Optimal Dividends under Reinsurance

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Abstract

We investigate whether the expected present value of net income to shareholders when there is a constant dividend barrier can be increased by the insurer effecting reinsurance. We analyse the problem numerically, using both De Vylder's approximation to a classical surplus process and Dickson and Waters' discrete time model. We find that excess of loss reinsurance can be used to increase the expected present value of net income to shareholders.

Key words: dividends, reinsurance, De Vylder's approximation, discrete time model

1 Introduction

The study of optimal dividends problems goes back to de Finetti (1957), and in recent years there have been many papers in the actuarial literature on optimal dividends problems. See, for example, Gerber and Shiu (2004, 2006), Dickson and Waters (2004), and references therein. Much of this literature concerns a model with a constant dividend barrier, and under such a barrier, the insurer's ultimate ruin is certain. Thus, the insurance operation is essentially being used to generate dividend income for the insurance company's shareholders, and a natural question of interest is how to maximise the expected present value of the shareholders' income. By 'income' we shall mean net income, as we shall apply the modification to de Finetti's idea introduced by Dickson and Waters (2004), and further studied by Gerber et al (2006), under which the shareholders are required both to provide the initial surplus and to cover the deficit at ruin.

In this paper we investigate whether the expected present value of net income to the shareholders can be increased by effecting reinsurance. We consider two forms of reinsurance, namely proportional and excess of loss. Our approach is to conduct an empirical investigation rather than a mathematical one. The reason for this is that it is difficult to obtain analytical solutions to our problem, especially in the case of excess of loss reinsurance. Therefore, we shall apply two approximation methods.

We first apply De Vylder's (1978) approximation to (net of reinsurance) surplus processes. There are three main reasons for applying this approximation. First, the computational time involved in applying this procedure is negligible. This is important as conclusions in this paper are based on grid searches over a range of values for retention levels and dividend barriers. Second, it has been shown to be a remarkably accurate approximation in other studies of dividend problems. See Højgaard (2002) and Dickson and Drekic (2006). Third, it has the advantage over other types of approximation to a surplus process (such as a Brownian motion approximation) that it produces a non-zero deficit at ruin. The importance of this can be seen in formula (2) below. However, like any approximation, De Vylder's approximation has limitations. It appears to work best when the moment generating function of the individual claim amount distribution (without reinsurance) exists. This was certainly the case in the original setting of the approximation, as illustrated by the numerical examples in De Vylder (1978).

Our second approximation is the discrete time model described and tested in Section 5 of Dickson and Waters (2004). Based on the numerical results presented in that paper, we view this model as providing very accurate numerical solutions to dividends problems. In the context of this paper, however, its drawback is that it is a computationally intensive tool for grid searches, particularly when compared with applying De Vylder's approximation. In the remainder of this paper we will refer to this approach as the DW approach.

2 Model and notation

In this paper we aim to draw conclusions about a classical risk model modified both by reinsurance and by the inclusion of a constant dividend barrier. We assume that claims occur as a Poisson process with parameter λ, and let $N(t)$ denote the number of claims up to time t, so that $N(t)$ is distributed as Poisson(λt). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed positive random variables, independent of the Poisson process, where X_i denotes the amount of the ith claim (before reinsurance). Let X represent a generic claim, and let f denote the density function of X. The insurer's premium income per unit time is $(1+\theta)\lambda E[X]$ where $\theta > 0$, and we assume that this is received continuously.

Now let h denote a reinsurance arrangement that applies to an individual claim, so that if the individual claim amount is x, the insurer pays

$h(x)$ where $0 \leq h(x) \leq x$. The reinsurance premium per unit time is $(1+\theta_R)\lambda E[X - h(X)]$, and we assume that this is paid continuously. In our numerical illustrations we further assume that $\theta_R \geq \theta$. Then the insurer's net of reinsurance premium income per unit time is

$$c^* = (1+\theta)\lambda E[X] - (1+\theta_R)\lambda E[X - h(X)].$$

If we further assume that reinsurance recoveries are made as soon as claims occur, the insurer's net of reinsurance surplus process, starting from initial surplus u, is

$$U(t) = u + c^* t - \sum_{i=1}^{N(t)} h(X_i), \tag{1}$$

where the sum is zero if $N(t) = 0$.

We now modify the net of reinsurance surplus process by a dividend barrier, which we denote b. When the surplus attains b, dividends are paid out to shareholders at rate c^* per unit time until the next claim occurs, so that the modified surplus process remains at b until the next claim occurs, then falls by the (net of reinsurance) amount of that claim. On any subsequent occasion that the net of reinsurance surplus process attains b, dividends are again payable at rate c^*. Ruin occurs when the surplus process falls below zero, and no dividends are payable after the time of ruin.

Our analysis in this paper is based on the function denoted $L(u, b)$ by Dickson and Waters (2004) and defined by

$$L(u, b) = V(u, b) - u - E[e^{-\delta T_u} Y_u] \tag{2}$$

where δ is the force of interest per unit time, $V(u, b)$ is the expected present value of dividend payments to the shareholders, T_u is the time of ruin and Y_u is the deficit at ruin.

Dickson and Waters (2004) give a formula for $L(u, b)$ for a classical risk model without reinsurance with individual claim amounts being exponentially distributed. Therefore, we can apply De Vylder's approximation to the surplus process given by (1), and as the individual claim amount distribution is exponential in De Vylder's approximation, we can thus apply Dickson and Waters' results to calculate $L(u, b)$ for the approximating surplus process. To compute our second approximation to $L(u, b)$, we apply the approach described in Section 5 of Dickson and Waters (2004), with a simple modification to the Poisson parameter in their discrete time model to ensure that the premium income per unit time, net of reinsurance, is 1.

