On the discounted penalty function in a discrete time renewal risk model with general interclaim times

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Abstract

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In this paper a discrete time renewal risk model with arbitrary interclaim times is discussed. We show that the expected discounted penalty function satisfies a recursive formula. In particular, the probability generating function of the time of ruin, as a function of the initial surplus, has a compound geometric tail. When the claim amounts follow a geometric distribution, explicit expression for the Gerber-Shiu function can be obtained for the specially chosen penalty function. The constant claim amounts and mixed geometric claim amounts are also examined.

Keywords: Sparre Andersen risk process, Gerber-Shiu function, recursive formula, generating function, geometric distribution, mixture of geometric distributions.

1 Introduction

In this paper we consider a discrete time Sparre Andersen risk process

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad n = 1, 2, \ldots, \tag{1.1}$$

where $u \in \mathbb{N}$ is the initial surplus, and X_i's are i.i.d. positive integer-valued random variables with common probability function (p.f.) $p(x) = \mathbb{P}\{X_1 = x\}$, for $x = 1, 2, \ldots$, denoting individual claim sizes. Let $P(x) = 1 - \bar{P}(x)$ be the distribution function of X_1, μ be its mean and $\hat{p}(z) = \sum_{x=1}^{\infty} z^x p(x), z \in \mathbb{C}$, be the probability generating function (p.g.f.).

The counting process $\{N(n); n \in \mathbb{N}\}$ denotes the number of claims up to time n and is defined as $N(n) = \max\{k : W_1 + W_2 + \cdots + W_k \leq n\}$, where the interclaim times W_i's are assumed to be i.i.d. positive integer-valued random variables with common p.f. $k(t) = \mathbb{P}\{W_1 = t\}$, for $t = 1, 2, \ldots$, distribution function $K(t)$, mean $\mathbb{E}[W_1] < \infty$, and p.g.f. $\hat{k}(z) = \sum_{t=1}^{\infty} z^t k(t), t \in \mathbb{C}$.

Further we assume that $\{W_i; i \in \mathbb{N}^+\}$ and $\{X_i; i \in \mathbb{N}^+\}$ are mutually independent, and $\mathbb{E}[W_1] = (1+\theta)\mathbb{E}[X_1] = (1+\theta)\mu, \theta > 0$, in order to have a positive safety loading.

For risk model (1.1), let random variable $T = \min\{n \in \mathbb{N}^+ : U(n) < 0\}$ be the time of ruin. Then

$$\psi(u) = \mathbb{P}\{T < \infty | U(0) = u\}, \quad u \in \mathbb{N},$$

is the ultimate ruin probability.

Let $w(x, y), x, y = 0, 1, 2, \ldots$, be a non-negative penalty function. For $0 < v < 1$, we define

$$\phi_v(u) = \mathbb{E}[v^T w(U(T-1), |U(T)|) I(T < \infty) | U(0) = u], \qquad u \in \mathbb{N}. \tag{1.2}$$

The quantity $w(U(T-1), |U(T)|)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T-1)$ and deficit $|U(T)|$. Then $\phi_v(u)$ is the expected discounted penalty (Gerber-Shiu) function with the discount factor v. When the penalty function depends on the deficit only, that is, with $w(x, y) = w_1(y)$, (1.2) becomes

$$\phi_{v,1}(u) = \mathbb{E}[v^T w_1(|U(T)|) I(T < \infty) | U(0) = u], \qquad u \in \mathbb{N}. \tag{1.3}$$

Also we consider a generalization of (1.3) with $w(x, y) = s^x w_1(y)$, namely

$$\phi_{v,s}(u) = \mathbb{E}[v^T s^{U(T-1)} w_1(|U(T)|) I(T < \infty) | U(0) = u], \qquad u \in \mathbb{N}. \tag{1.4}$$

Recently, much work has been done to analyze the Gerber-Shiu function under a continuous time Sparre Andersen model. For most of the results, assumptions were made about the distribution of the interclaim times. See Li and Garrido (2005) and references therein.

For the discrete time risk models, Cheng et al. (2000) study the expected discounted penalty function and quantities associated with time of ruin in the compound binomial risk model. For the discrete time Sparre Andersen models, Pavlova and Willmot (2004) give an expression of the expected discounted penalty function in the discrete time stationary renewal risk model in terms of that in the corrsponding ordinary renewal risk model; Li (2005a, b) study the expected discounted penalty function in the Sparre Andersen risk model with discrete K_n interclaim times.

Assuming an arbitrary distribution for the interclaim times, Willmot (2006) and Landriault and Willmot (2007) derive the defective renewal equation satisfied by the Gerber-Shiu discounted penalty function in a continuous time renewal risk model.

Malinovskii (1998) and Wang and Liu (2002) discuss the Laplace transform of finite time ruin probabilities with same assumptions.

In this paper, a discrete time Sparre Andersen model with an arbitrary interclaim times distribution $K(t)$ is considered and some general analytic properties of $\phi_v(u)$ are explored. When individual claim amounts follow some specific distributions, including geometric, degenerate distribution at constant size, and the mixture of geometric, explicit results for the expected discounted penalty function with an appropriately chosen $w(x, y)$ can be obtained.

2 A recursive formula for $\phi_v(u)$

We consider, for $x \in \mathbb{N}$ and $y \in \mathbb{N}^+$,

$$f_3(x, y, t|u) = \mathbb{P}\{U(T-1) = x, |U(T)| = y, T = t|U(0) = u\},$$

which is the joint p.f. of the surplus just before ruin, the deficit at ruin, and the time of ruin. And we define

$$f_2(x, t|u) = \sum_{y=1}^{\infty} f_3(x, y, t|u)$$

to be the joint p.f. of $U(T-1)$ and T. Let $v \in (0, 1)$ be the discount factor over one time period, then we have $f_1(x|u) = \sum_{t=1}^{\infty} v^t f_2(x, t|u)$ as a discounted p.f. of $U(T-1)$. Further, we define

$$p_x(y) = \frac{p(x+y+1)}{\bar{P}(x+1)}, \qquad y \in \mathbb{N}^+,$$

to be the conditional p.f. of the deficit (y), given both the surplus prior to ruin (x) and the time of ruin (t). Thus, we have

$$f_3(x, y, t|u) = p_x(y) f_2(x, t|u), \qquad x \in \mathbb{N}, \quad y \in \mathbb{N}^+. \tag{2.1}$$

