delta(0,x) - \frac{1}{1+\theta}F_1(x)\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} + \lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}(F_1(x)-F_1(t))dt. \tag{4.13}$$

Proof. By equations (4.8), (2.8), and (4.9), we have

$$\begin{aligned}
& F_\delta(u,x) \\
= & \frac{c\,F_\delta(0,x)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u\wedge x) + \int_0^u K_\delta(u,t)\left[\frac{c\,F_\delta(0,x)}{c+\delta t} - \frac{\lambda\mu}{c+\delta t}F_1(t\wedge x)\right]dt \\
= & \frac{c\,F_\delta(0,x)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u\wedge x) + c\,F_\delta(0,x)\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt \\
& -\lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t\wedge x)dt \\
= & \frac{c\,F_\delta(0,x)}{c+\delta u} - \frac{\lambda\mu}{c+\delta u}F_1(u\wedge x) + c\,F_\delta(0,x)\left[\frac{\bar{\psi}_\delta(u)}{c\bar{\psi}_\delta(0)} - \frac{1}{c+\delta u}\right] \\
& -\lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t\wedge x)dt \\
= & F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(u\wedge x) - \lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t\wedge x)dt. \qquad (4.14)
\end{aligned}$$

Thus, when $u \le x$, by equations (4.14) and (4.11), we have,

$$\begin{aligned}
F_\delta(u,x) &= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(u) - \lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt \\
&= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(u) - \left[\frac{\psi_\delta(0)\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \psi_\delta(u) - \frac{\lambda\mu}{c+\delta u}F_1(u)\right] \\
&= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\psi_\delta(0)\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} + \psi_\delta(u) \\
&= \psi_\delta(u) - \frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)}(\psi_\delta(0) - F_\delta(0,x)),
\end{aligned}$$

which implies that equation (4.12) holds.

On the other hand, when $u > x$, by equations (4.14) and (4.9), we have,

$$\begin{aligned}
F_\delta(u,x) &= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(x) - \lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt - \lambda\mu\int_x^u \frac{K_\delta(u,t)}{c+\delta t}F_1(x)dt \\
&= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(x) - \lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}F_1(t)dt \\
&\quad -F_1(x)\left[\lambda\mu\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt - \lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}dt\right] \\
&= F_\delta(0,x)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda\mu}{c+\delta u}F_1(x) - \lambda\mu F_1(x)\left[\frac{\bar{\psi}_\delta(u)}{c\bar{\psi}_\delta(0)} - \frac{1}{c+\delta u}\right] \\
&\quad +\lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}(F_1(x) - F_1(t))dt \\
&= \left(F_\delta(0,x) - \frac{1}{1+\theta}F_1(x)\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} + \lambda\mu\int_0^x \frac{K_\delta(u,t)}{c+\delta t}(F_1(x) - F_1(t))dt,
\end{aligned}$$

which gives equation (4.13).

Therefore, differentiating the equations for $F_\delta(u,x)$ in Theorem 4.1 with respect to x, we get the following generalisation of Dickson's formulae when $\delta > 0$.

Theorem 4.2 When $u \le x$,

$$f_\delta(u,x) = \frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} f_\delta(0,x).$$

When $u > x$,

$$f_\delta(u,x) = \left[f_\delta(0,x) - \frac{\lambda}{c}\bar{F}(x)\right] \frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} + \lambda \bar{F}(x) \int_0^x \frac{K_\delta(u,t)}{c+\delta t} dt. \tag{4.15}$$

Remark 4.1 When $\delta = 0$, we have $f_\delta(0,x) = \frac{\lambda}{c}\bar{F}(x)$ and $K_\delta(u,t) = K_0(u-t)$. Thus, by equation (4.10), we find for $u > x$,

$$\int_0^x \frac{K_\delta(u,t)}{c+\delta t} dt = \frac{1}{c}\int_0^x K_0(u-t)dt = \frac{\psi(u-x)-\psi(u)}{c(1-\psi(0))}.$$

Therefore Theorem 4.2 generalises Dickson's formulae (4.5) and (4.6). However, we point out that for $u > x$, we cannot express $\int_0^x (K_\delta(u,t)/(c+\delta t))dt$ in terms of $\psi_\delta(u)$. In this sense, Dickson's formula holds when $\delta > 0$ only when $x \ge u$. Dickson interpreted his formulae using dual events, but the duality argument does not hold when $\delta > 0$.

Next, we give an analytical proof of equation (4.4) based on the representation of solution for $h_\delta(u,x,y)$ and Theorem 4.2.

