MINIMAL PARTIAL PROXY SIMULATION SCHEMES FOR GENERIC AND ROBUST MONTE-CARLO GREEKS

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Abstract. In this paper, we present a generic framework known as the minimal partial proxy simulation scheme. This framework allows stable computation of the Monte-Carlo Greeks for financial products with trigger features via finite difference approximation. The minimal partial proxy simulation scheme can be considered as a special case of the partial proxy simulation scheme (Fries and Joshi, 2008b) as a measure change (weighted Monte Carlo) is performed to prevent path-wise discontinuities. However, our approach differs in term of how the measure change is performed. Specifically, we select the measure change optimally such that it minimises the variance of the Monte-Carlo weight. Our method can be applied to popular classes of trigger products including digital caplets, autocaps and target redemption notes. While the Monte-Carlo Greeks obtained using the partial proxy simulation scheme can blow up in certain cases, these Monte-Carlo Greeks remain stable under the minimal partial proxy simulation scheme. Standard errors for Vega are also significantly lower under the minimal partial proxy simulation scheme.

1. Introduction

Monte Carlo simulation is a standard methodology for computing prices of derivatives in high-dimensional models such as the LIBOR market model (LMM). Whilst pricing is simple, there are many challenges to be met when trying to compute sensitivities. These challenges are particularly hard when the product has discontinuities and the density of the underlying discretised model is not smooth. A particular example is the computation of Greeks for target redemption notes in the LMM, see Piterbarg (2004). The discontinuous pay-off means that the path-wise method is not applicable, and finite differences lead to high variances. Several methods such as the likelihood ratio method (see, Broadie and Glasserman (1996) and Glasserman (2003)) and the proxy simulation scheme (Fries and Kampen, 2007), have been proposed to address this problem. However, these methods have limited practical application. The likelihood ratio method and the proxy simulation scheme can only be applied to full-factor models, while in practice, most practitioners prefer to use a reduced-factor model since Monte-Carlo simulation using a full-factor model can be very time consuming.

A breakthrough came when the partial proxy simulation scheme was introduced by Fries and Joshi (2008b) followed by the introduction of the conditional analytic Monte-Carlo pricing scheme (see, Fries and Joshi (2008a)

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and a similar approach to conditional analytic was also proposed by Piterbarg (2004)). Both approaches can be easily applied to factor-reduced models and they involve modifying the underlying numerical scheme in such a way that a small perturbation to the initial data will not result in the trigger index crossing the trigger level. Hence, finite differences are effectively being applied to a smooth function and stable Monte-Carlo Greeks can be obtained.

Although the Monte-Carlo Greeks calculated using these methods display substantial variance reduction, both methods still have shortcomings. The conditional analytic pricing scheme relies heavily on the existence of an analytic pricing formula for the next period's redemption payment; however, not all financial products with trigger features satisfy this requirement. One obvious example is an autocap which can trigger several times. The conditional analytic pricing scheme also requires a lot of handcrafting. Specifically, this pricing scheme is model dependent, it requires a major reformulation of the product payoff and from a programming perspective, specific product classes have to be created for this pricing scheme. Therefore, implementation of the conditional analytic pricing scheme can be time consuming. Whilst the partial proxy simulation scheme can be easily applied to all financial products with trigger features without much handcrafting, this simulation scheme does not work well for Vega and short-dated Monte-Carlo Greeks. Under certain circumstances, the Vega and short-dated Monte-Carlo Greeks can have high variance.

The key to the partial proxy scheme was that the rate triggering a change in behaviour was constrained to have the same value before and after bumping. This was at the cost of a likelihood ratio. However, it is not necessary for the rate to have the same value: it need only lie on the same side of the trigger. Relaxing this constraint yields a one-dimensional class of schemes. The key idea of this paper is to select from this class in such a way that variance is minimized.

We thus present a new partial proxy simulation scheme which we call the *minimal partial proxy simulation scheme* that is in a certain sense optimal. Under this simulation scheme, the measure change is performed optimally in such a way that it minimises the variance of the Monte-Carlo weight. Our motivation of minimising the variance of the Monte-Carlo weight is driven by the fact that a naive measure change can result in an unstable Monte-Carlo weight and this might lead to an increase in the variance of the Monte-Carlo Greeks (see Glasserman, 2003). Therefore, by minimising the variance of the Monte-Carlo weight, we can potentially further reduce the variance of the Monte-Carlo Greeks for a trigger product. Our approach requires an Euler-like discretisation and can be applied to popular classes of trigger products including digital caplets, autocaps and target redemptions notes.

We note an alternate approach due to Fries (2007a) where a selection between pathwise and likelihood ratio methods is made on a path by path basis. That method is known as the localized partial proxy.

The paper is organised as follow: in section 2, we present the numerical scheme and product specification. A brief review of the partial proxy simulation scheme is provided in section 3. In section 4, a detailed explanation of the minimal partial proxy simulation scheme is presented. Section 5 is used to discuss the benchmark products and the model for our numerical tests. Numerical results are presented in section 6, with the conclusion in section 7. In Appendix A, an algorithm to decompose the correlation matrix is presented while in Appendix B, the proof that the minimal partial proxy simulation scheme always minimises the variance of the Monte-Carlo weight is presented.

2. Numerical Scheme and Trigger Product Specification

2.1. **Numerical Scheme Specification.** Application of the minimal partial proxy simulation scheme requires an Euler-like discretisation. Suppose that the underlying quantities of a financial products after a change of coordinates (e.g in log-coordinates), $K = (K_1, K_2, \ldots, K_m)$, satisfy the following stochastic differential equation (SDE),

$$dK_j(t) = \mu_j(K,t)dt + \sigma_j(K,t)dB_j(t), \tag{2.1}$$

where $B_j(t)$ is a standard Brownian motion with instantaneous correlation of $\rho_{j,k}$ between $B_j(t)$ and $B_k(t)$. Under an Euler-like discretisation, we have (in vector notation)

$$K(t_{i+1}) = K(t_i) + \mu(K,t_i)\Delta t_i + \Sigma(K,t_i)\Delta B(t_i), \tag{2.2}$$

where $\Sigma(K,t_i)$ is a $m \times m$ matrix while $\mu(K,t_i)$ and $\Delta B(t_i)$ are a m-dimensional column vector such that

$$\begin{aligned} \Sigma(K,t_i) &= diag(\sigma_1(K,t_i), \ldots, \sigma_m(K,t_i)) \\ \mu(K,t_i) &= (\mu_1(K,t_i), \ldots, \mu_m(K,t_i)) \\ \Delta B(t_i) &= (\Delta B_1(t_i), \ldots, \Delta B_m(t_i)) \end{aligned}$$

with $\Delta B_j(t_i) = B_j(t_{i+1}) - B_j(t_i)$.