Similar to the one in Li (2005a), conditioning on the first time when the surplus process drops below the initial surplus u, we have the following equation for $\phi_v(u)$:

$$\begin{aligned} \phi_v(u) &= \sum_{y=1}^{u} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t \phi_v(u-y) f_3(x, y, t|0) \\ &\quad + \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t w(x+u, y-u) f_3(x, y, t|0), \qquad u \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that when $u = 0$, the summation over y from 1 to u is 0. Following from (2.1) and the definition of $f_1(x|u)$, we have

$$\sum_{t=1}^{\infty} v^t f_3(x, y, t|0) = p_x(y) f_1(x|0). \tag{2.3}$$

Substituting (2.3) into (2.2) yields

$$\begin{aligned}\phi_v(u) &= \sum_{y=1}^{u}\sum_{x=0}^{\infty}\phi_v(u-y)p_x(y)f_1(x|0)\\ &\quad+\sum_{y=u+1}^{\infty}\sum_{x=0}^{\infty}w(x+u,y-u)p_x(y)f_1(x|0), \qquad u\in\mathbb{N}. \end{aligned}\tag{2.4}$$

Let $\xi_v=\sum_{x=0}^{\infty}f_1(x|0)$ and

$$g_1(y)=\sum_{x=0}^{\infty}p_x(y)f_1(x|0)/\xi_v, \qquad y\in\mathbb{N}^+. \tag{2.5}$$

The definitions of $p_x(y)$ and ξ_v show that $g_1(y)$ is a proper p.f.. Then we obtain the following recursive equation for $\phi_v(u)$:

$$\phi_v(u)=\xi_v\sum_{y=1}^{u}\phi_v(u-y)g_1(y)+\sum_{y=u+1}^{\infty}\sum_{x=0}^{\infty}w(x+u,y-u)p_x(y)f_1(x|0), u\in\mathbb{N}^+, \tag{2.6}$$

where

$$\phi_v(0)=\sum_{y=1}^{\infty}\sum_{x=0}^{\infty}w(x,y)p_x(y)f_1(x|0).$$

If the penalty function only involves the deficit so that $w(x,y)=w_1(y)$, then (1.2) reduces to (1.3), and then from (2.6), we have

$$\phi_{v,1}(u) = \xi_v\sum_{y=1}^{u}\phi_{v,1}(u-y)g_1(y)+\xi_v\sum_{y=u+1}^{\infty}w_1(y-u)g_1(y), \qquad u\in\mathbb{N}^+, \tag{2.7}$$

$$\phi_{v,1}(0) = \xi_v\sum_{y=1}^{\infty}w_1(y)g_1(y). \tag{2.8}$$

Furthermore, if $w_1(y)=1$, we denote $\bar{D}_v(u)=\phi_{v,1}(u)$, with definition

$$\bar{D}_v(u)=\mathbb{E}\Big[v^T I(T<\infty)|U(0)=u\Big]$$

and satisfying

$$\bar{D}_v(u)=\xi_v\sum_{y=1}^{u}\bar{D}_v(u-y)g_1(y)+\xi_v\bar{G}(u), \qquad u\in\mathbb{N}, \tag{2.9}$$

where $\bar{G}(u)=1-G(u)=\sum_{y=u+1}^{\infty}g_1(y)$. When the discount factor v goes to 1, the limit of $\bar{D}_v(u)$ is just the probability of ruin $\psi(u)$. It is known that the recursive equation (2.9) has the following solution

$$\bar{D}_v(u)=(1-\xi_v)\sum_{n=1}^{\infty}\xi_v^n\bar{G}^{*n}(u), \qquad u\in\mathbb{N}, \tag{2.10}$$

where $G^{*n}(u) = 1 - \bar{G}^{*n}(u)$ is the d.f. of the n-fold convolution of $G(u)$. Particularly, when $u = 0$, $\bar{D}_v(0) = \xi_v$. Since the support of $g_1(y)$ is $\mathbb{N}^+$, then $G^{*n}(u) = 0$, if $n > u$. Therefore, the right-hand side of (2.10) can be reduced to a sum of finite terms as

$$\bar{D}_v(u) = 1 - (1 - \xi_v) \sum_{n=0}^{u} \xi_v^n G^{*n}(u), \qquad u \in \mathbb{N}. \tag{2.11}$$

A similar explicit expression as (2.11) will be derived for $\phi_v(u)$ in next section.

3 Explicit expression for $\phi_v(u)$

In this section, we will derive an explicit expression for $\phi_v(u)$ in terms of a compound geometric distribution function. Similar work is done by Li (2005b).
Now we define a compound geometric d.f. as

$$a(u) = (1 - \xi_v) \sum_{n=0}^{\infty} \xi_v^n g_1^{*n}(u), \qquad u \in \mathbb{N},$$

where $g_1^{*n}(u)$ is the n-fold convolution of d.f. $g_1(u)$, and $a(0) = 1 - \xi_v$. By a similar argument to the one at the end of above section, we can reduce $a(u)$ to a sum of finite terms as

$$a(u) = (1 - \xi_v) \sum_{n=0}^{u} \xi_v^n g_1^{*n}(u), \qquad u \in \mathbb{N}.$$

We then have the following result showing that, for general penalty function $w(x, y)$, the Gerber-Shiu function $\phi_v(u)$ can be expressed explicitly in terms of the compound geometric d.f. $a(u)$.

Theorem 1

$$\phi_v(u) = \frac{1}{1 - \xi_v} \sum_{y=0}^{u} H(u - y) a(y), \qquad u \in \mathbb{N}, \tag{3.1}$$

where $H(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x + u, y - u) p_x(y) f_1(x|0)$.

Proof. Using the newly defined function $H(u)$, (2.6) can be rewritten as

$$\phi_v(u) = \xi_v \sum_{y=1}^{u} \phi_v(u - y) g_1(y) + H(u), \qquad u \in \mathbb{N}^+. \tag{3.2}$$

Let $\hat{\phi}_v(z) = \sum_{u=0}^{\infty} \phi_v(u) z^u$, $\hat{g}_1(z) = \sum_{y=1}^{\infty} g_1(y) z^y$, $\hat{a}(z) = \sum_{y=0}^{\infty} a(y) z^y$, and $\hat{H}(z) = \sum_{u=0}^{\infty} H(u) z^u$ be the generating functions of $\phi_v(u), g_1(y), a(y)$ and $H(u)$, respectively. Then multiplying both sides of (3.2) by z^u and summing over u from 0 to ∞ yields

$$\hat{\phi}_v(z) = \frac{\hat{H}(z)}{1 - \xi_v \hat{g}_1(z)}, \qquad |\Re(z)| < 1. \tag{3.3}$$

Since $\hat{a}(z) = (1-\xi_v)/[1-\xi_v\hat{g}_1(z)]$, then $\hat{\phi}_v(z) = \hat{a}(z)\hat{H}(z)/(1-\xi_v)$. Inverting this generating function gives (3.1).