Theorem 4.3 For any $u \ge 0$,

$$h_\delta(u,x,y) = \frac{f(x+y)}{\bar{F}(x)} f_\delta(u,x).$$

Proof. By equations (4.7), (2.8) and (4.9), we have

$$\begin{aligned}
&h_\delta(u,x,y)\\
=\;& \frac{c\,h_\delta(0,x,y)}{c+\delta u} - \frac{\lambda I(u>x)}{c+\delta u} f(x+y) + \int_0^u K_\delta(u,t)\left[\frac{c\,h_\delta(0,x,y)}{c+\delta t} - \frac{\lambda I(t>x)}{c+\delta t} f(x+y)\right] dt\\
=\;& \frac{c\,h_\delta(0,x,y)}{c+\delta u} - \frac{\lambda I(u>x)}{c+\delta u} f(x+y) + c\,h_\delta(0,x,y)\left[\frac{\bar{\psi}_\delta(u)}{c\bar{\psi}_\delta(0)} - \frac{1}{c+\delta u}\right]\\
&-\lambda f(x+y)\int_0^u \frac{K_\delta(u,t)}{c+\delta t} I(t>x)dt\\
=\;& h_\delta(0,x,y)\frac{\bar{\psi}_\delta(u)}{\bar{\psi}_\delta(0)} - \frac{\lambda I(u>x)}{c+\delta u} f(x+y) - \lambda f(x+y)\int_0^u \frac{K_\delta(u,t)}{c+\delta t} I(t>x)dt. \qquad (4.16)
\end{aligned}$$

Thus, when $u \le x$, by equations (4.16), (4.3), and Theorem 4.2, we have

$$h_\delta(u,x,y) = h_\delta(0,x,y)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} = f_\delta(0,x)\,\frac{f(x+y)}{\bar F(x)}\,\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} = f_\delta(u,x)\frac{f(x+y)}{\bar F(x)}.$$

When $u > x$, by equations (4.16), (4.9), and (4.15), we have

$$\begin{aligned}
h_\delta(u,x,y) &= h_\delta(0,x,y)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}f(x+y) - \lambda f(x+y)\int_x^u \frac{K_\delta(u,t)}{c+\delta t}dt \\
&= h_\delta(0,x,y)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}f(x+y) - \lambda f(x+y)\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt \\
&\quad +\lambda f(x+y)\int_0^x \frac{K_\delta(u,t)}{c+\delta t}dt \\
&= h_\delta(0,x,y)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}f(x+y) - \lambda f(x+y)\left[\frac{\bar\psi_\delta(u)}{c\bar\psi_\delta(0)} - \frac{1}{c+\delta u}\right] \\
&\quad +\lambda f(x+y)\int_0^x \frac{K_\delta(u,t)}{c+\delta t}dt \\
&= \left[h_\delta(0,x,y) - \frac{\lambda}{c}f(x+y)\right]\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} + \lambda f(x+y)\int_0^x \frac{K_\delta(u,t)}{c+\delta t}dt \\
&= \left(h_\delta(0,x,y) - \frac{\lambda}{c}f(x+y)\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} \\
&\quad +\frac{f(x+y)}{\bar F(x)}\left\{f_\delta(u,x) - \left(f_\delta(0,x) - \frac{\lambda}{c}\bar F(x)\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)}\right\} \\
&= \frac{f(x+y)}{\bar F(x)}f_\delta(u,x) + \left(h_\delta(0,x,y) - \frac{f(x+y)}{\bar F(x)}f_\delta(0,x)\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} \\
&= \frac{f(x+y)}{\bar F(x)}f_\delta(u,x)
\end{aligned}$$

which completes the proof of Theorem 4.3.

In general, we can express Φ_δ as a function of ψ_δ and K_δ as follows.

Theorem 4.4 For any $u \ge 0$,

$$\Phi_\delta(u) = \left(\Phi_\delta(0) - \frac{\lambda}{c}\int_0^u A(s)ds\right)\frac{1-\psi_\delta(u)}{1-\psi_\delta(0)} + \lambda\int_0^u A(t)\int_0^t \frac{K_\delta(u,s)}{c+\delta s}ds\,dt. \tag{4.17}$$