For practical applications, it is common to use a factor-reduced model to reduce the computational time (Joshi, 2003b). Under a factor-reduced model, the correlated Brownian increments can be rewritten as

$$\Delta B(t_i) = A(t_i)\Delta W(t_i), \tag{2.3}$$

where $\Delta W(t_i)$ is a F-dimensional column vector of uncorrelated Brownian increments from t_i to t_{i+1} and F represents the number of driving factors. The pseudo-root $A(t_i) = (a_{jl}(t_i))$ is a $m \times F$ matrix such that

$$A(t_i)A(t_i)^T = C(t_i),$$

where $C(t_i)$ represents the correlation matrix for $\Delta B(t_i)$ with entry c_{ij} representing the correlation between $\Delta B_i(t_i)$ and $\Delta B_j(t_i)$. In general, there are many different $A(t_i)$'s and there are many ways to obtain them (see Joshi, 2003a and Appendix A). We can therefore rewrite the equation (2.2) as

$$K(t_{i+1}) = K(t_i) + \mu(K,t_i)\Delta t_i + \Sigma(K,t_i)A(t_i)\Delta W(t_i), \tag{2.4}$$

and we also have

$$K_j(t_{i+1}) = K_j(t_i) + \mu_j(K, t_i)\Delta t_i + \sigma_j(K, t_i) \sum_{l=1}^{F} a_{jl}(t_i)\Delta W_l(t_k). \tag{2.5}$$

2.2. **Trigger Product Specification.** Consider a financial product which depends on the underlying quantities of $K = (K_1, K_2, \ldots, K_m)$ and pays a sequence of cash-flows C_i at time t_i. We assume that C_i is $\mathcal{F}_{t_i}$ measurable and is a continuous function of the inputs. We shall say that the product triggers at time t_i if some event E_i occurs such that a rebate R_i ($\mathcal{F}_{t_i}$ measurable) is received and no further cash flows are received.

Application of the minimal partial proxy simulation scheme requires E_i to be defined either as

$$K_s(t_i) > H(t_i), \quad \text{or} \quad K_s(t_i) < H(t_i),$$

for some rate $H(t_i)$ and index s determined by the product and both $H(t_i)$ and s are $\mathcal{F}_{t_{i-1}}$ measurable. Although, there is a restrictive requirement on how the event E_i has to be defined, many path-dependent trigger products can be rewritten in the above form as $H(t_i)$ is a $\mathcal{F}_{t_{i-1}}$-measurable function. We will also consider more general products such as autocaps where the product can trigger several times and on triggering the product becomes another product of the same type but with one fewer trigger left (e.g the rebate is another autocap with one fewer trigger left).

3. Partial Proxy Simulation Scheme

The partial proxy simulation scheme was introduced by Fries and Joshi (2008b). Their approach uses importance sampling to fix quantities that give rise to path-wise discontinuities. Therefore, finite differences are effectively applied to a smooth function, hence the variance of the Monte Carlo Greeks is reduced.

Let G denote a generalised numerical scheme driven by Brownian increments and define G^0 to be the unperturbed (generalised) scheme such that

$$K^0(t_{i+1}) = G^0(t_{i+1}, K^0(t_i), \Delta W(t_i)), \tag{3.1}$$

and G^B denote the perturbed scheme of G^0 (e.g a scheme with a different calibration) such that

$$K^B(t_{i+1}) = G^B(t_{i+1}, K^B(t_i), \Delta W(t_i)). \tag{3.2}$$

Under the usual bump-and-revalue approach, finite differences will be applied to prices obtained under G^0 and G^B where both G^0 and G^B are generated using the same Brownian increment $\Delta W(t_i)$.

Instead of using G^B, Fries and Joshi introduced a third scheme, G^1, known as the partial proxy simulation scheme. The partial proxy simulation scheme [1] has the following properties

[1]Here, we will only discuss a specific form of $\nu(t_i)$ where $\nu(t_i) = (1 - \alpha(t_i))\Delta W(t_i) - \beta(t_i)$

$$K^1(t_0) = K^B(t_0), \tag{3.3}$$
$$K^1(t_{i+1}) = G^1(t_{i+1}, K^1(t_i), \Delta W(t_i)), \tag{3.4}$$
$$= G^B(t_{i+1}, K^1(t_i), \Delta W(t_i) - \nu(t_i)), \tag{3.5}$$

where $\nu(t_i)$ is a F-dimensional column vector $(\nu_1(t_i), \nu_2(t_i), \ldots, \nu_F(t_i))$ with $\nu_l(t_i) = (1-\alpha_l(t_i))\Delta W_l(t_i) - \beta_l(t_i)$ for $l = 1, 2, \ldots, F$. Both $\alpha_l(t_i)$ and $\beta_l(t_i)$ are determined by the condition

$$f(t_{i+1}, K^1(t_{i+1})) = f(t_{i+1}, K^0(t_{i+1})) \tag{3.6}$$

where f is the proxy constraint ensuring quantities that give rise to path-wise discontinuities are the same under G^0 and G^1. The Monte-Carlo weight for the partial proxy simulation scheme, $w(t_{i+1})$, is given by

$$w(t_{i+1}) = \prod_{l=1}^{F} \alpha_l(t_i) \exp\left(-\frac{(\Delta W(t_i) - \nu(t_i))^2 - (\Delta W(t_i))^2}{2\Delta t_i} \right). \tag{3.7}$$

Monte-Carlo pricing under G^B is, in the Monte-Carlo limit, equivalent to pricing under G^1 using Monte-Carlo weights of $\prod w(t_{i+1})$.

Under the specific class of trigger product discussed in the previous section, the proxy constraint is a simple linear function given by

$$f(t_{i+1}, K(t_{i+1})) = K_s(t_{i+1}) \tag{3.8}$$

Therefore, solutions for $\alpha_l(t_i)$ and $\beta_l(t_i)$ must satisfy

$$K_s^1(t_{i+1}) = K_s^0(t_{i+1}) \tag{3.9}$$

As explained by Fries and Joshi, the proxy constraint will hold only if the diffusion coefficient for $K_s(t_{i+1})$ in G^1 is exactly the same as in G^0. As a result, under an Euler-like discretisation, the solution for $\alpha_l(t_i)$ must satisfy the following equation

$$\sigma_s^B(t_i) a_{sl}^B(t_i) \alpha_l(t_i) = \sigma_s^0(t_i) a_{sl}^0(t_i) \tag{3.10}$$

and consequently,

$$\alpha_l(t_i) = \frac{\sigma_s^0(t_i) a_{sl}^0(t_i)}{\sigma_s^B(t_i) a_{sl}^B(t_i)} \tag{3.11}$$

for $l = 1, \ldots, F$.