The introduction of reinsurance will have a number of effects. First, as the rate of premium income is reduced under reinsurance, the time at which dividends can first be payable increases for $u < b$. Second, the rate of dividend payment is reduced as $c^* < c$. These two effects should act to reduce the expected present value of dividend payments. However, as reinsurance reduces the amount of individual claims, we might expect the time of ruin to be later under a reinsurance arrangement, resulting in a larger number of dividend streams, and a smaller expected present value of the deficit at ruin. We will see in Section 4 that these contrasting effects produce outcomes which are not uniform across different claim size distributions and which can be difficult to interpret.

3 Proportional reinsurance

We first consider proportional reinsurance, with $h(x) = ax$ where $0 < a \leq 1$. Our methodology in applying the De Vylder approximation, which also applies in the next section (with appropriate adjustments), is as follows. For given values of a and θ_R, we calculate both c^* and the moments of the net of reinsurance individual claim amount distribution. Using these, we then calculate the parameters of De Vylder's approximating surplus process. Given the parameters of this approximating surplus process, we can then use the final formula on page 64 of Dickson and Waters (2004) to calculate the value of b, which we denote by b^*, that maximises $L(u, b)$. We can then calculate $L(u, b^*)$ using formulae (2.7) and (4.4) in Dickson and Waters (2004). We performed this calculation for values of a that are integer multiples of 0.01 which also satisfy the condition $c^* \geq \lambda E[h(X)]$. From this set of values for a, we can find (numerically) the value of a which maximises the expected present value of net income to shareholders, and if a higher degree of accuracy is required, we can apply the same approach with a larger set of values for a, say integer multiples of 0.001. We remark that the optimal barrier varies with a, but for brevity we write b^* rather than $b^*(a)$ (or $b^*(M)$ in the next section).

From Dickson and Waters (2004) we note that the optimal value of b is independent of u when the individual claim amount distribution is exponential. In cases where the computed value of b^* was less than u, we assumed that the excess $u - b^*$ was immediately returned to the shareholders as dividends, resulting in an expected present value of net income to the shareholders of

$$V(b^*, b^*) - b^* - E[e^{-\delta T_{b^*}} Y_{b^*}].$$

Our study of proportional reinsurance, which included calculations by both approaches, led us to the conclusion that proportional reinsurance cannot be used to increase the expected present value of net income to shareholders. Figure 1 is typical of our findings and illustrates the situation when $u = 10$, the individual claim amount distribution is mixed exponential with density function

$$f(x) = \frac{2}{3}\left(2e^{-2x}\right) + \frac{1}{3}\left(\frac{1}{2}e^{-x/2}\right)$$

(so that $E[X] = 1$), $\lambda = 100$, $\delta = 0.1$ and $\theta = 0.1$. We have plotted $L(10, b^*)$ based on De Vylder approximations for three values of θ_R, namely 0.1, 0.15 and 0.2. Throughout this paper we have limited our investigations to situations where $c^* \geq \lambda E[h(X)]$, although as ultimate ruin is certain, this restriction is not of great importance. However, it accounts for the different minimum values of proportion retained for the different values of θ_R. We can see in Figure 1 that for each value of θ_R, the expected present value of net income is an increasing function of the proportion retained which is maximised when $a = 1$, i.e. when there is no reinsurance.

This general pattern emerged in all calculations we performed using different claim amount distributions and different parameter values, so we will not analyse proportional reinsurance further.

4 Excess of loss reinsurance

We now consider excess of loss reinsurance with retention level M, so that the insurer pays the lesser of an individual claim amount and M. Our empirical findings are illustrated below for different individual claim amount distributions, namely mixed exponential and lognormal distributions. Unlike the case of proportional reinsurance, we observe that the insurer can increase the expected present value of net income to shareholders by effecting excess of loss reinsurance, and the increase can be substantial.

In each of the numerical illustrations below, the mean individual claim amount (before reinsurance) is 1, and we have set $\lambda = 100$, $\delta = 0.1$, and $\theta = 0.1$. Results are given for a range of values for θ_R. For a given set of parameters, we found the optimal retention level and barrier under De Vylder's approximation as described in the previous section, searching over values of M rather than a. In the case of Tables 1 to 3, the values of M considered were integer multiples of 0.1, with the minimum value of M being determined by the condition $c^* \geq \lambda E[h(X)]$.

4.1 A mixture of two exponential distributions

We first consider the same mixed exponential individual claim amount distribution as in the previous section. Figures 2 and 3 show De Vylder approximations to $L(u, b^*)$ as a function of M when $u = 10$ (Figure 2) and when

$u = 30$ (Figure 3) for five different values of θ_R, and we see from these figures that reinsurance can increase the expected present value of net income to shareholders. The expected present value with no reinsurance is shown as a horizontal dotted line in each figure. Table 1 shows some numbers relating to these figures. These include the optimal value of M and the corresponding value of b^*, the expected amount of dividends payable until ruin, denoted by $E[D_u]$ and the expected time of ruin, denoted by $E[T_u]$. The former can be calculated from the results given for the distribution of the total amount of dividends by Dickson and Waters (2004, Section 3) and the latter from Lin et al (2003, p.562). The final column (headed %) shows the percentage increase in the expected present value of net income as a result of effecting reinsurance, and the final row for each value of u shows values when there is no reinsurance.