4 Particular claim size distributions

In this section we will consider several particular distributions for individual claim sizes X_i, $i = 1, 2, \ldots$. These claim amounts distributions include a degenerate distribution at a constant size, the geometric distribution and a mixture of geometric distributions.

4.1 Constant Claim Sizes

In this subsection, we assume that the claim amounts are of constant sizes of 2, i.e., $p(2) = 1$. Then

$$p_x(y) = \frac{p(x+y+1)}{\bar{P}(x+1)} = \begin{cases} 1, & x = 0, y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (2.5) that $g_1(y) = 1$, if $y = 1$, and $g_1(y) = 0$, for $y = 2, 3, \ldots$. Equation (2.9) shows that

$$\begin{aligned} \bar{D}_v(0) &= \xi_v, \\ \bar{D}_v(u) &= \xi_v \bar{D}_v(u-1), \quad u \in \mathbb{N}^+. \end{aligned}$$

Solving the above recursive formulas give

$$\bar{D}_v(u) = \xi_v^{u+1}, \qquad u \in \mathbb{N}.$$

To determine ξ_v, we have by conditioning on the time and the amount of the first claim that, for $u \in \mathbb{N}^+$,

$$\begin{aligned} \bar{D}_v(u) &= \mathbb{E}\left[v^{W_1}\bar{D}_v(u+W_1-2)\right] \\ &= \sum_{t=1}^{\infty} v^t k(t)\bar{D}_v(u+t-2) = \sum_{t=1}^{\infty} v^t k(t)\xi_v^{u+t-1} \\ &= \frac{\bar{D}_v(u)}{\xi_v^2}\sum_{t=1}^{\infty}(v\xi_v)^t k(t). \end{aligned} \tag{4.1}$$

Formula (4.1) shows that ξ_v $(0 < \xi_v < 1)$ is the solution of the following equation:

$$\xi_v^2 = \hat{k}(v\xi_v). \tag{4.2}$$

In particular,

$$\psi(u) = \lim_{v\to 1^-}\mathbb{E}\left[v^T I(T<\infty)|U(0)=u\right] = \lim_{v\to 1^-}\xi_v^{u+1} = \xi_1^{u+1}, \quad u \in \mathbb{N},$$

where $\xi_1 = \psi(0)$ is the solution of the equation $\xi_1^2 = \hat{k}(\xi_1)$.

Define $M(u) = \mathbb{E}\left[TI(T < \infty)|U(0) = u\right]$. Then

$$M(u) = \left.\frac{\partial \bar{D}_v(u)}{\partial v}\right|_{v=1} = (u+1)\xi_1^u \left.\frac{\mathrm{d}\xi_v}{\mathrm{d}v}\right|_{v=1} = (u+1)\xi_1^u M(0). \tag{4.3}$$

To determine $M(0)$, one can differentiate Eq. (4.2) with respect to v and set $v = 1$, i.e.,

$$2\xi_1 M(0) = \hat{k}'(\xi_1)[\xi_1 + M(0)]. \tag{4.4}$$

Solving it gives

$$M(0) = \frac{\xi_1 \hat{k}'(\xi_1)}{2\xi_1 - \hat{k}'(\xi_1)}. \tag{4.5}$$

Substituting (4.5) into (4.3) yields

$$M(u) = (u+1)\xi_1^{u+1}\frac{\hat{k}'(\xi_1)}{2\xi_1 - \hat{k}'(\xi_1)} = \frac{(u+1)\hat{k}'(\xi_1)}{2\xi_1 - \hat{k}'(\xi_1)}\psi(u).$$

Then we have for $u \in \mathbb{N}$,

$$\mathbb{E}[T|T < \infty, U(0) = u] = \frac{\mathbb{E}\left[TI(T < \infty)|U(0) = u\right]}{\mathbb{E}\left[I(T < \infty)|U(0) = u\right]} = \frac{(u+1)\hat{k}'(\xi_1)}{2\xi_1 - \hat{k}'(\xi_1)}.$$

4.2 Geometric Claim Amounts Distributions

We assume that claim amounts are geometrically distributed with $p(x) = (1-q)q^{x-1}$, for $x \in \mathbb{N}^+$, and $0 < q < 1$. It can be easily verified that $p_x(y) = (1-q)q^{y-1} = p(y)$, for $y \in \mathbb{N}^+$. Then $g_1(y) = \sum_{x=0}^{\infty} p_x(y) f_1(x|0)/\xi_v = p(y), y \in \mathbb{N}^+$, because $\xi_v = \sum_{x=0}^{\infty} f_1(x|0)$.

Equation (2.9) shows that $\bar{D}_v(u) = \mathbb{E}\left[v^T I(T < \infty)|U(0) = u\right]$ with $\bar{D}_v(0) = \xi_v$ satisfies the following recursive formula:

$$\begin{aligned} \bar{D}_v(u) &= \xi_v \sum_{y=1}^{u} \bar{D}_v(u-y)p(y) + \xi_v \bar{P}(u) \\ &= \xi_v \sum_{y=1}^{u} \bar{D}_v(u-y)(1-q)q^{y-1} + \xi_v q^u, \qquad u \in \mathbb{N}. \end{aligned} \tag{4.6}$$

The solution to (4.6) is

$$\bar{D}_v(u) = \xi_v[q + \xi_v(1-q)]^u, \qquad u \in \mathbb{N}, \tag{4.7}$$

where ξ_v is to be determined later on in this section.