4. Minimal Partial Proxy Simulation Scheme

4.1. **Main Idea.** The partial proxy simulation scheme eliminates payoff discontinuities, however it comes with a price; Monte-Carlo weights are needed to offset the impact of the measure change. A naive measure change may result in an unstable Monte-Carlo weight (i.e sensitive to the outcome of the Brownian increments) and hence, an increase in the variance of the Monte-Carlo Greeks (Glasserman, 2003). However, if we only focus on Euler-like discretisation and the specific class of trigger products discussed in section 2, the solution for $\alpha_l(t_i)$ no longer need to satisfy the equation (3.10). As a result, we can select $\alpha_l(t_i)$ optimally to lower the variance of the Monte-Carlo Greeks. In particular, we no longer select the proxy constraint to make the rate equal to the unbumped rate at the end of the step, but instead make the weaker requirement that the bumped and unbumped rates are on the same side of the trigger. This gives us an extra degree of freedom and so we are able to selected the measure change which minimizes the variance of the likelihood weight across the step.

4.2. **Detailed Explanations of The Minimal Partial Proxy Simulation Scheme.** Suppose Q denotes an Euler-like discretisation scheme and define Q^0 to be the unperturbed scheme such that

$$\begin{aligned} K^0(t_{i+1}) &= Q^0(t_{i+1}, K^0(t_i), \Delta W(t_i)), \\ &= K^0(t_i) + \mu^0(K^0, t_i)\Delta t_i + \Sigma^0(K^0, t_i)A^0(t_i)\Delta W(t_i), \quad (4.1) \end{aligned}$$

and the perturbed scheme, Q^B, is given by

$$\begin{aligned} K^B(t_{i+1}) &= Q^B(t_{i+1}, K^B(t_i), \Delta W(t_i)), \\ &= K^B(t_i) + \mu^B(K^B, t_i)\Delta t_i + \Sigma^B(K^B, t_i)A^B(t_i)\Delta W(t_i). \end{aligned} \quad (4.2)$$

Instead of having the partial proxy simulation scheme, we consider the minimal partial proxy simulation scheme, $\tilde{Q}$, with the following properties

$$\tilde{K}(t_0) = K^B(t_0), \quad (4.3)$$

$$\begin{aligned} \tilde{K}(t_{i+1}) &= \tilde{Q}(t_{i+1}, \tilde{K}(t_i), \Delta W(t_i)), \\ &= Q^B(t_{i+1}, \tilde{K}(t_i), \Delta W(t_i) - \nu(t_i)), \\ &= \tilde{K}(t_i) + \mu^B(\tilde{K}, t_i)\Delta t_i + \Sigma^B(\tilde{K}, t_i)A^B(t_i)(\Delta W(t_i) - \nu(t_i)), \end{aligned} \quad (4.4)$$

where $\nu(t_i)$ is a F-dimensional column vector $(\nu_1(t_i), \nu_2(t_i), \dots, \nu_F(t_i))$ with $\nu_l(t_i) = (1-\alpha_l(t_i))\Delta W_l(t_i) - \beta_l(t_i)$ for $l = 1, 2, \dots, F$. Observe that, if we set $\alpha_l(t_i) = 1$ and $\beta_l(t_i) = 0$ for all i and l, the minimal partial proxy simulation scheme is effectively the perturbed scheme. The essential difference from the original partial proxy scheme is that we pick $\nu_l(t_i)$ in an optimal way.

Instead of defining a proxy constraint, path-wise discontinuities are eliminated at t_{i+1} as long as the following condition [2]

$$K_s^0(t_{i+1}) > H^0(t_{i+1}) \iff \tilde{K}_s(t_{i+1}) > \tilde{H}(t_{i+1}), \quad (4.5)$$

is satisfied. Essentially, the trigger event, E_{i+1}, occurs at the same time in both numerical schemes. However, solving for $\nu(t_i)$ such that the constraint above holds can be complicated as both $K_s^0(t_{i+1})$ and $\tilde{K}_s(t_{i+1})$ are driven by F uncorrelated Brownian increments with different diffusion coefficients. Fortunately, as discussed earlier in section 2, many different $A^0(t_i)$'s and $A^B(t_i)$'s exist. If we have $A^0(t_i)$ and $A^B(t_i)$ such that

$$a_{s1}^0 = a_{s1}^B = 1, \quad (4.6)$$

$$a_{sl}^0 = a_{sl}^B = 0, \quad (4.7)$$

for $l = 2, 3, \dots, F$ then solving for $\nu_l(t_i)$ is straight-forward. We define $A_s^0(t_i)$ and $A_s^B(t_i)$ to represent such pseudo square roots. An algorithm to obtain

[2] Here, we are assuming that trigger events, E_{i+1} is defined to be $K_s(t_{i+1}) > H(t_{i+1})$. For the case where trigger event, E_{i+1}, is defined to be $K_s(t_{i+1}) < H(t_{i+1})$, the exact same result is obtained by using the same arguments except for the equation (4.8), (4.9) and (4.10) where we will have less than instead of greater than.