u	θ_R	b^*	$L(u, b^*)$	M	$E[D_u]$	$E[T_u]$	%
10	0.1	31.22	37.47	2.1	387.00	49.42	23.4
10	0.125	39.21	33.69	3.4	282.53	32.06	11.0
10	0.15	44.12	32.01	4.7	244.89	25.84	5.5
10	0.175	47.16	31.22	6.0	227.93	23.00	2.9
10	0.2	49.03	30.83	7.3	219.29	21.53	1.6
10		51.79	30.36	∞	208.06	19.62	
30	0.1	42.68	47.71	4.2	441.60	44.71	6.2
30	0.125	46.71	46.37	5.7	393.59	38.07	3.3
30	0.15	48.75	45.72	7.0	373.19	35.24	1.8
30	0.175	49.91	45.38	8.2	362.84	33.78	1.1
30	0.2	50.59	45.19	9.3	357.19	32.98	0.6
30		51.79	44.91	∞	348.13	31.63	

Table 1: Mixture of two exponential distributions

For each value of u in Table 1, we see that as θ_R increases, the percentage increase in the expected present value of net income decreases, as do both the expected amount of dividends and the expected time to ruin. However, as θ_R increases, the optimal barrier level and retention level both increase. It is perhaps not surprising that the optimal retention level increases as θ_R increases since this feature appears in other reinsurance problems such as maximising the adjustment coefficient under reinsurance. (See, for example, Dickson (2005).) Indeed, we observed this feature in all examples where it was optimal to effect reinsurance. Given that the optimal value of M increases with θ_R, it seems that increased net individual claim amounts are causing a decrease in the expected time to ruin.

Although reinsurance does increase the expected present value of net income, the percentage increase is not significant for most combinations of u and θ_R considered in Table 1. We remark that results obtained by application of the DW approach are virtually identical. Our methodology in applying this approach was to use the same scaling factor of 100 to the individual claim amount distribution without reinsurance, as in Dickson and Waters (2004), and to consider values of M and b which are integer multiples of 0.1 and 0.01 respectively, so that in the approximating discrete time surplus process, the surplus always took an integer value. (We have chosen these two numbers to match our search under the De Vylder approach, i.e. searching for M to one decimal place and recording b^* to two decimal places.)

4.2 A mixture of three exponential distributions

As a second illustration, we consider the mixture of three exponential distributions introduced by Wikstad (1971), namely

$$f(x) = \sum_{i=1}^{3} \alpha_i \, \beta_i \exp\{-\beta_i x\}$$

where $\alpha_1 = 0.0039793$, $\alpha_2 = 0.1078392$, $\alpha_3 = 0.8881815$, $\beta_1 = 0.014631$, $\beta_2 = 0.190206$ and $\beta_3 = 5.51451$. This distribution contrasts the distribution in the previous section in that it has a much larger variance (of 42.2) and is more heavily skewed. Consequently, we might expect excess of loss reinsurance to eliminate large claims and hence increase both the expected time to ruin and the expected present value of net income. However, in our numerical experiments with this distribution, this was generally not the case. Figure 4 shows both approximations to $L(20, b^*)$ for $\theta_R = 0.1$, 0.125 and 0.15 – results from the De Vylder approximation are given by the bold lines – with calculations being performed at integer values of M only. In the case when $\theta_R = 0.1$, reinsurance can increase the expected present value of net income, but it does not for $\theta_R = 0.125$ and 0.15. For the values of θ_R illustrated, the expected present value of net income is not a monotonically increasing function of M, although it is for larger values such as $\theta_R = 0.2$. It is difficult to explain this behaviour and we can only attribute it to the differing effects of reinsurance described at the end of Section 2. We also observed this behaviour for small values of u for the individual claim amount distribution of Section 4.1.

In terms of the approximations, we see that for values of M less than about 50, both approximations perform similarly. For larger values of M we view the DW approach as providing the more accurate approximation. An interesting feature of the results produced by the DW approach is that as a function of M, $L(u, b^*)$ has a local minimum around 60 for each value of θ_R. What appears to underlie this feature is that there is a considerable change in the value of the optimal barrier. For example, when $\theta_R = 0.1$, we observe that the optimal barrier increases with M up to 67.9 when $M = 62$, but then drops to 48.6 when $M = 63$. It is not clear why this should happen. For each integer value of M up to 62, we found that the optimal barrier was above M, so that reinsurance could effect how much the surplus fell in each time period in the discrete model. However, for each value of M above 62, we found that the optimal barrier was below M, so that the effect of reinsurance is limited to the premium income, and the amount of the deficit at ruin.

4.3 Lognormal distributions

We now consider the situation when the individual claim amount distribution is lognormal. Results obtained by applying the DW approach are shown in Table 2 when the individual claim amount distribution has variance 3 and in Table 3 when the variance is 6, in each case for the same values of u and θ_R as in Table 1. Each distribution has a greater variance than the mixture of two exponential distributions from Section 4.1, but a considerably smaller variance than the mixture of three exponential distributions from Section 4.2.