In what follows we consider the Gerber-Shiu function $\phi_{v,s}(u)$. For the geometric p.f. $p(x)$, we have $a(u) = (1-\xi_v)\sum_{n=0}^{u}\xi_v^n g_1^{*n}(u) = (1-\xi_v)\sum_{n=0}^{u}\xi_v^n p^{*n}(u), u \in \mathbb{N}$, and

$$\begin{aligned} H(u) &= \sum_{y=u+1}^{\infty}\sum_{x=0}^{\infty} s^{x+u} w_1(y-u)p(y)f_1(x|0) \\ &= s^u \sum_{y=1}^{\infty} w_1(y)p(y+u)\sum_{x=0}^{\infty} s^x f_1(x|0) \\ &= s^u q^u \eta_v(s), \end{aligned} \tag{4.8}$$

where $\eta_v(s) = \sum_{y=1}^{\infty} w_1(y)p(y)\sum_{x=0}^{\infty} s^x f_1(x|0) = \mathbb{E}[w_1(X_1)]\sum_{x=0}^{\infty} s^x f_1(x|0)$. Then the generating function of $H(u)$ is

$$\hat{H}(z) = \sum_{u=0}^{\infty} H(u)z^u = \eta_v(s)\sum_{u=0}^{\infty}(qs)^u z^u = \frac{\eta_v(s)}{1-qsz}.$$

Substituting $\hat{H}(z)$ into (3.3), with $\hat{g}_1(z) = \hat{p}(z)$, gives

$$\begin{aligned} \hat{\phi}_{v,s}(z) &= \frac{1}{1-\xi_v\frac{(1-q)z}{1-qz}}\frac{\eta_v(s)}{1-qsz}, \\ &= \eta_v(s)\frac{1-qz}{[1-(q+\xi_v(1-q))z](1-qsz)} \\ &= \eta_v(s)\left[\frac{\alpha}{1-qsz} + \frac{1-\alpha}{1-(q+\xi_v(1-q))z}\right], \end{aligned}$$

where

$$\alpha = \frac{q(1-s)}{q(1-s)+\xi_v(1-q)}.$$

Inverting $\hat{\phi}_{v,s}(z)$ yields

$$\phi_{v,s}(u) = \eta_v(s)\left\{\alpha(qs)^u + (1-\alpha)[q+\xi_v(1-q)]^u\right\}, \qquad u \in \mathbb{N}. \tag{4.9}$$

Now we are going to determine ξ_v and $\eta_v(s)$ in (4.9). Conditioning on the time and amount of the first claim for the surplus process, one obtains that for $u \in \mathbb{N}$,

$$\begin{aligned} &\phi_{v,s}(u) \\ =\; &\mathbb{E}\left[v^{W_1}\phi_{v,s}(u+W_1-X_1)\right] = \sum_{t=1}^{\infty} v^t k(t)\mathbb{E}[\phi_{v,s}(u+t-X_1)] \\ =\; &\sum_{t=1}^{\infty} v^t k(t)\left[\sum_{x=1}^{u+t}\phi_{v,s}(u+t-x)p(x) + s^{u+t-1}\sum_{x=u+t+1}^{\infty} w_1(x-u-t)p(x)\right] \end{aligned} \tag{4.10}$$

Let $\tau_{v,s}(t) = \sum_{x=1}^{t} \phi_{v,s}(t-x)p(x), t \in \mathbb{N}^+$. Working on its generating function yields the following results:

$$\begin{aligned}
\hat{\tau}_{v,s}(z) &= \hat{\phi}_{v,s}(z)\hat{p}(z) \\
&= \eta_v(s)\frac{(1-q)z}{[1-(q+\xi_v(1-q))z](1-qsz)} \\
&= \eta_v(s)\frac{1-\alpha}{\xi_v}\Big[\frac{1}{1-(q+\xi_v(1-q))z} - \frac{1}{1-qsz}\Big],
\end{aligned}$$

and therefore

$$\tau_{v,s}(t) = \eta_v(s)\frac{1-\alpha}{\xi_v}\Big[(q+\xi_v(1-q))^t - (qs)^t\Big], \quad t \in \mathbb{N}^+. \tag{4.11}$$

Substituting (4.11) into (4.10) gives

$$\begin{aligned}
\phi_{v,s}(u) &= \sum_{t=1}^{\infty} v^t k(t)\left[\tau_{v,s}(u+t) + q^{u+t}s^{u+t-1}\sum_{x=1}^{\infty} w_1(x)p(x)\right] \\
&= \eta_v(s)\frac{1-\alpha}{\xi_v}\Big\{[q+\xi_v(1-q)]^u\hat{k}(v[q+\xi_v(1-q)]) - (qs)^u\hat{k}(qsv)\Big\} \\
&\quad +(qs)^u\mathbb{E}[w_1(X_1)]\hat{k}(qsv)/s. \qquad (4.12)
\end{aligned}$$

Comparing the right-hand side of (4.9) and (4.12), we obtain the following equations of ξ_v and $\eta_v(s)$:

$$\begin{aligned}
\xi_v &= \hat{k}\{v[q+\xi_v(1-q)]\}, \qquad (4.13) \\
\eta_v(s) &= \frac{\xi_v\mathbb{E}[w_1(X_1)]\hat{k}(vqs)s^{-1}}{\alpha\xi_v + (1-\alpha)\hat{k}(vqs)}. \qquad (4.14)
\end{aligned}$$

It can be shown that equation (4.13) has a unique solution between 0 and 1 as follows. Define $h(x) = \hat{k}\{v[q+\xi_v(1-q)]\} - x$. Since $h(0) = \hat{k}(vq) > 0, h(1) = \hat{k}(v) - 1 < 0$, and $h(x)$ is convex, then $h(x) = 0$ has a unique solution in $(0,1)$.