$A_s^0(t_i)$ and $A_s^B(t_i)$ is discussed in Appendix A. Using $A_s^0(t_i)$ and $A_s^B(t_i)$ and replacing $\nu_1(t_i)$ with $(1-\alpha_1(t_i))\Delta W_1(t_i) - \beta_1(t_i)$, the constraint in (4.5) becomes

$$K_s^0(t_i) + \mu_s^0(K^0, t_i)\Delta t_i + \sigma_s^0(K^0, t_i)\Delta W_1(t_i) > H^0(t_{i+1}) \tag{4.8}$$
$$\Longleftrightarrow$$
$$\tilde{K}_s(t_i) + \mu_s^B(\tilde{K}, t_i)\Delta t_i + \sigma_s^B(\tilde{K}, t_i)(\alpha_1(t_i)\Delta W_1(t_i) + \beta_1(t_i)) > \tilde{H}(t_{i+1}). \tag{4.9}$$

Assume without loss of generality that $\sigma_s^0(K^0, t_i)$, $\sigma_s^B(K^0, t_i)$ and $\alpha_1(t_i)$ are positive, the inequality in (4.8) will hold if

$$\Delta W_1(t_i) > \Delta W_1^*(t_i), \tag{4.10}$$

where

$$\Delta W_1^*(t_i) = \frac{H^0(t_{i+1}) - K_s^0(t_i) + \mu_s^0(K^0, t_i)\Delta t_i}{\sigma_s^0(K^0, t_i)}. \tag{4.11}$$

Therefore, under the same condition, i.e $\Delta W_1(t_i) > \Delta W_1^*(t_i)$, we must ensure that the inequality in (4.9) holds to prevent path-wise discontinuities. Clearly, when $\Delta W_1(t_i) = \Delta W_1^*(t_i)$, we must then have

$$K_s^0(t_i) + \mu_s^0(K^0, t_i)\Delta t_i + \sigma_s^0(K^0, t_i)\Delta W_1^*(t_i) = H^0(t_{i+1}) \tag{4.12}$$
$$\Longleftrightarrow$$
$$\tilde{K}_s(t_i) + \mu_s^B(\tilde{K}, t_i)\Delta t_i + \sigma_s^B(\tilde{K}, t_i)(\alpha_1(t_i)\Delta W_1^*(t_i) + \beta_1(t_i)) = \tilde{H}(t_{i+1}). \tag{4.13}$$

If we solve for $\alpha_l(t_i)$ and $\beta_l(t_i)$ for $l = 1, 2, \ldots, F$ such that the equation (4.13) holds then the constraint in (4.5) will hold. Since the equation (4.13) does not depend on $\alpha_l(t_i)$ and $\beta_l(t_i)$ for $l = 2, \ldots, F$, we set $\alpha_l(t_i) = 1$ and $\beta_l(t_i) = 0$ (i.e. no measure change) for $l = 2, \ldots, F$ to reduce the impact of measure change. Hence, we are left with 2 variables and 1 equation.

We select the solution of $\alpha_1(t_i)$ and $\beta_1(t_i)$ dynamically. Observe that, if the probability of the product triggering at next time step is close to 0 or 1, the trigger effectively does not exist and there are no path-wise discontinuities. Under such scenarios, no measure change will be performed as performing a measure change will generally increase the variance of the Monte Carlo Greeks. Therefore, whenever both numerical schemes require $\Delta W_1(t_i)$ to be greater than six standard deviation away from the mean to hit the trigger or not to hit the trigger, we set $\alpha_1(t_i) = 1$ and $\beta_1(t_i) = 0$.

Otherwise, we select $\alpha_1(t_i)$ such that it minimises the variance of the Monte-Carlo weight and the solution for $\beta_1(t_i)$ is determined by the equation (4.13). Here, we let $\alpha_1^*(t_i)$ denote the $\alpha_1(t_i)$ that minimises the variance of the Monte-Carlo weight and based on the setup above, the Monte Carlo weight at t_{i+1} is given by

$$w(t_{i+1}) = \alpha_1(t_i) \exp\left(-\frac{(\alpha_1(t_i)\Delta W_1(t_i) + \beta_1(t_i))^2 - (\Delta W_1(t_i))^2}{2\Delta t_i}\right), \tag{4.14}$$

where

$$\beta_1(t_i) = \frac{\tilde{H}(t_{i+1}) - \tilde{K}_s(t_i) - \mu_s^B(\tilde{K}, t_i)\Delta t_i}{\sigma_s^B(\tilde{K}, t_i)} - \alpha_1(t_i)\Delta W_1^*(t_i), \tag{4.15}$$

and the variance of the Monte Carlo weight is

$$\mathrm{Var}(w(t_{i+1})) = \frac{\alpha_1(t_i)^2}{\sqrt{2\alpha_1(t_i)^2 - 1}} \exp\left(\frac{\beta_1(t_i)^2}{\Delta t_i(2\alpha_1(t_i)^2 - 1)}\right) - 1, \quad (4.16)$$

for $\alpha_1(t_i) > \frac{1}{\sqrt{2}}$ (see Appendix B1). (For $\alpha_1(t_i) < \frac{1}{\sqrt{2}}$, the variance will actually be infinite.) It turns out that $\alpha_1^*(t_i)$ satisfies the following equation

$$a\alpha_1^4 + b\alpha_1^3 + c\alpha_1^2 + d\alpha_1 + e = 0, \quad (4.17)$$

where

$$\begin{aligned} a &= 2\Delta t_i, \\ b &= 2X(t_i)\Delta W_1^*(t_i), \\ c &= -(3\Delta t_i + 2X(t_i)^2 + \Delta W_1^*(t_i)^2), \\ d &= X(t_i)\Delta W_1^*(t_i), \\ e &= \Delta t_i, \\ X(t_i) &= \frac{\tilde{H}(t_{i+1}) - \tilde{K}_s(t_i) - \mu_s^B(\tilde{K}, t_i)\Delta t_i}{\sigma_s^B(\tilde{K}, t_i)}. \end{aligned}$$

Four distinct real roots always exist for the equation (4.17), only one real root is greater than $\frac{1}{\sqrt{2}}$ and this root is the solution for $\alpha_1^*(t_i)$ which minimises $\mathrm{Var}(w(t_{i+1}))$ (see Appendix B2-3). As closed form solutions exist for the roots of a 4^{th} order polynomial, there is no need to perform a numerical root search, hence adopting our approach to calculating Monte-Carlo Greeks will not increase the computational time substantially.

5. Benchmark Model and Benchmark Products

5.1. **Benchmark Model - The LIBOR Market Model.**

5.1.1. *Model Setup.* The LIBOR market model, also known as BGM (Brace *et al* (1997), and Brace (2007)) is a very popular model for the pricing of exotic interest rate derivatives. This model assumes that there are $n+1$ zero coupon bonds, each with maturity associated to one of the $n+1$ dates

$$0 < T_0 < T_1 < \ldots < T_n,$$

However, instead of evolving bond prices directly, dynamics are assigned to n contiguous forward rates spanning across the above dates. Let $\tau_j = T_{j+1} - T_j$, $P(t,T)$ denote the price at time t of a zero-coupon bond paying one at its maturity, T, and $f_j(t)$ denote the forward rate from T_j to T_{j+1} as observed at time t. Using the no-arbitrage argument, the forward rate, $f_j(t)$, is determined by

$$f_j(t) = \frac{\frac{P(t,T_j)}{P(t,T_{j+1})} - 1}{\tau_j},$$

Under the LIBOR market model, the forward rate $f_j(t)$ is assumed to follow the following process

$$df_j(t) = \mu_j(f,t)f_j(t)dt + \sigma_j(t)f_j(t)dB_j(t), \quad (5.1)$$

where σ_j is a time dependent deterministic volatility function, $B_j(t)$ is a standard Brownian motion with instantaneous correlation of $\rho_{j,k}$ between $B_j(t)$ and $B_k(t)$ and $\mu_j(f,t)$ is the drift of the SDE to be determined uniquely based on the no-arbitrage requirement.