u	θ_R	b^*	$L(u, b^*)$	M	$E[D_u]$	$E[T_u]$	%
10	0.1	29.46	37.72	1.98	406.22	53.72	44.6
10	0.125	36.12	33.12	3.12	283.05	34.34	27.0
10	0.15	41.10	30.55	4.45	234.13	26.60	17.1
10	0.175	44.45	28.99	5.85	206.51	22.41	11.1
10	0.2	47.01	27.98	7.38	190.04	19.91	7.3
10		55.66	26.09	∞	144.86	12.97	
30	0.1	39.77	46.59	3.96	448.48	47.71	24.7
30	0.125	43.77	44.21	5.40	381.59	39.07	18.3
30	0.15	46.43	42.64	6.83	345.11	34.38	14.1
30	0.175	48.31	41.53	8.24	322.36	31.45	11.2
30	0.2	49.71	40.72	9.63	307.00	29.48	9.0
30		55.66	37.36	∞	239.68	20.37	

Table 2: Lognormal distribution with variance 3

The pattern in Table 2 is basically the same as in Table 1. However, the main difference between Tables 1 and 2 is that the percentage increase in

u	θ_R	b^*	$L(u,b^*)$	M	$E[D_u]$	$E[T_u]$	%
10	0.1	28.05	32.04	2.01	310.36	45.84	61.7
10	0.125	36.85	26.26	3.57	192.06	25.12	32.6
10	0.15	43.13	23.19	5.51	146.22	17.36	17.1
10	0.175	48.09	21.43	7.92	124.55	13.68	8.2
10	0.2	50.92	20.31	10.01	112.91	11.88	2.5
10		60.49	19.81	∞	93.73	7.36	
30	0.1	39.78	39.66	4.20	343.04	38.67	42.2
30	0.125	45.24	36.10	6.13	271.49	28.80	29.4
30	0.15	48.97	33.75	8.17	234.79	23.75	21.0
30	0.175	51.69	32.12	10.31	212.80	20.71	15.2
30	0.2	53.73	30.92	12.51	198.39	18.72	10.9
30		60.49	27.89	∞	156.63	11.45	

Table 3: Lognormal distribution with variance 6

expected present value of net income is much greater in Table 2. Indeed, the percentage increases in this table are substantial, and this is also a feature of Table 3. Figure 5 shows $L(10, b^*)$ as a function of M when the individual claim amount distribution has variance 3, with the horizontal dotted line again showing the expected present value of net income without reinsurance. This figure is very similar to Figure 2, and highlights the greater percentage increase in expected present value of net income than was observed in Figure 2.

Figure 6 shows both approximations to $L(10, b^*)$ when the individual claim amount distribution has variance 6 and when $\theta_R = 0.175$. The bold lines show De Vylder approximations, with the horizontal dotted lines showing $L(10, b^*)$ as $M \to \infty$. This figure illustrates a weakness of the De Vylder approximation. Whilst we see that both approximations give similar results for small values of M, and essentially identify the same retention level as optimal, as M increases, the De Vylder approximation becomes less accurate. Although it correctly identifies $L(10, b^*)$ as being an increasing function of M for M greater than about 60, it produces an approximation in the case $M \to \infty$ that results in the conclusion that no reinsurance is the optimal strategy. We see from the results of the DW approach that this is not the case. Thus, the De Vylder approximation is very useful in that it gives a very good indication of the optimal retention level and barrier – which is very helpful when we have to perform a grid search using the DW approach – but it ultimately provides the wrong conclusion. This is perhaps not surprising as the examples in De Vylder's original paper suggest it is not a great approximation when the individual claim amounts are lognormally distributed.

5 Concluding remarks

This has been an empirical study, and as such it is difficult to draw firm conclusions. One conclusion is clear and important – it is possible to increase the expected present value of net income to shareholders by effecting reinsurance. Our investigations have ranged over a greater set of distributions and over a wider range of parameter values than reported here. From these wider investigations, no clear picture has emerged as to the circumstances under which excess of loss reinsurance can increase the expected present value of net income. Although De Vylder's approximation can lead to incorrect conclusions, it has the merit that it is generally good at identifying the value of M at which $L(u, b^*)$ has a local maximum. Compared with the DW approach, De Vylder's approximation is computationally much more efficient and is therefore a very useful tool in helping to identify an optimal reinsurance strategy.