From (4.9) and (4.14) we obtain

$$\begin{aligned}
\phi_{v,s}(u) &= \frac{\xi_v\mathbb{E}[w_1(X_1)]\hat{k}(vqs)s^{-1}}{\alpha\xi_v + (1-\alpha)\hat{k}(vqs)}\{\alpha(qs)^u + (1-\alpha)[q+\xi_v(1-q)]^u\} \\
&= \frac{\mathbb{E}[w_1(X_1)]\hat{k}(vqs)s^{-1}}{\alpha\xi_v + (1-\alpha)\hat{k}(vqs)}\left[\xi_v\alpha(qs)^u + (1-\alpha)\bar{D}_v(u)\right], \quad u \in \mathbb{N}.
\end{aligned}$$

In particular, if $v = 1$ and $w_1(y) = 1$, then

$$\begin{aligned}
\phi_{1,s}(u) &= \mathbb{E}\left[s^{U(T-1)}I(T<\infty)|U(0)=u\right] \\
&= \frac{\hat{k}(qs)s^{-1}}{\alpha\psi(0) + (1-\alpha)\hat{k}(qs)}\left[\psi(0)\alpha(qs)^u + (1-\alpha)\psi(u)\right], \quad u \in \mathbb{N},
\end{aligned}$$

where $\psi(u)=\bar{D}_1(u)=\xi_1[q+\xi_1(1-q)]^u$, $\psi(0)=\xi_1$ can be obtained by solving the following equation:

$$\xi_1=\hat{k}\{q+\xi_1(1-q)\}.$$

Since $\frac{\mathrm{d}\alpha}{\mathrm{d}s}|_{s=1}=-q/[\xi_1(1-q)]$, then we have

$$\begin{aligned}\mathbb{E}\left[U(T-1)I(T<\infty)|U(0)=u\right] &= \lim_{s\to 1^-}\frac{\partial\phi_{1,s}(u)}{\partial s}\\ &= \frac{q}{1-q}\left[\frac{\psi(u)}{\hat{k}(q)}-q^u\right]-\psi(u), \quad u\in\mathbb{N}\quad(4.15)\end{aligned}$$

To calculate $M(u)=\mathbb{E}\left[TI(T<\infty)|U(0)=u\right]$, we differentiate Eq. (4.7) with respect to v and set $v=1$, then

$$\begin{aligned}M(u) &= M(0)[q+\xi_1(1-q)]^u+\xi_1 u[q+\xi_1(1-q)]^{u-1}(1-q)M(0)\\ &= M(0)[q+\xi_1(1-q)(1+u)][q+\xi_1(1-q)]^{u-1}.\end{aligned}\qquad(4.16)$$

To determine $M(0)$, we differentiate Eq. (4.13) with respect to v and set $v=1$, then

$$M(0)=\hat{k}'\{q+\xi_1(1-q)\}[q+\xi_1(1-q)+(1-q)M(0)].$$

Solving it gives

$$M(0)=\frac{[q+\xi_1(1-q)]\hat{k}'\{q+\xi_1(1-q)\}}{1-(1-q)\hat{k}'\{q+\xi_1(1-q)\}}.\qquad(4.17)$$

Substituting (4.17) into (4.16) yields

$$M(u)=\frac{\hat{k}'\{q+\xi_1(1-q)\}[q/\xi_1+(1-q)(1+u)]}{1-(1-q)\hat{k}'\{q+\xi_1(1-q)\}}\psi(u).$$

Then

$$\begin{aligned}\mathbb{E}[T|T<\infty,U(0)=u] &= \frac{\mathbb{E}\left[TI(T<\infty)|U(0)=u\right]}{\mathbb{E}\left[I(T<\infty)|U(0)=u\right]}\\ &= \frac{\hat{k}'\{q+\xi_1(1-q)\}[q/\xi_1+(1-q)(1+u)]}{1-(1-q)\hat{k}'\{q+\xi_1(1-q)\}}.\end{aligned}$$

4.3 Mixed Geometric Claim Amounts

We assume that claim amounts have a mixed geometric distribution with coefficients $0<\alpha_j<1, \sum_{j=1}^n\alpha_j=1$, i.e.,

$$p(x)=\sum_{j=1}^n\alpha_j\rho_j(x), \qquad x\in\mathbb{N}^+,\qquad(4.18)$$

where $\rho_j(x) = (1-q_j)\, q_j^{x-1}, j = 1, 2, \ldots, n$, is a geometric p.f. with parameter $0 < q_j < 1$. We also denote by Y_j a random variable following the distribution $\rho_j(x)$. For this distribution, its survival function is $\bar{P}(x) = \sum_{i=x+1}^{\infty} p(i) = \sum_{j=1}^{n} \alpha_j\, q_j^x$, and its p.g.f. is $\hat{p}(z) = \sum_{x=1}^{\infty} p(x) z^x = \sum_{j=1}^{n} \alpha_j \hat{\rho}_j(z)$, where $\hat{\rho}_j(z) = \sum_{x=1}^{\infty} \rho_j(x) z^x = (1-q_j)z/(1-q_j z)$ is the generating function of $\rho_j(x)$. It can be shown that $p(x)$ satisfies the following property

$$p(x+y) = \sum_{j=1}^{n} \zeta_j(x)\, \rho_j(y), \qquad x \in \mathbb{N}, \quad y \in \mathbb{N}^+,$$

where $\zeta_j(x) = \alpha_j q_j^x, j = 1, 2, \ldots, n$. Continuous claim size distributions, which admit the above factorization, are also considered by Willmot (2006) and Landriault and Willmot (2007). Since $\sum_{j=1}^{n} \zeta_j(y) = \bar{P}(y)$, then we have, for $x \in \mathbb{N}, y \in \mathbb{N}^+$,

$$\begin{aligned}
p_x(y) &= \frac{p(x+y+1)}{\bar{P}(x+1)} = \frac{\sum_{j=1}^{n} \zeta_j(x+1)\rho_j(y)}{\bar{P}(x+1)} \\
&= \sum_{j=1}^{n} \zeta_j^*(x+1)\rho_j(y), \\
p_x(y+u) &= \frac{\sum_{j=1}^{n} \zeta_j(x+1)\rho_j(y+u)}{\bar{P}(x+1)} \\
&= \sum_{j=1}^{n} \zeta_j^*(x+1)\rho_j(y+u), \qquad u \geq 1, \qquad (4.19)
\end{aligned}$$

where $\zeta_j^*(x+1) = \zeta_j(x+1)/\bar{P}(x+1)$, so $p_x(y)$ is a mixture of $\rho_1(y), \ldots, \rho_n(y)$ with weights $\zeta_j^*(x+1), j = 1, 2, \ldots, n$.

The roots of the following generalised Lundberg's equation plays an important role in the rest of this paper.

It follows from Li (2005b, Section 4) that the following equation

$$\hat{k}(v/z)\hat{p}(z) = 1 \qquad (4.20)$$

is called the generalised Lundberg equation.

Using the same arguments as in Landriault and Willmot (2007), we can prove that the generalised Lundberg equation (4.20) has the same roots as the following equation:

$$\sum_{x=0}^{\infty} \sum_{y=1}^{\infty} z^y f_1(x|0) p_x(y) = 1. \qquad (4.21)$$

Lemma 1 If the claim amounts have a mixture of geometric distribution with the density function being given in (4.18), the generalised Lundberg's equation (4.20) has n roots, say $R_1, R_2, \ldots, R_n$, with $|R_i| > 1$.