If spot LIBOR measure, which corresponds to using the discretely-compounded money market account as numeraire, is used, the drift term in (5.1) is given by

$$\mu_j(f,t) = \sum_{h=\eta(t)}^{j} \frac{f_h(t)\tau_h}{1+f_h(t)\tau_h}\sigma_j(t)\sigma_h(t)\rho_{j,h}, \tag{5.2}$$

and the value of the numeraire at time t is,

$$N(t) = P(t,\eta(t)) \prod_{j=1}^{\eta(t)-1} (1+\tau_j f_j(T_j)), \tag{5.3}$$

where $\eta(t)$ is the unique integer satisfying

$$T_{\eta(t)-1} \leqslant t < T_{\eta(t)}.$$

5.1.2. *Monte-Carlo Pricing Under LMM.* Due to the state-dependent drift in the equation (2.1), the true process of forward rates cannot be solved. One popular approximation is to discretise the log of the forward-rate process by assuming the drift term is constant across each time step (see Joshi, 2003a). Under such a discretisation, the log-forward-rate process is given by

$$\log f_j(T_{i+1}) = \log f_j(T_i) + \bar{\mu}_j(f(T_i),t_i)\Delta T_i + \sqrt{C_{jj}(T_i)}\sum_{l=1}^{F} a_{jl}\Delta W_l(T_i) \tag{5.4}$$

where

$$\begin{aligned} \Delta T_i &= T_{i+1} - T_i, \\ C_{kl}(T_i) &= \frac{1}{\Delta T_i}\int_{T_i}^{T_{i+1}} \rho_{kl}\sigma_k(t)\sigma_l(t)dt, \\ \bar{\mu}_j(f(T_i),T_i) &= \sum_{h=\eta(t)}^{j} \frac{f_h(T_i)\tau_h}{1+f_h(T_i)\tau_h}C_{jh}(T_i) - 0.5C_{jj}(T_i), \end{aligned} \tag{5.5}$$

and, as usual, a_{jl} is the element on the j^{th} row and l^{th} column of the pseudo square root, A, of the forward rate correlation matrix and $\Delta W_l(t_i)$ is a F-dimensional row vector of uncorrelated Brownian increments with F being the number of driving factors

The price of an interest rate derivative with payoff $V(T_k, f(T_0), ..., f(T_k))$ at time T_k, depending on the realization of f at $T_0, \ldots, T_k$, is given by:

$$V(0) = N(0)\mathbb{E}\left(\frac{V(T_k, f(T_0), f(T_1), ..., f(T_k))}{N(T_k)}\middle|\mathcal{F}_0\right). \tag{5.6}$$

5.2. **Benchmark Products.** For our numerical tests, we will consider digital caplets, autocaps, and target redemption notes with a reverse LIBOR floater. A brief description of these products is provided below (For the precise mathematical definition of these products, see Fries (2007b)).

5.2.1. *Digital Caplets.* A digital caplet will pay 1 if the underlying forward rate finishes above the strike or zero otherwise. A small perturbation of the forward rate can shift a digital caplet from finishing out-of-money to in-the-money resulting a discontinuous path-wise value.

5.2.2. *Autocaps.* Autocaps are very similar to caps. However, buyers must exercise when a caplet finishes in-the-money and only limited number of caplets (e.g. 5 out of 10) can be exercised throughout the life of the contract. This product has a strong discontinuity feature as a small perturbation of the forward rate can shift an out-of-money caplet to in-the-money causing another caplet to have zero value.

5.2.3. *Target Redemption Notes(TARNs).* These have the following features,

(1) similar to a callable bond
(2) pays large initial coupon followed by inverse floating coupons (e.g. $\text{Max}(10\% - 2f_j(T_j), 0)$)
(3) will be redeemed once the total coupon paid reaches the target coupon or at maturity whichever earlier.

After the initial coupon, subsequent coupons will only be paid if the underlying forward rate is low and no coupon payments will be made if the underlying forward rate is high. Hence, in a upward sloping forward curve environment, TARN will either be redeemed very early in the life of the contact or at maturity. A small bump to forward rates can change the timing of redemption resulting a discontinuous path-wise value.

6. Numerical Test and Results

For our numerical tests, we use digital caplets, autocaps and TARNs as our benchmark products and the LIBOR market model as our benchmark model. we model the semi-annual forward rates with a flat volatility of 0.2 and we assume that the Brownian driver of LMM has an exponentially decaying correlation, $\rho_{ij} = \exp(-0.2|T_i - T_j|)$ and reduced to the first 5 factors. Means and standard errors of the Monte-Carlo Greeks are calculated using simulations with 5000 paths. We compare the Monte-Carlo Greeks obtained using the direct simulation, the partial proxy simulation scheme and the minimal partial proxy simulation scheme. The scaling of the sensitivities is as follows: Delta and gamma are normalized as price change per 1% parallel shift in the forward curve. Vega is normalized as price change per 1% parallel shift in the forward rate volatility curve times 100.