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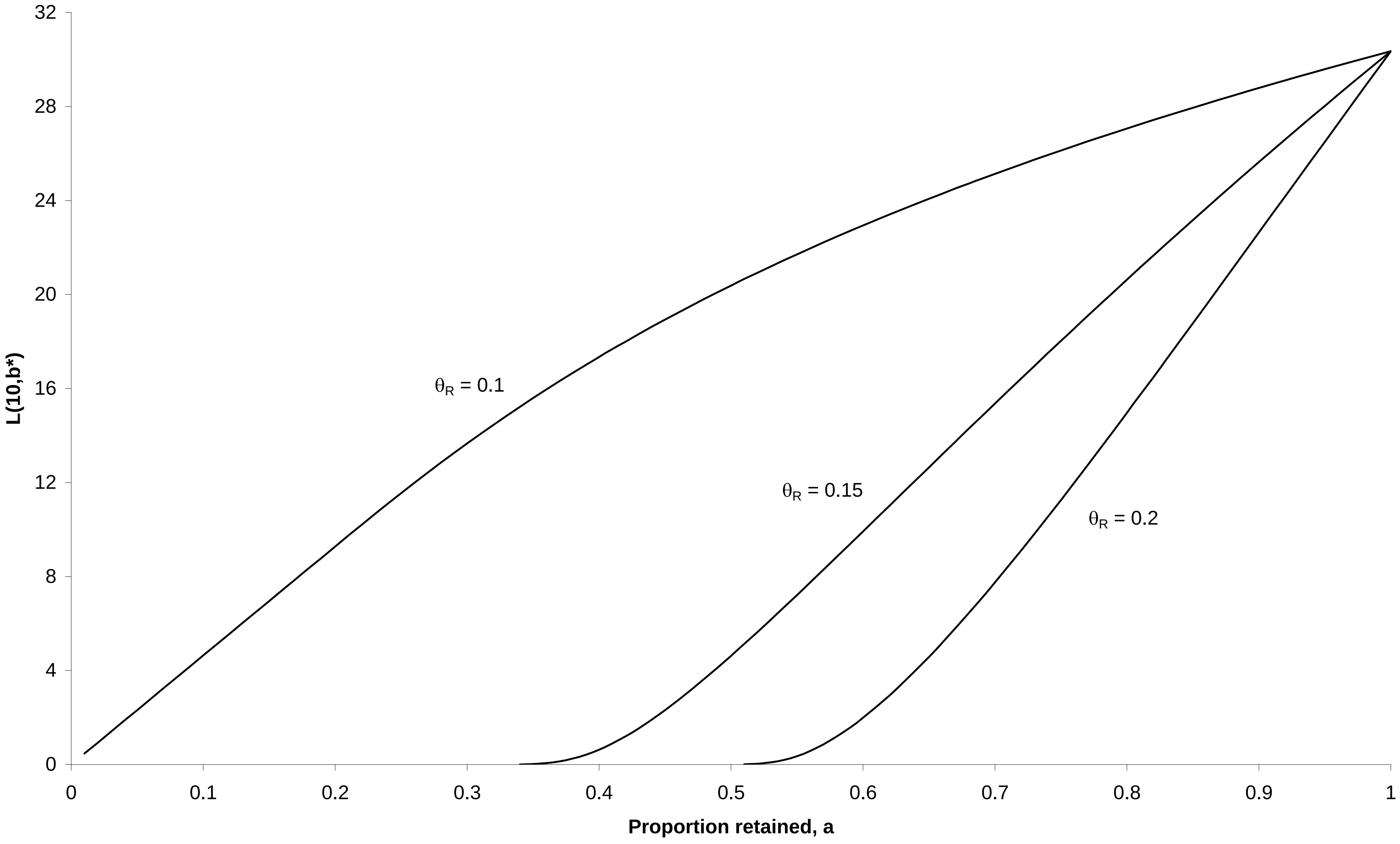

Figure 1: L(10,b*), mixture of two exponential distributions, proportional reinsurance