Proof: It is suficient to prove that the equaion

$$\hat{k}(vz)\hat{p}(1/z) = 1$$

has n roots, say $r_1, r_2, \ldots, r_n$, with $0 < |r_i| < 1$. Then $R_i = 1/r_i$, for $i = 1, 2, \ldots, n$. Consider a unit contour $\Gamma = \{z : |z| = 1\}$ in $\mathbb{C}$. Since

$$\left|\frac{1}{\hat{p}(1/z)}\right| = \frac{1}{|\hat{p}(1/z)|} \geq \frac{1}{\hat{p}(1/|z|)} = 1 = \hat{k}(1) \geq \hat{k}(v) = \hat{k}(|vz|) \geq |\hat{k}(vz)|,$$

on the contour Γ, then Rouché's Theorem shows that equation $1/\hat{p}(1/z) = 0$ and $\hat{k}(vz)\hat{p}(1/z) = 1$ have the same number of roots within the unit circle, while

$$\frac{1}{\hat{p}(1/z)} = \frac{\prod_{j=1}^{n}(z - q_j)}{\sum_{j=1}^{n}\left[\alpha_j(1 - q_j)\prod_{i=1, i\neq j}^{n}(z - q_i)\right]} = 0$$

has n roots within the unit circle. □

To derive an explicit expression for $\phi_{v,s}(u)$, we will go back to equation (2.6), where $\phi_v(u)$ is replaced by $\phi_{v,s}(u)$ and $w(x, y) = s^x w_1(y)$, i.e.,

$$\begin{aligned} \phi_{v,s}(u) &= \xi_v \sum_{y=1}^{u} \phi_{v,s}(u - y) g_1(y) \\ &\quad + \xi_v \sum_{y=u+1}^{\infty} s^u w_1(y - u) \sum_{x=0}^{\infty} s^x p_x(y) \frac{f_1(x|0)}{\xi_v}, \qquad u \in \mathbb{N}. \end{aligned} \tag{4.22}$$

We define $g_s(y) = \sum_{x=0}^{\infty} s^x p_x(y) f_1(x|0)/\xi_v$, and (4.22) turns to be

$$\phi_{v,s}(u) = \xi_v \sum_{y=1}^{u} \phi_{v,s}(u - y) g_1(y) + \xi_v \sum_{y=1}^{\infty} s^u w_1(y) g_s(y + u), \quad u \in \mathbb{N}. \tag{4.23}$$

Combining (4.19) and the definition of $g_s(y)$ gives

$$g_s(y + u) = \sum_{x=0}^{\infty} s^x p_x(y + u) f_1(x|0)/\xi_v = \sum_{j=1}^{n} \delta_j(s) \rho_j(y + u), \tag{4.24}$$

where $\delta_j(s) = \sum_{x=0}^{\infty} s^x \zeta_j^*(x + 1) f_1(x|0)/\xi_v$. A special case of (4.24),

$$g_1(y) = \sum_{x=0}^{\infty} p_x(y) \frac{f_1(x|0)}{\xi_v} = \sum_{j=1}^{n} \delta_j(1) \rho_j(y), \tag{4.25}$$

is a mixture of the same geometric distributions as in $p_x(y)$ with different weights $\delta_j(1), j = 1, 2, \ldots, n$. Recursion (4.23) can be rewritten as, by using (4.25) and (4.24),

$$\phi_{v,s}(u) = \xi_v \sum_{j=1}^{n} \delta_j(1) \sum_{y=1}^{u} \phi_{v,s}(u - y) \rho_j(y) + \xi_v \sum_{j=1}^{n} \delta_j(s) \sum_{y=1}^{\infty} w_1(y) \rho_j(y + u) s^u. \tag{4.26}$$

Thus the generating function of $\phi_{v,s}(u)$ can be calculated as

$$\begin{aligned}\hat{\phi}_{v,s}(z) &= \sum_{u=0}^{\infty} \phi_{v,s}(u) z^u \\ &= \xi_v \sum_{j=1}^{n} \delta_j(1)\hat{\phi}_{v,s}(z)\hat{\rho}_j(z) + \xi_v \sum_{j=1}^{n} \delta_j(s)\mathbb{E}[w_1(Y_j)]\frac{1}{1-q_j s z}. \qquad (4.27)\end{aligned}$$

Therefore,

$$\hat{\phi}_{v,s}(z) = \frac{\xi_v \sum_{j=1}^{n} \delta_j(s)\mathbb{E}[w_1(Y_j)]/(1-q_j s z)}{1-\xi_v \sum_{j=1}^{n} \delta_j(1)\hat{\rho}_j(z)}. \qquad (4.28)$$

Note that the denominator in the right-hand side of (4.28) has the same zeros as equation (4.21), because that $\sum_{x=0}^{\infty}\sum_{y=1}^{\infty} z^y f_1(x|0)p_x(y) = \xi_v \sum_{j=1}^{n} \delta_j(1)\hat{\rho}_j(z)$. Therefore, the denominator has the same n zeros $R_1, \ldots, R_n$ as equation (4.20).

Next we will try to find an explicit expression for $\phi_{v,s}(u)$ by inverting (4.28). Since $\hat{\rho}_j(z) = (1-q_j)z/(1-q_j z), j = 1, 2, \ldots, n$, we have, from (4.28), that

$$\begin{aligned}\hat{\phi}_{v,s}(z) &= \frac{\xi_v \sum_{j=1}^{n} \delta_j(s)\mathbb{E}[w_1(Y_j)]/(1-q_j s z)}{1-\xi_v \sum_{j=1}^{n} \delta_j(1)(1-q_j)z/(1-q_j z)} \\ &= \frac{\hat{\alpha}_{v,s}(z)}{\hat{\alpha}_v(z)} \cdot \prod_{j=1}^{n} \frac{1-q_j z}{1-q_j s z}, \qquad (4.29)\end{aligned}$$

where

$$\hat{\alpha}_{v,s}(z) = \xi_v \sum_{j=1}^{n} \delta_j(s) A_{-j}(sz)\mathbb{E}[w_1(Y_j)], \qquad (4.30)$$

$$\hat{\alpha}_v(z) = \prod_{j=1}^{n}(1-q_j z) - \xi_v z \sum_{j=1}^{n} \delta_j(1)(1-q_j)A_{-j}(z), \qquad (4.31)$$

and $A_{-j}(z) = \prod_{i=1, i\neq j}^{n}(1-q_i z)$. Assume the roots $R_1, \ldots, R_n$ are distinct, then using the Lagrange interpolating polynomial, (4.30) and (4.31) can be rewritten as

$$\begin{aligned}\hat{\alpha}_{v,s}(z) &= \sum_{j=1}^{n} \hat{\alpha}_{v,s}(R_j) \prod_{i=1, i\neq j}^{n} \frac{(R_i - z)}{R_i - R_j}, \\ \hat{\alpha}_v(z) &= \prod_{j=1}^{n} \frac{R_j - z}{R_j}.\end{aligned}$$