6.1. **Sensitivities of Digital Caplets.** A digital caplet with maturity, $t = 4$ and strike of 5% is used for our numerical tests. We assume that the LIBOR curve is flat at 5%. The numerical results are given by Figure 1 and Table 6.1. For Delta and Gamma, the partial proxy and the minimal partial proxy simulation scheme give similar standard errors. However, Vega calculated using the minimal partial proxy simulation scheme has a significant variance

	Delta						Gamma						Vega					
Shift	Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP	
in bp	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
0.010	16.74%	52.23%	15.70%	1.02%	15.63%	0.69%	-15710.22%	718786.18%	-2.77%	2.09%	-2.80%	2.00%	-34.79%	422.88%	-24.73%	10.47%	-24.76%	0.65%
0.025	17.40%	34.44%	15.70%	1.02%	15.63%	0.69%	-8.07%	197755.11%	-2.77%	2.09%	-2.80%	2.00%	-29.48%	239.05%	-24.73%	10.47%	-24.76%	0.65%
0.050	16.23%	23.99%	15.70%	1.02%	15.63%	0.69%	-1119.03%	68730.58%	-2.77%	2.09%	-2.80%	2.00%	-21.53%	134.37%	-24.73%	10.47%	-24.76%	0.65%
0.100	15.53%	16.41%	15.69%	1.03%	15.63%	0.69%	-964.30%	23331.34%	-2.77%	2.09%	-2.80%	2.00%	-21.52%	94.47%	-24.73%	10.47%	-24.76%	0.65%
0.500	15.85%	7.69%	15.69%	1.03%	15.62%	0.69%	-93.10%	2187.88%	-2.77%	2.09%	-2.80%	2.00%	-24.18%	44.52%	-24.73%	10.47%	-24.76%	0.65%
1.000	15.82%	5.39%	15.68%	1.03%	15.61%	0.69%	-6.63%	793.03%	-2.77%	2.09%	-2.80%	2.00%	-24.75%	33.21%	-24.73%	10.47%	-24.76%	0.65%
1.500	15.65%	4.25%	15.67%	1.03%	15.61%	0.69%	-9.93%	423.83%	-2.77%	2.09%	-2.80%	2.00%	-24.21%	25.97%	-24.73%	10.47%	-24.76%	0.65%
2.000	15.59%	3.69%	15.67%	1.03%	15.60%	0.69%	-10.75%	269.92%	-2.77%	2.09%	-2.80%	2.00%	-23.55%	22.78%	-24.73%	10.47%	-24.76%	0.65%
2.500	15.59%	3.26%	15.66%	1.03%	15.59%	0.69%	-6.31%	190.39%	-2.77%	2.09%	-2.80%	2.00%	-24.18%	20.43%	-24.73%	10.47%	-24.76%	0.65%
3.000	15.57%	2.93%	15.65%	1.04%	15.59%	0.69%	-8.22%	138.69%	-2.77%	2.09%	-2.80%	2.00%	-25.22%	19.51%	-24.73%	10.47%	-24.76%	0.65%
4.000	15.65%	2.59%	15.64%	1.04%	15.57%	0.70%	-3.60%	91.23%	-2.77%	2.09%	-2.80%	1.99%	-25.09%	16.39%	-24.73%	10.47%	-24.76%	0.65%
5.000	15.66%	2.30%	15.62%	1.05%	15.56%	0.70%	0.22%	65.69%	-2.77%	2.09%	-2.80%	1.99%	-25.33%	15.08%	-24.73%	10.47%	-24.76%	0.65%
6.000	15.56%	2.07%	15.61%	1.05%	15.54%	0.70%	-1.67%	51.88%	-2.77%	2.09%	-2.80%	1.98%	-25.12%	13.52%	-24.73%	10.47%	-24.76%	0.65%
8.000	15.51%	1.80%	15.58%	1.06%	15.51%	0.70%	-1.50%	33.00%	-2.77%	2.09%	-2.80%	1.96%	-25.37%	12.33%	-24.73%	10.47%	-24.76%	0.65%
10.000	15.49%	1.56%	15.55%	1.07%	15.48%	0.71%	-2.04%	23.08%	-2.77%	2.09%	-2.80%	1.94%	-25.51%	10.86%	-24.73%	10.47%	-24.76%	0.65%
15.000	15.40%	1.29%	15.47%	1.09%	15.40%	0.72%	-2.55%	12.57%	-2.76%	2.09%	-2.80%	1.87%	-25.16%	9.09%	-24.73%	10.47%	-24.76%	0.65%
20.000	15.34%	1.12%	15.39%	1.12%	15.32%	0.73%	-2.66%	8.21%	-2.76%	2.08%	-2.80%	1.78%	-25.19%	7.78%	-24.73%	10.47%	-24.76%	0.65%
25.000	15.25%	0.97%	15.31%	1.15%	15.24%	0.74%	-2.69%	5.73%	-2.75%	2.08%	-2.80%	1.70%	-25.13%	7.08%	-24.73%	10.47%	-24.76%	0.65%
30.000	15.17%	0.88%	15.22%	1.18%	15.15%	0.75%	-2.68%	4.37%	-2.74%	2.08%	-2.79%	1.62%	-24.93%	6.38%	-24.73%	10.47%	-24.76%	0.65%
35.000	15.08%	0.91%	15.13%	1.22%	15.06%	0.76%	-2.71%	3.70%	-2.73%	2.07%	-2.78%	1.54%	-24.99%	5.96%	-24.73%	10.47%	-24.76%	0.65%
40.000	14.99%	0.92%	15.04%	1.25%	14.97%	0.78%	-2.73%	3.20%	-2.72%	2.07%	-2.77%	1.47%	-24.97%	5.48%	-24.73%	10.47%	-24.76%	0.65%
45.000	14.90%	0.91%	14.94%	1.29%	14.88%	0.79%	-2.72%	2.75%	-2.70%	2.07%	-2.75%	1.42%	-24.95%	5.19%	-24.73%	10.47%	-24.76%	0.65%
50.000	14.80%	0.90%	14.84%	1.34%	14.78%	0.81%	-2.67%	2.40%	-2.69%	2.06%	-2.73%	1.37%	-25.04%	4.84%	-24.73%	10.47%	-24.76%	0.65%

TABLE 6.1. The Monte-Carlo Greeks for digital caplets. Data corresponding to Figure 1