Proof. By equations (2.5) and (2.8), and (4.9) we have

$$\begin{aligned}
&\Phi_\delta(u) \\
= &\frac{c\,\Phi_\delta(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds + \int_0^u K_\delta(u,t)\left(\frac{c\,\Phi_\delta(0)}{c+\delta t} - \frac{\lambda}{c+\delta t}\int_0^t A(s)ds\right)dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c\,\Phi_\delta(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds + c\,\Phi_\delta(0)\int_0^u \frac{K_\delta(u,t)}{c+\delta t}dt - \lambda\int_0^u \frac{K_\delta(u,t)}{c+\delta t}\int_0^t A(s)dsdt \\
&= \frac{c\,\Phi_\delta(0)}{c+\delta u} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds + c\,\Phi_\delta(0)\left(\frac{\bar\psi_\delta(u)}{c\bar\psi_\delta(0)} - \frac{1}{c+\delta u}\right) - \lambda\int_0^u \frac{K_\delta(u,t)}{c+\delta t}\int_0^t A(s)dsdt \\
&= \Phi_\delta(0)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds - \lambda\int_0^u \frac{K_\delta(u,t)}{c+\delta t}\int_0^t A(s)dsdt \\
&= \Phi_\delta(0)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds - \lambda\int_0^u\int_0^t A(s)ds\, d\left(\int_0^t \frac{K_\delta(u,s)}{c+\delta s}ds\right) \\
&= \Phi_\delta(0)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds \\
&\quad -\lambda\left(\int_0^u A(s)ds\int_0^u \frac{K_\delta(u,s)}{c+\delta s}ds - \int_0^u A(t)\int_0^t \frac{K_\delta(u,s)}{c+\delta s}dsdt\right) \\
&= \Phi_\delta(0)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} - \frac{\lambda}{c+\delta u}\int_0^u A(s)ds \\
&\quad -\lambda\left(\int_0^u A(s)ds\left(\frac{\bar\psi_\delta(u)}{c\bar\psi_\delta(0)} - \frac{1}{c+\delta u}\right) - \int_0^u A(t)\int_0^t \frac{K_\delta(u,s)}{c+\delta s}dsdt\right) \\
&= \left(\Phi_\delta(0) - \frac{\lambda}{c}\int_0^u A(s)ds\right)\frac{\bar\psi_\delta(u)}{\bar\psi_\delta(0)} + \lambda\int_0^u A(t)\int_0^t \frac{K_\delta(u,s)}{c+\delta s}dsdt,
\end{aligned}
$$

which completes the proof of Theorem 4.4.

In particular, when $\delta = 0$, we can obtain an expression for $\Phi = \Phi_{\delta=0}$, the expected value of the penalty function in the classical risk model, as a function of ψ as follows.

Corollary 4.1 For any $u \geq 0$,

$$
\Phi(u) = \frac{\lambda}{c}\left(\int_u^\infty A(s)ds\right)\frac{1-\psi(u)}{1-\psi(0)} + \frac{\lambda}{c}\int_0^u A(t)\frac{\psi(u-t)-\psi(u)}{1-\psi(0)}dt. \tag{4.18}
$$

Proof. By equations (4.17), (3.2), and (4.10), we have

$$
\begin{aligned}
\Phi(u) &= \left(\Phi(0) - \frac{\lambda}{c}\int_0^u A(s)ds\right)\frac{1-\psi(u)}{1-\psi(0)} + \frac{\lambda}{c}\int_0^u A(t)\int_0^t K_0(u-s)dsdt \\
&= \left(\frac{\lambda}{c}\int_0^\infty A(s)ds - \frac{\lambda}{c}\int_0^u A(s)ds\right)\frac{1-\psi(u)}{1-\psi(0)} - \frac{\lambda}{c}\int_0^u A(t)\frac{\psi(u-t)-\psi(u)}{1-\psi(0)}dt,
\end{aligned}
$$

which gives equation (4.18).

Example 4.1 For $r \geq 0$, let $w(x_1, x_2) = e^{-r(x_1+x_2)}$. Then

$$
\Phi(u) = E(e^{-r(U(T^-)+|U(T)|)}I(T<\infty)) = \tilde D(u,r),
$$

the Laplace transform of the amount of the claim causing ruin when ruin occurs, where T is the time of ruin when $\delta = 0$. Then

$$A(t) = \int_t^\infty w(t, x-t)dF(x) = \int_t^\infty e^{-rx}dF(x)$$

and Corollary 4.1 gives

$$\tilde{D}(u,r) = \frac{\lambda}{c}\left(\int_u^\infty \int_t^\infty e^{-rx}dF(x)dt\right)\frac{1-\psi(u)}{1-\psi(0)} + \frac{\lambda}{c}\int_0^u \left(\int_t^\infty e^{-rx}dF(x)\right)\frac{\psi(u-t)-\psi(u)}{1-\psi(0)}dt.$$