Substituting them into (4.29), one finds

$$\hat{\phi}_{v,s}(z) = \frac{\hat{\beta}_{v,s}(z)\prod_{l=1}^{n} R_l}{\prod_{j=1}^{n}(R_j - z)\prod_{i=1}^{n}(1-q_i s z)}, \qquad (4.32)$$

where

$$\hat{\beta}_{v,s}(z)=\sum_{j=1}^{n}\hat{\alpha}_{v,s}(R_j)\prod_{i=1,i\neq j}^{n}\frac{R_i-z}{R_i-R_j}\prod_{l=1}^{n}(1-q_lz),$$

is a polynomial of degree $2n-1$. Using partial fractions, (4.32) can be rewritten as

$$\hat{\phi}_{v,s}(z)=\sum_{j=1}^{n}\gamma_{v,s}(j)\frac{R_j}{R_j-z}+\sum_{j=1}^{n}\kappa_{v,s}(j)\frac{1}{1-q_jsz}, \tag{4.33}$$

where

$$\begin{aligned}\gamma_{v,s}(j) &= \frac{\hat{\beta}_{v,s}(R_j)\prod_{l=1,l\neq j}^{n}R_l}{\prod_{i=1,i\neq j}^{n}(R_i-R_j)\prod_{k=1}^{n}(1-q_ksR_j)},\\ \kappa_{v,s}(j) &= \frac{\hat{\beta}_{v,s}((q_js)^{-1})\prod_{l=1}^{n}R_l}{\prod_{i=1}^{n}[R_i-(q_js)^{-1}]\prod_{i=1,i\neq j}^{n}(1-q_i/q_j)}.\end{aligned}$$

The inversion of (4.33) yields

$$\phi_{v,s}(u)=\sum_{j=1}^{n}\gamma_{v,s}(j)R_j^{-u}+\sum_{j=1}^{n}\kappa_{v,s}(j)(q_js)^u, \qquad u\in\mathbb{N}. \tag{4.34}$$

Within the rest of the subsection, we will develop a system of linear equations which can be used to determine the unknown coefficients $\gamma_{v,s}(j)$ and $\kappa_{v,s}(j)$, $j=1,2,\ldots,n$.

Conditioning on the time and amount of the first claim for the surplus process, one can see that for $u\in\mathbb{N}$, equation (4.10) still holds, which is

$$\begin{aligned}\phi_{v,s}(u) &= \sum_{t=1}^{\infty}v^tk(t)\left[\sum_{x=1}^{u+t}\phi_{v,s}(u+t-x)p(x)+s^{u+t-1}\sum_{x=u+t+1}^{\infty}w_1(x-u-t)p(x)\right]\\ &= \sum_{t=1}^{\infty}v^tk(t)\left[\nu_{v,s}(u+t)+s^{u+t-1}\sum_{y=1}^{\infty}w_1(y)p(y+u+t)\right],\end{aligned} \tag{4.35}$$

where $\nu_{v,s}(t)=\sum_{y=0}^{t-1}\phi_{v,s}(y)p(t-y)$. The generating function of $\nu_{v,s}(t)$ then has the form $\hat{\nu}_{v,s}(z)=\hat{\phi}_{v,s}(z)\hat{p}(z)$. From (4.33) we have

$$\begin{aligned}\hat{\nu}_{v,s}(z) &= \sum_{j=1}^{n}\gamma_{v,s}(j)\frac{R_j}{R_j-z}\hat{p}(z)+\sum_{j=1}^{n}\kappa_{v,s}(j)\frac{\hat{p}(z)}{1-q_jsz}\\ &= \sum_{j=1}^{n}\gamma_{v,s}(j)\sum_{i=1}^{n}\alpha_i\frac{(1-q_i)R_jz}{(R_j-z)(1-q_iz)}+\sum_{j=1}^{n}\kappa_{v,s}(j)\sum_{i=1}^{n}\alpha_i\frac{(1-q_i)z}{(1-q_jsz)(1-q_iz)}\\ &= \sum_{j=1}^{n}\gamma_{v,s}(j)\left[\hat{p}(R_j)\frac{R_j}{R_j-z}-\sum_{i=1}^{n}\frac{\alpha_i}{1-q_iz}\hat{\rho}_i(R_j)\right]\\ &\quad+\sum_{j=1}^{n}\kappa_{v,s}(j)\sum_{i=1}^{n}\alpha_i\frac{1-q_i}{q_js-q_i}\left[\frac{1}{1-q_jsz}-\frac{1}{1-q_iz}\right],\end{aligned} \tag{4.36}$$

and inverting it gives, for $t \in \mathbb{N}^+$,

$$\nu_{v,s}(t) = \sum_{j=1}^{n} \gamma_{v,s}(j) \Big[\hat{p}(R_j) R_j^{-t} - \sum_{i=1}^{n} \alpha_i \hat{\rho}_i(R_j) q_i^t\Big] + \sum_{j=1}^{n} \kappa_{v,s}(j) \sum_{i=1}^{n} \alpha_i \frac{1-q_i}{q_j s - q_i} \left[(q_j s)^t - q_i^t\right]. \quad (4.37)$$