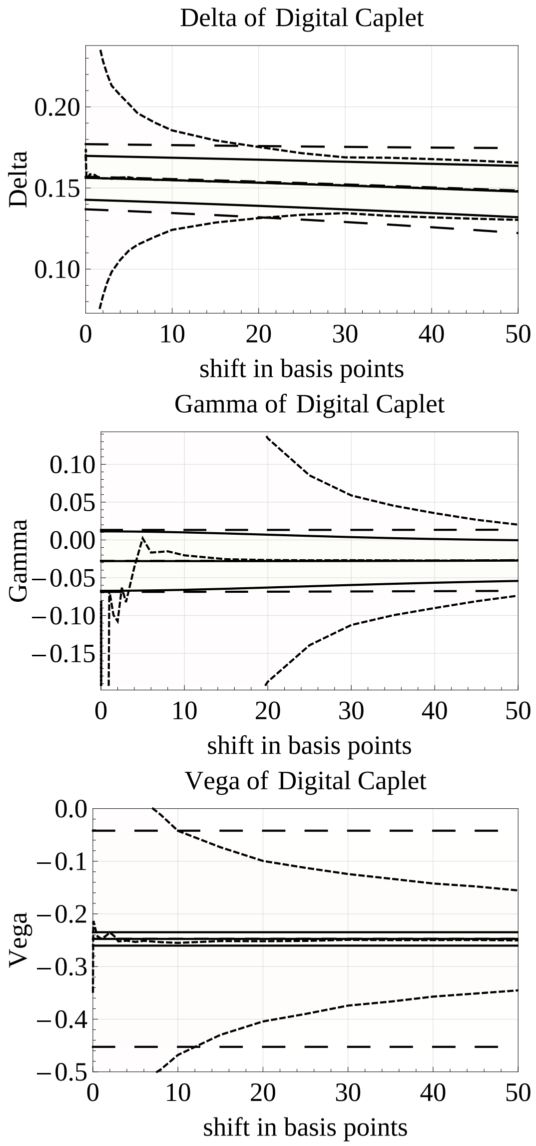

FIGURE 1. Digital Caplet: Finite differences are applied to the direct simulation (fine dotted lines), the partial proxy simulation scheme (large dotted lines) and the minimal partial proxy simulation scheme (solid lines). The upper and lower lines represent the 95% confidence interval for the Monte-Carlo Greeks and the middle line represents the mean of the Monte-Carlo Greeks.

reduction compared to the partial proxy simulation scheme. Based on the result of our numerical tests, the standard error of Vega calculated using the minimal partial proxy simulation scheme is approximately 16 times lower than the standard error obtained using the partial proxy simulation scheme.

6.2. **Sensitivities of Autocap.** A five-year autocap with semi-annual fixings and a 5% flat LIBOR curve are used for our numerical tests. We assume

	Delta						**Gamma**						**Vega**					
Shift	Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP	
in bp	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
0.010	1.15%	1.08%	1.15%	0.05%	1.14%	0.05%	-1067.61%	15873.20%	0.47%	0.14%	0.47%	0.15%	7.93%	5.86%	7.87%	1.15%	7.91%	0.09%
0.025	1.12%	0.70%	1.15%	0.05%	1.14%	0.05%	-51.60%	4034.32%	0.47%	0.14%	0.47%	0.15%	7.75%	2.49%	7.87%	1.15%	7.91%	0.09%
0.050	1.14%	0.47%	1.15%	0.05%	1.14%	0.05%	-22.53%	1473.44%	0.47%	0.14%	0.47%	0.15%	7.80%	2.20%	7.87%	1.15%	7.91%	0.09%
0.100	1.13%	0.37%	1.15%	0.05%	1.15%	0.05%	1.84%	513.82%	0.47%	0.14%	0.47%	0.15%	7.82%	1.76%	7.87%	1.15%	7.91%	0.09%
0.500	1.16%	0.16%	1.15%	0.05%	1.15%	0.05%	1.91%	46.41%	0.47%	0.14%	0.47%	0.15%	7.84%	1.06%	7.87%	1.15%	7.91%	0.09%
1.000	1.15%	0.12%	1.15%	0.05%	1.15%	0.05%	0.95%	16.78%	0.47%	0.14%	0.47%	0.15%	7.85%	0.78%	7.87%	1.15%	7.91%	0.09%
1.500	1.15%	0.09%	1.15%	0.05%	1.15%	0.05%	0.49%	9.14%	0.47%	0.14%	0.47%	0.15%	7.86%	0.63%	7.87%	1.15%	7.91%	0.09%
2.000	1.16%	0.08%	1.15%	0.05%	1.15%	0.05%	0.60%	5.86%	0.47%	0.14%	0.47%	0.15%	7.89%	0.56%	7.87%	1.15%	7.91%	0.09%
2.500	1.16%	0.07%	1.15%	0.05%	1.15%	0.05%	0.53%	4.05%	0.47%	0.14%	0.47%	0.15%	7.89%	0.52%	7.87%	1.15%	7.91%	0.09%
3.000	1.16%	0.06%	1.16%	0.05%	1.15%	0.05%	0.51%	3.06%	0.47%	0.14%	0.47%	0.15%	7.90%	0.49%	7.87%	1.15%	7.91%	0.09%
4.000	1.16%	0.06%	1.16%	0.05%	1.15%	0.05%	0.51%	2.02%	0.47%	0.14%	0.47%	0.15%	7.91%	0.44%	7.87%	1.15%	7.91%	0.09%
5.000	1.16%	0.05%	1.16%	0.05%	1.16%	0.05%	0.47%	1.47%	0.47%	0.14%	0.47%	0.15%	7.90%	0.39%	7.87%	1.15%	7.91%	0.09%
6.000	1.16%	0.04%	1.16%	0.05%	1.16%	0.05%	0.48%	1.12%	0.47%	0.14%	0.47%	0.15%	7.90%	0.36%	7.87%	1.16%	7.91%	0.09%
8.000	1.17%	0.04%	1.17%	0.05%	1.16%	0.05%	0.47%	0.74%	0.47%	0.14%	0.47%	0.14%	7.90%	0.33%	7.87%	1.16%	7.91%	0.09%
10.000	1.18%	0.04%	1.17%	0.05%	1.17%	0.05%	0.47%	0.52%	0.47%	0.14%	0.47%	0.14%	7.91%	0.30%	7.87%	1.16%	7.91%	0.09%
15.000	1.19%	0.03%	1.18%	0.05%	1.18%	0.04%	0.46%	0.28%	0.47%	0.14%	0.47%	0.13%	7.92%	0.25%	7.87%	1.17%	7.91%	0.09%
20.000	1.20%	0.03%	1.20%	0.06%	1.19%	0.04%	0.46%	0.18%	0.47%	0.14%	0.47%	0.11%	7.91%	0.22%	7.87%	1.18%	7.91%	0.09%
25.000	1.21%	0.02%	1.21%	0.06%	1.20%	0.04%	0.46%	0.13%	0.47%	0.14%	0.47%	0.10%	7.91%	0.20%	7.88%	1.19%	7.91%	0.09%
30.000	1.22%	0.02%	1.22%	0.06%	1.22%	0.04%	0.46%	0.10%	0.47%	0.14%	0.47%	0.09%	7.92%	0.19%	7.88%	1.19%	7.91%	0.09%
35.000	1.23%	0.02%	1.23%	0.07%	1.23%	0.04%	0.46%	0.08%	0.47%	0.14%	0.46%	0.09%	7.91%	0.18%	7.88%	1.20%	7.91%	0.09%
40.000	1.24%	0.02%	1.25%	0.07%	1.24%	0.04%	0.46%	0.06%	0.46%	0.14%	0.46%	0.08%	7.91%	0.17%	7.88%	1.21%	7.91%	0.09%
45.000	1.26%	0.02%	1.26%	0.08%	1.25%	0.04%	0.46%	0.05%	0.46%	0.14%	0.46%	0.07%	7.91%	0.16%	7.88%	1.22%	7.91%	0.09%
50.000	1.27%	0.02%	1.27%	0.08%	1.26%	0.05%	0.46%	0.04%	0.46%	0.15%	0.46%	0.07%	7.91%	0.16%	7.88%	1.23%	7.91%	0.09%