Thus, given that ruin occurs, if the $(n+1)$-th moment of F exists, we get the n-th moment of the claim causing ruin, namely,

$$E((U(T^-)+|U(T)|)^n \,|\, T<\infty) = \frac{(-1)^n}{\psi(u)}\left(\frac{d^n}{dr^n}\tilde{D}(u,r)|_{r=0}\right)$$

$$= \frac{\lambda}{c\psi(u)}\int_u^\infty \int_t^\infty x^n dF(x)dt\frac{1-\psi(u)}{1-\psi(0)} + \frac{\lambda}{c\psi(u)}\int_0^u \int_t^\infty x^n dF(x)\frac{\psi(u-t)-\psi(u)}{1-\psi(0)}dt.$$

Corollary 4.1 is a very important result because we can use it to obtain many well known results from the literature. These include expressions for the ruin probability ψ, for the joint and marginal (defective) distributions of the surplus prior to ruin and the deficit at ruin, and for moments of these marginal distributions. For example, we can use Corollary 4.1 to obtain formula (4.5) of Lin and Willmot (2000) for the moments of the deficit at ruin, given that ruin occurs.

Finally, we can apply the methods of this section to the expected value of the discounted penalty function of Gerber and Shiu (1998). We will derive a formula similar to that in Corollary 4.1 in the next section.

5 Gerber and Shiu's discounted penalty function revisited

Gerber and Shiu (1998) introduced the function

$$\phi_\alpha(u) = \Phi_{\delta=0,\alpha}(u) = E(w(U(T^-), |U(T)|)e^{-\alpha T}I(T<\infty)),$$

where T is as defined in Example 4.1. Through this function we can study the joint distribution of surplus prior to ruin, the deficit at ruin and the time of ruin. They defined the following ruin function:

$$\Psi_\alpha(u) = E(e^{-\alpha T+\rho U(T)}I(T<\infty))$$

where ρ is the unique non-negative root of Lundberg's fundamental equation for the classical risk model, i.e.

$$-\alpha + c\xi + \lambda(\int_0^{\infty} e^{-\xi x} f(x)dx - 1) = 0.$$

They showed that $\phi_\alpha(u)$ and $\Psi_\alpha(u)$ satisfy the following integral equations:

$$\phi_\alpha(u) = h(u) + \int_0^u \phi_\alpha(x)g(u-x)dx$$

and

$$\Psi_\alpha(u) = \int_u^{\infty} e^{-\rho(x-u)}g(x)dx + \int_0^u \Psi_\alpha(x)g(u-x)dx \tag{5.1}$$

where

$$h(x) = \frac{\lambda}{c}\int_x^{\infty} e^{-\rho(t-x)}A(t)dt$$

and

$$g(x) = \frac{\lambda}{c}\int_x^{\infty} e^{-\rho(t-x)}f(t)dt.$$

They also considered the following ruin function:

$$\bar{\Psi}_\alpha(u) = e^{\rho u} - \Psi_\alpha(u). \tag{5.2}$$

It is easy to verify from equations (5.1) and (5.2) that $\bar{\Psi}_\alpha(u)$ satisfies the following integral equation:

$$\bar{\Psi}_\alpha(u) = e^{\rho u}\bar{\Psi}_\alpha(0) + \int_0^u \bar{\Psi}_\alpha(x)g(u-x)dx$$

where

$$\bar{\Psi}_\alpha(0) = 1 - \Psi_\alpha(0) = 1 - \int_0^{\infty} e^{-\rho x}g(x)dx.$$

Thus, letting

$$K_\alpha(x) = \sum_{n=1}^{\infty} g^{*n}(x),$$

we have by equations (2.7) to (2.9)

$$\bar{\Psi}_\alpha(u) = e^{\rho u}\bar{\Psi}_\alpha(0) + \bar{\Psi}_\alpha(0)\int_0^u e^{\rho x}K_\alpha(u-x)dx, \tag{5.3}$$

which is equation (4.12) of Gerber and Shiu (1997), an equation they obtained using Laplace transforms.

Equation (5.3) implies that

$$\int_0^u K_\alpha(u-t)e^{\rho t}dt = \frac{\bar{\Psi}_\alpha(u)}{\bar{\Psi}_\alpha(0)} - e^{\rho u},$$

or, equivalently,

$$\int_0^u K_\alpha(y)e^{-\rho y}dy = \frac{e^{-\rho u}\bar{\Psi}_\alpha(u) - \bar{\Psi}_\alpha(0)}{\bar{\Psi}_\alpha(0)}.$$

Thus, using arguments similar to those in the previous section, we can easily obtain an expression for $\phi_\alpha(u)$ as follows.