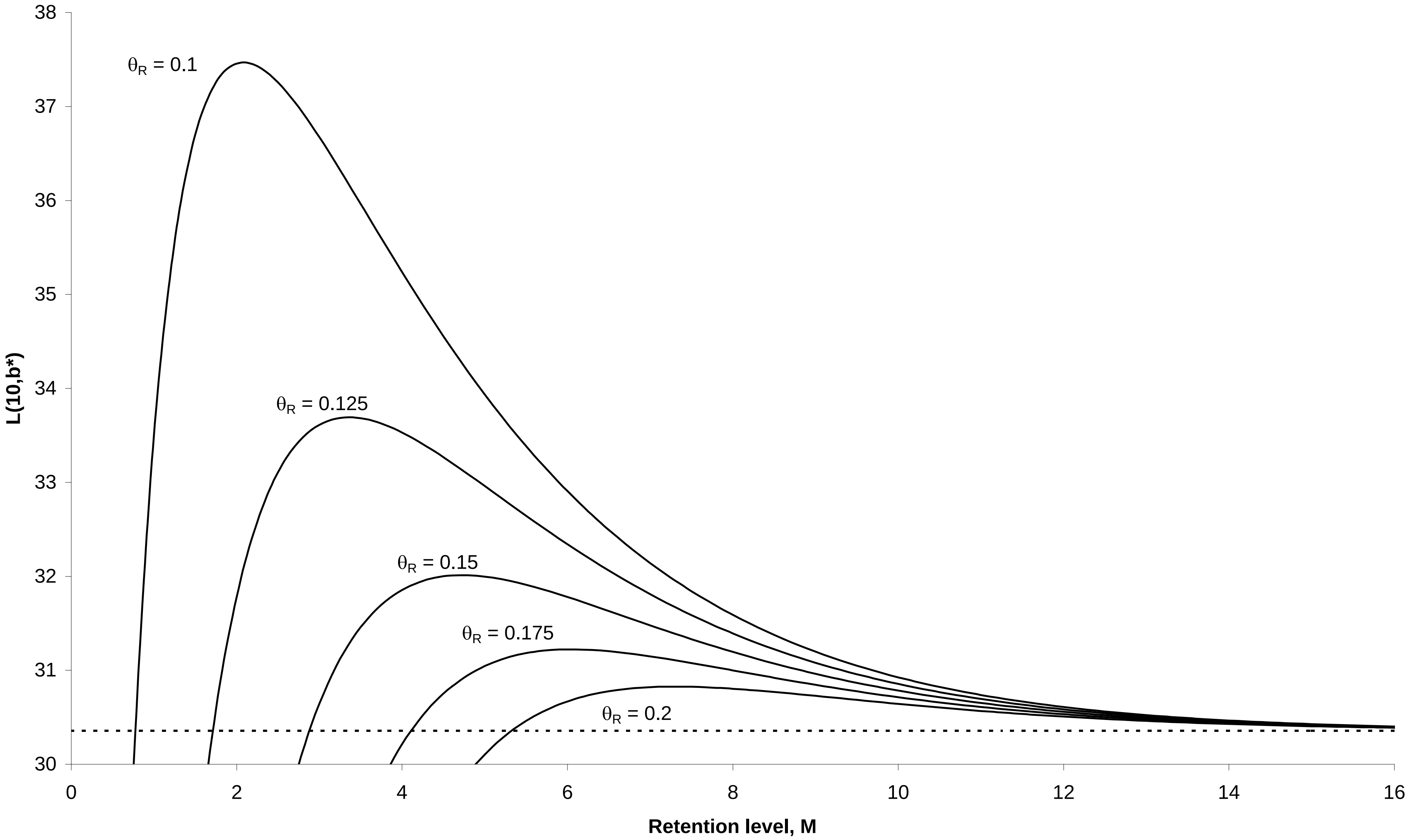

Figure 2: L(10,b*), mixture of two exponential distributions, excess of loss

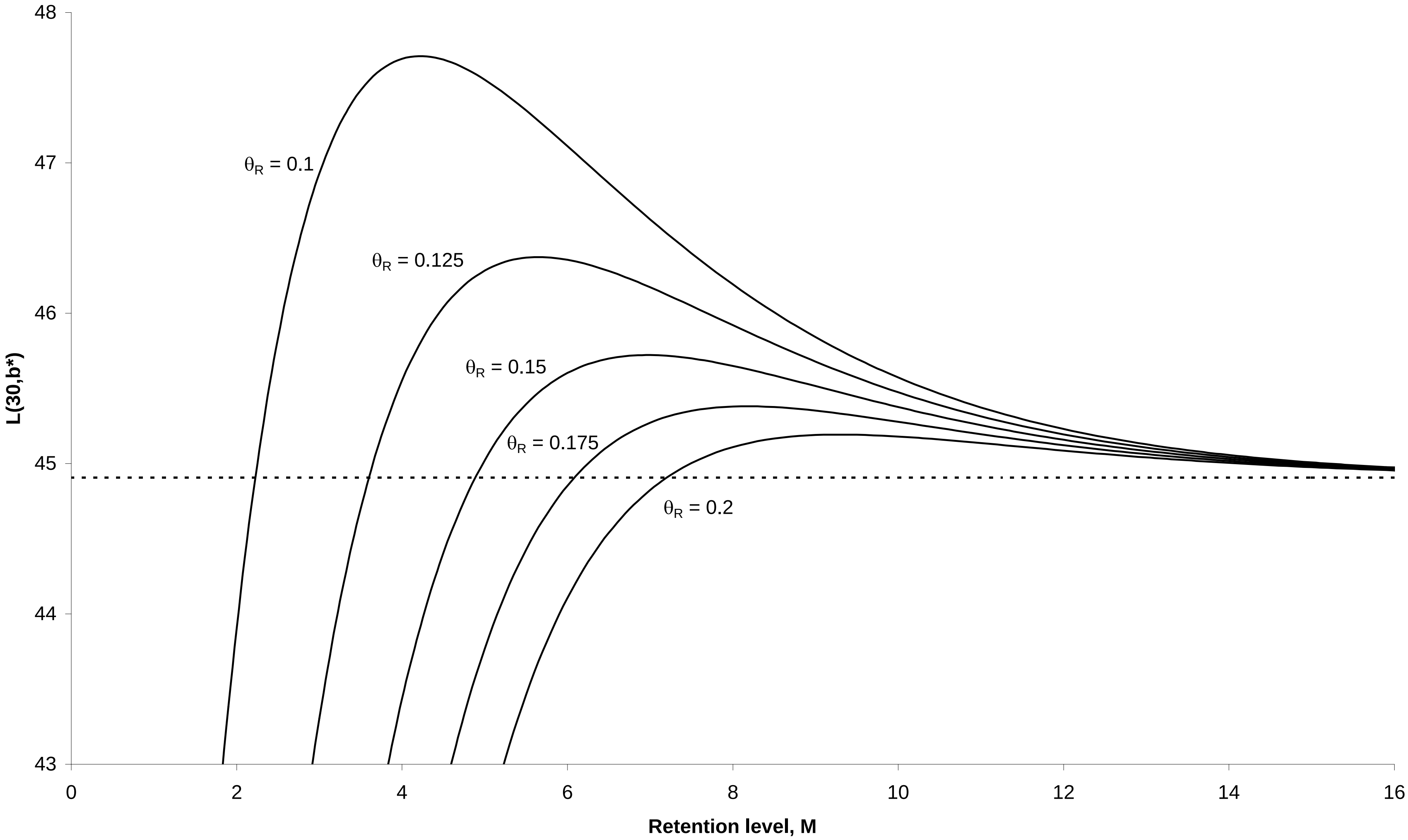

Figure 3: L(30,b*), mixture of