Combining (4.37) and (4.35) yields

$$\begin{aligned}
\phi_{v,s}(u) &= \sum_{t=1}^{\infty} v^t k(t) \Bigg\{ \sum_{j=1}^{n} \gamma_{v,s}(j) \Big[\hat{p}(R_j) R_j^{-(u+t)} - \sum_{i=1}^{n} \alpha_i \hat{\rho}_i(R_j) q_i^{u+t}\Big] \\
&\quad + \sum_{j=1}^{n} \kappa_{v,s}(j) \sum_{i=1}^{n} \alpha_i \frac{1-q_i}{q_j s - q_i} \left[(q_j s)^{u+t} - q_i^{u+t}\right] \\
&\quad + s^{u+t-1} \sum_{y=1}^{\infty} w_1(y) p(y+u+t) \Bigg\} \\
&= \sum_{j=1}^{n} \gamma_{v,s}(j) \hat{p}(R_j) \hat{k}(\frac{v}{R_j}) R_j^{-u} - \sum_{j=1}^{n} \gamma_{v,s}(j) \sum_{i=1}^{n} \alpha_i \hat{\rho}_i(R_j) \hat{k}(v q_i) q_i^u \\
&\quad + \sum_{j=1}^{n} \kappa_{v,s}(j) \sum_{i=1}^{n} \alpha_i \frac{1-q_i}{q_j s - q_i} \hat{k}(v\, q_j\, s)(q_j s)^u \\
&\quad - \sum_{j=1}^{n} \kappa_{v,s}(j) \sum_{i=1}^{n} \alpha_i \frac{1-q_i}{q_j s - q_i} \hat{k}(v q_i) q_i^u \\
&\quad + \sum_{j=1}^{n} \alpha_j \hat{k}(v q_j s) \mathbb{E}[w_1(Y_j)] (q_j s)^u / s. \quad (4.38)
\end{aligned}$$

Since R_j satisfies the generalised Lundberg equation (4.20), i.e., $\hat{p}(R_j)\hat{k}(v/R_j) = 1$, comparing (4.34) and (4.38) gives a system of linear equations for $\gamma_{v,s}(j)$ and $\kappa_{v,s}(j)$:

$$\begin{aligned}
&\kappa_{v,s}(j) - \kappa_{v,s}(j) \sum_{i=1}^{n} \alpha_i \frac{1-q_i}{q_j s - q_i} \hat{k}(v q_j s) - \alpha_j \hat{k}(v q_j s) \mathbb{E}[w_1(Y_j)] s^{-1} = 0, \quad j = 1, \ldots, n, \\
&\sum_{j=1}^{n} \gamma_{v,s}(j) \hat{\rho}_i(R_j) + \sum_{j=1}^{n} \kappa_{v,s}(j) \frac{1-q_i}{q_j s - q_i} = 0, \quad i = 1, \ldots, n. \quad (4.39)
\end{aligned}$$

Let $\mathbf{A} = (a_{i,j})_{i,j=1,\ldots,n}$ and $\mathbf{B}(s) = (b_{i,j}(s))_{i,j=1,\ldots,n}$ be two matrixes with $a_{i,j} = \hat{\rho}_i(R_j)$ and $b_{i,j}(s) = (1-q_i)/(q_i - q_j s)$, $\overrightarrow{\gamma}_{v,s} = (\gamma_{v,s}(1), \ldots, \gamma_{v,s}(n))^T$ and $\overrightarrow{\kappa}_{v,s} = (\kappa_{v,s}(1), \ldots, \kappa_{v,s}(n))^T$ are two n-dimensional vectors to be solved. Then the second line of equations in (4.39) can be rewritten in a matrix form as

$$\mathbf{A} \overrightarrow{\gamma}_{v,s} = \mathbf{B}(s) \overrightarrow{\kappa}_{v,s}. \quad (4.40)$$

Assume that the inverse of matrix $\mathbf{A}$ exists, and is denoted by $\mathbf{A}^{-1} = (a^*_{i,j})_{i,j=1,\dots,n}$, then from (4.39) and (4.40) we obtain the following solutions for $\gamma_{v,s}(j)$ and $\kappa_{v,s}(j)$, $j = 1, 2, \dots, n$,

$$\kappa_{v,s}(j) = \frac{\alpha_j \hat{k}(vq_j s)\mathbb{E}[w_1(Y_j)]/s}{1 - \sum_{i=1}^n \alpha_i(1-q_i)/(q_j s - q_i)\hat{k}(vq_j s)},$$

$$\gamma_{v,s}(j) = \sum_{i=1}^n a^*_{j,i} \sum_{l=1}^n b_{i,l}(s)\kappa_{v,s}(l)$$

$$= \sum_{i=1}^n a^*_{j,i} \sum_{l=1}^n \frac{\alpha_l\,(1-q_i)\hat{k}(q_l s v)\mathbb{E}[w_1(Y_l)]/s}{(q_i - q_l\,s)\Big[1 - \hat{k}(v\,q_l\,s)\sum_{m=1}^n \alpha_m(1-q_m)/(q_l\,s - q_m)\Big]}. \quad (4.41)$$

References

[1] Cheng, S., Gerber, H.U. and Shiu, E.S.W. (2000). Discounted probabilities and ruin theory in the compound binomial model. *Insurance: Mathematics and Economics,* **26**, 239-250.

[2] Landriault, D. and Willmot, G. (2007). On the Gerber-Shiu discounted penalty function in the Sparre Andersen model with an arbitrary interclaim time distribution. Unpublished observation(s).

[3] Li, S. (2005a). On a class of discrete time renewal risk models. *Scandinavian Actuarial Journal*, **4**, 241-260.

[4] Li, S. (2005b). Distributions of the surplus before ruin, the deficit at ruin and the claim causing ruin in a class of discrete time risk models. *Scandinavian Actuarial Journal,* **4**, 271-284.

[5] Li, S. and Garrido, J. (2005). On a general class of renewal risk process: analysis of the Gerber-Shiu function. *Advances in Applied Probability,* **37**, 836-856.

[6] Malinovskii, V. (1998). Non-Poissonian claims' arrivals and calculation of the probability of ruin. *Insurance: Mathematics and Economics,* **22**, 123-138.

[7] Pavlova, K. and Willmot, G.E. (2004). The discrete stationary renewal risk model and the Gerber-Shiu discounted penalty function. *Insurance: Mathematics and Economics,* **2**, 267-277.

[8] Wang, R. and Liu, H. (2002). On the ruin probability under a class of risk processes. *ASTIN Bulletin,* **32**, 81-90.

[9] Willmot, G.E. (2007). On the discounted penalty function in the renewal risk model with general interclaim times. *Insurance: Mathematics and Economics,* **41**, 17-31.

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