Theorem 5.1 For any $u \geq 0$,

$$\phi_\alpha(u) = \frac{\lambda}{c}\int_u^\infty e^{-\rho t}A(t)dt\,\frac{e^{\rho u} - \Psi_\alpha(u)}{1-\Psi_\alpha(0)} + \frac{\lambda}{c}\int_0^u A(t)\frac{\Psi_\alpha(u-t) - e^{-\rho t}\Psi_\alpha(u)}{1-\Psi_\alpha(0)}dt.$$

Proof. We have by equations (2.7) to (2.9)

$$\begin{aligned}
\phi_\alpha(u) &= h(u) + \int_0^u K_\alpha(x)h(u-x)dx \\
&= h(u) + \int_0^u e^{\rho x}h(u-x)\,d\left(\int_0^x e^{-\rho t}K_\alpha(t)dt\right) \\
&= h(u) + h(0)e^{\rho u}\int_0^u e^{-\rho t}K_\alpha(t)dt - \frac{\lambda}{c}\int_0^u e^{\rho x}A(u-x)\int_0^x e^{-\rho t}K_\alpha(t)dtdx \\
&= h(u) + h(0)\frac{\bar{\Psi}_\alpha(u) - e^{\rho u}\bar{\Psi}_\alpha(0)}{\bar{\Psi}_\alpha(0)} - \frac{\lambda}{c}\int_0^u A(u-x)\frac{\bar{\Psi}_\alpha(x) - e^{\rho x}\bar{\Psi}_\alpha(0)}{\bar{\Psi}_\alpha(0)}dx \\
&= h(u) + h(0)\frac{e^{\rho u}\Psi_\alpha(0) - \Psi_\alpha(u)}{1-\Psi_\alpha(0)} - \frac{\lambda}{c}\int_0^u A(u-x)\frac{e^{\rho x}\Psi_\alpha(0) - \Psi_\alpha(x)}{1-\Psi_\alpha(0)}dx \\
&= \frac{\lambda}{c}\int_u^\infty e^{-\rho(t-u)}A(t)dt + \left(\frac{\lambda}{c}\int_0^u e^{-\rho t}A(t)dt + \frac{\lambda}{c}\int_u^\infty e^{-\rho t}A(t)dt\right)\frac{e^{\rho u}\Psi_\alpha(0) - \Psi_\alpha(u)}{1-\Psi_\alpha(0)} \\
&\quad -\frac{\lambda}{c}\int_0^u A(t)\frac{e^{\rho(u-t)}\Psi_\alpha(0) - \Psi_\alpha(u-t)}{1-\Psi_\alpha(0)}dt \\
&= \frac{\lambda}{c}\int_u^\infty e^{-\rho t}A(t)dt\frac{e^{\rho u} - \Psi_\alpha(u)}{1-\Psi_\alpha(0)} + \frac{\lambda}{c}\int_0^u A(t)\frac{\Psi_\alpha(u-t) - e^{-\rho t}\Psi_\alpha(u)}{1-\Psi_\alpha(0)}dt.
\end{aligned}$$

The above result gives an expression for $\phi_\alpha(u)$ as a function of the ruin probability $\Psi_\alpha(u)$ defined by Gerber and Shiu (1998). In fact, Corollary 4.1 is a special case of this result when $\alpha = 0$. Also, Theorem 5.1 of Lin and Willmot (1999) can be obtained by taking $w(x_1, x_2) = I(x_1 \leq x)I(x_2 \leq y)$ in the above result.

6 Concluding remarks

We have derived the following: an integral equation for $\Phi_{\delta,\alpha}(u)$; the Laplace transform of an auxiliary function of $\Phi_\delta(u)$; an exact expression for $\Phi_\delta(0)$; and relationships between ruin functions and the ultimate ruin probability. We have also generalised Dickson's (1992) formulae from the case when $\delta = 0$ to the case when $\delta > 0$. Ruin functions are very complicated when $\delta > 0$. Although we have discussed some properties of ruin functions when $\delta > 0$ we have been unable to find many explicit results. Further research in ruin theory when $\delta > 0$ is clearly required. For example, it seems that we cannot apply both the probability measure transform technique of Gerber and Shiu (1998) and the Laplace transform technique of Sundt and Teugels (1995) to determine an expression for $\Phi_{\delta,\alpha}(0)$ when $\delta > 0$ and $\alpha > 0$. We leave this as an open question.

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