two exponential distributions, excess of loss

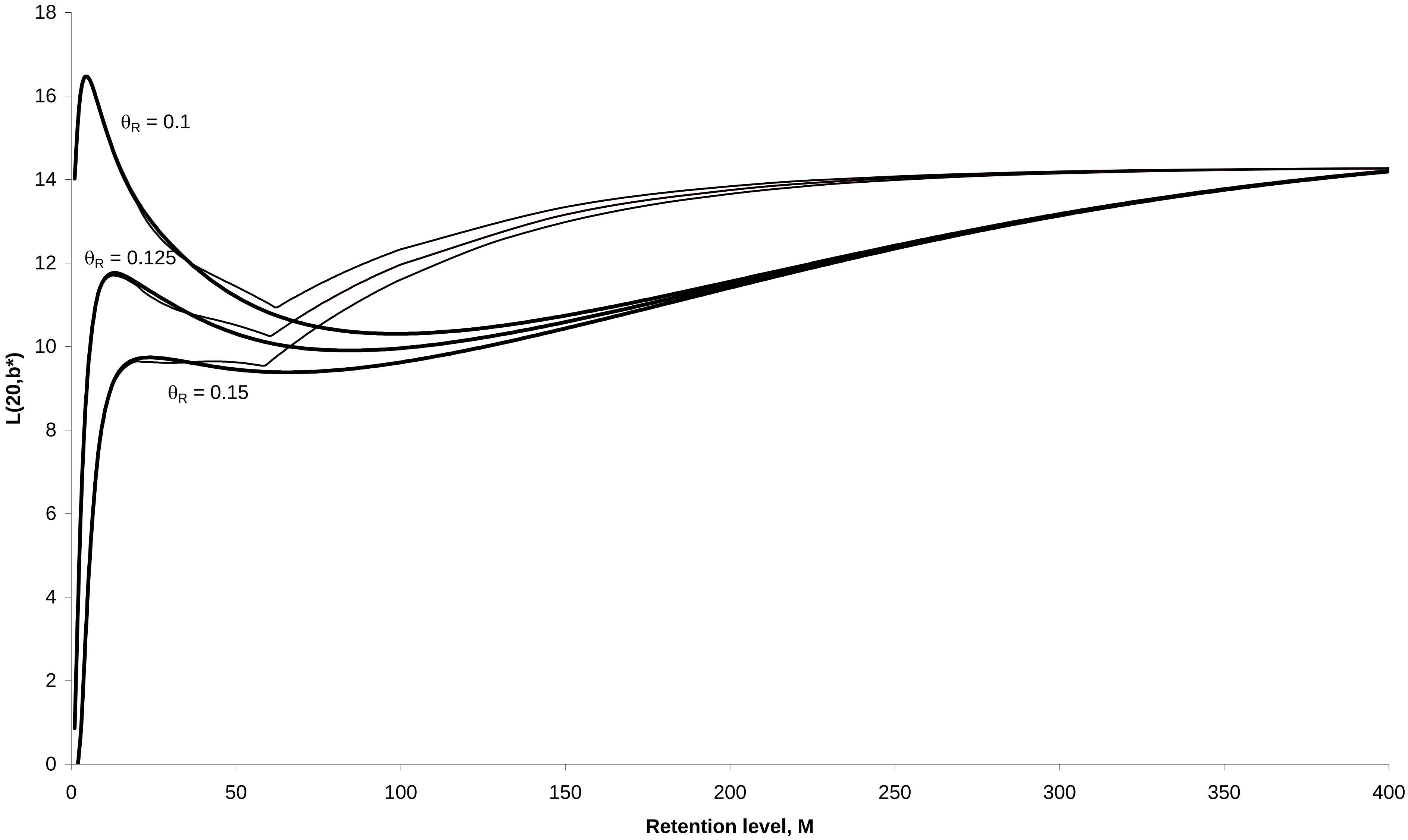

Figure 4: Both approximations to L(20,b*), mixture of three exponential distributions, excess of loss

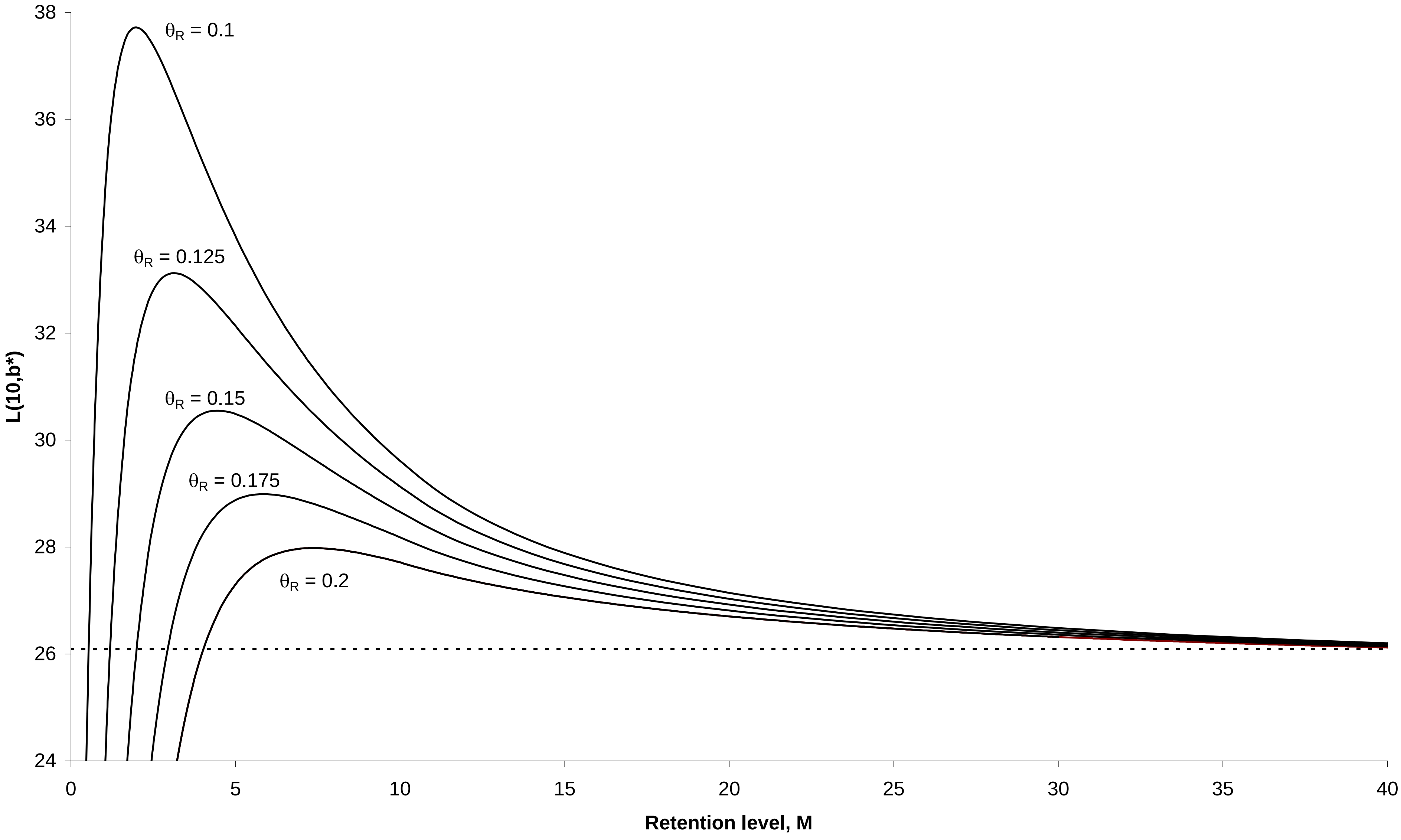

Figure 5: L(10,b*), lognormal claims with variance of 3, excess of loss

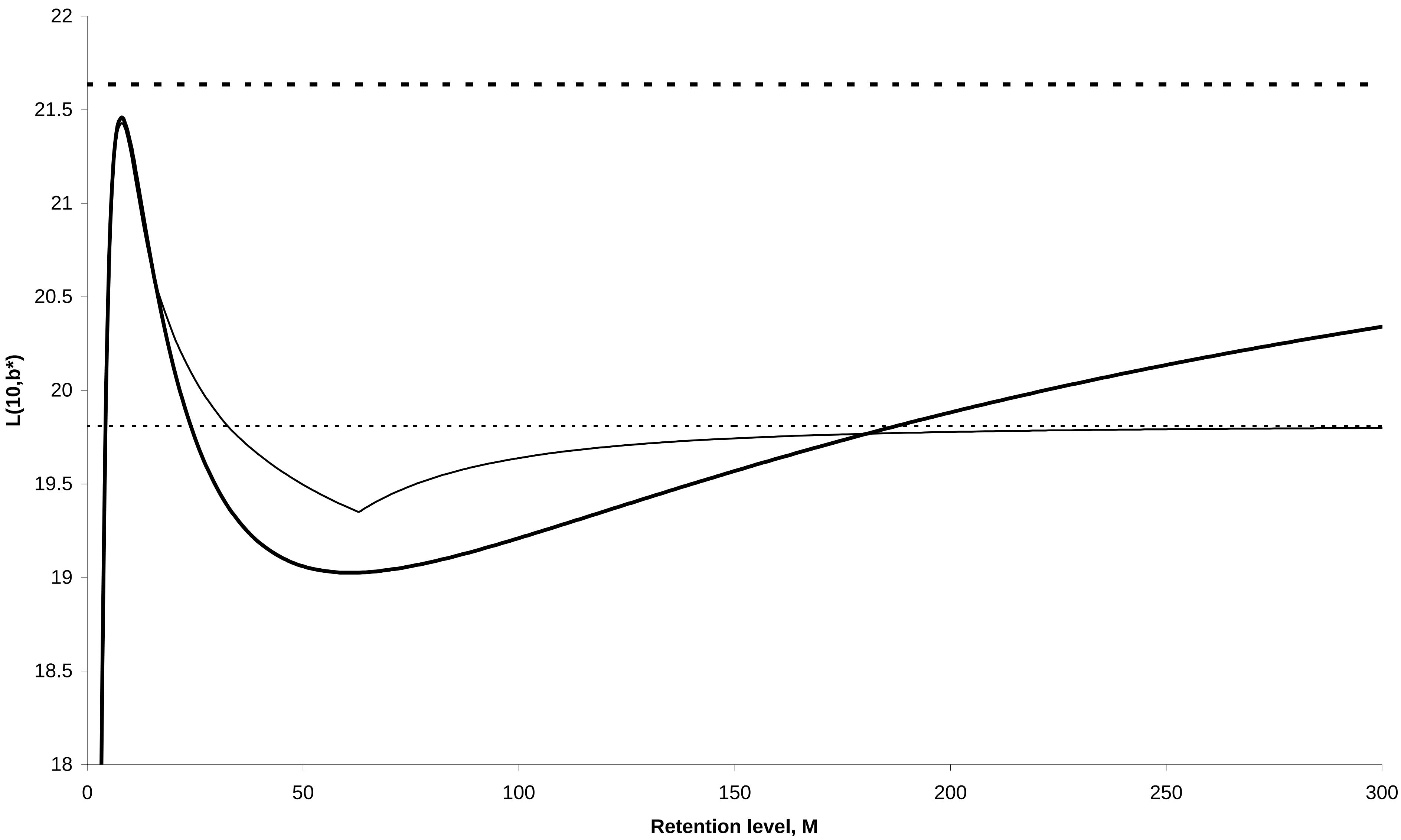

Figure 6: Both approximations to L(10,b*), lognormal claims with variance of 6, excess of loss, θ_R = 0.175