TABLE 6.2. The Monte-Carlo Greeks for autocaps. Data corresponding to Figure 2

	Delta						**Gamma**						**Vega**					
Shift	Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP	
in bp	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
0.010	0.49%	1.68%	0.46%	0.17%	0.47%	0.18%	-404.09%	24139.36%	0.83%	2.20%	0.83%	2.25%	6.63%	4.41%	6.71%	1.04%	6.74%	0.08%
0.025	0.52%	1.08%	0.46%	0.17%	0.47%	0.18%	47.88%	6062.59%	0.83%	2.20%	0.83%	2.25%	6.61%	2.55%	6.71%	1.04%	6.74%	0.08%
0.050	0.49%	0.77%	0.46%	0.17%	0.47%	0.18%	-26.22%	2114.34%	0.83%	2.20%	0.83%	2.25%	6.60%	1.60%	6.71%	1.04%	6.74%	0.08%
0.100	0.48%	0.53%	0.47%	0.17%	0.47%	0.18%	-3.50%	747.87%	0.83%	2.20%	0.83%	2.25%	6.75%	2.54%	6.71%	1.04%	6.74%	0.08%
0.500	0.47%	0.24%	0.47%	0.17%	0.47%	0.18%	0.55%	68.75%	0.83%	2.20%	0.83%	2.23%	6.73%	1.05%	6.71%	1.04%	6.74%	0.08%
1.000	0.47%	0.17%	0.47%	0.17%	0.47%	0.18%	0.05%	24.37%	0.83%	2.20%	0.83%	2.19%	6.74%	0.76%	6.71%	1.04%	6.74%	0.08%
1.500	0.47%	0.14%	0.47%	0.17%	0.47%	0.18%	0.64%	13.33%	0.83%	2.20%	0.83%	2.11%	6.73%	0.60%	6.71%	1.04%	6.74%	0.08%
2.000	0.47%	0.12%	0.48%	0.17%	0.48%	0.18%	0.68%	8.54%	0.83%	2.20%	0.83%	2.02%	6.72%	0.50%	6.71%	1.04%	6.74%	0.08%
2.500	0.48%	0.11%	0.48%	0.17%	0.48%	0.18%	0.69%	6.12%	0.83%	2.20%	0.83%	1.91%	6.73%	0.48%	6.71%	1.04%	6.74%	0.08%
3.000	0.48%	0.10%	0.48%	0.18%	0.48%	0.18%	0.69%	4.70%	0.83%	2.20%	0.83%	1.80%	6.72%	0.44%	6.71%	1.04%	6.74%	0.08%
4.000	0.49%	0.09%	0.49%	0.18%	0.49%	0.18%	0.79%	3.05%	0.83%	2.21%	0.82%	1.57%	6.74%	0.38%	6.71%	1.05%	6.74%	0.08%
5.000	0.50%	0.08%	0.50%	0.18%	0.50%	0.17%	0.79%	2.20%	0.83%	2.21%	0.82%	1.35%	6.74%	0.35%	6.71%	1.05%	6.74%	0.08%
6.000	0.51%	0.07%	0.51%	0.19%	0.51%	0.17%	0.78%	1.67%	0.83%	2.21%	0.82%	1.16%	6.74%	0.32%	6.71%	1.05%	6.74%	0.08%
8.000	0.53%	0.06%	0.54%	0.20%	0.54%	0.17%	0.79%	1.06%	0.83%	2.22%	0.81%	0.87%	6.73%	0.27%	6.71%	1.05%	6.74%	0.08%
10.000	0.56%	0.05%	0.57%	0.22%	0.57%	0.16%	0.79%	0.75%	0.83%	2.24%	0.81%	0.70%	6.73%	0.24%	6.71%	1.05%	6.74%	0.08%
15.000	0.65%	0.04%	0.65%	0.28%	0.65%	0.15%	0.79%	0.40%	0.82%	2.36%	0.79%	0.56%	6.74%	0.21%	6.71%	1.06%	6.74%	0.08%
20.000	0.74%	0.03%	0.75%	0.38%	0.74%	0.15%	0.78%	0.25%	0.82%	2.68%	0.78%	0.55%	6.74%	0.19%	6.71%	1.07%	6.74%	0.08%
25.000	0.83%	0.03%	0.84%	0.55%	0.83%	0.14%	0.76%	0.17%	0.82%	3.39%	0.76%	0.52%	6.74%	0.17%	6.72%	1.07%	6.74%	0.08%
30.000	0.91%	0.03%	0.91%	0.83%	0.92%	0.14%	0.75%	0.13%	0.83%	4.81%	0.75%	0.48%	6.74%	0.16%	6.72%	1.08%	6.74%	0.08%
35.000	0.99%	0.02%	0.96%	1.24%	0.99%	0.14%	0.74%	0.10%	0.83%	7.35%	0.74%	0.44%	6.74%	0.15%	6.72%	1.09%	6.74%	0.08%
40.000	1.05%	0.02%	0.98%	1.81%	1.05%	0.14%	0.73%	0.08%	0.82%	11.46%	0.73%	0.39%	6.74%	0.15%	6.72%	1.10%	6.74%	0.08%
45.000	1.10%	0.02%	0.94%	2.51%	1.11%	0.14%	0.72%	0.06%	0.78%	17.36%	0.72%	0.36%	6.74%	0.14%	6.72%	1.11%	6.74%	0.08%
50.000	1.15%	0.02%	0.83%	3.27%	1.15%	0.14%	0.71%	0.05%	0.68%	24.85%	0.71%	0.33%	6.74%	0.14%	6.72%	1.11%	6.74%	0.08%

TABLE 6.3. The Monte-Carlo Greeks for autocaps CTR. Data corresponding to Figure 2

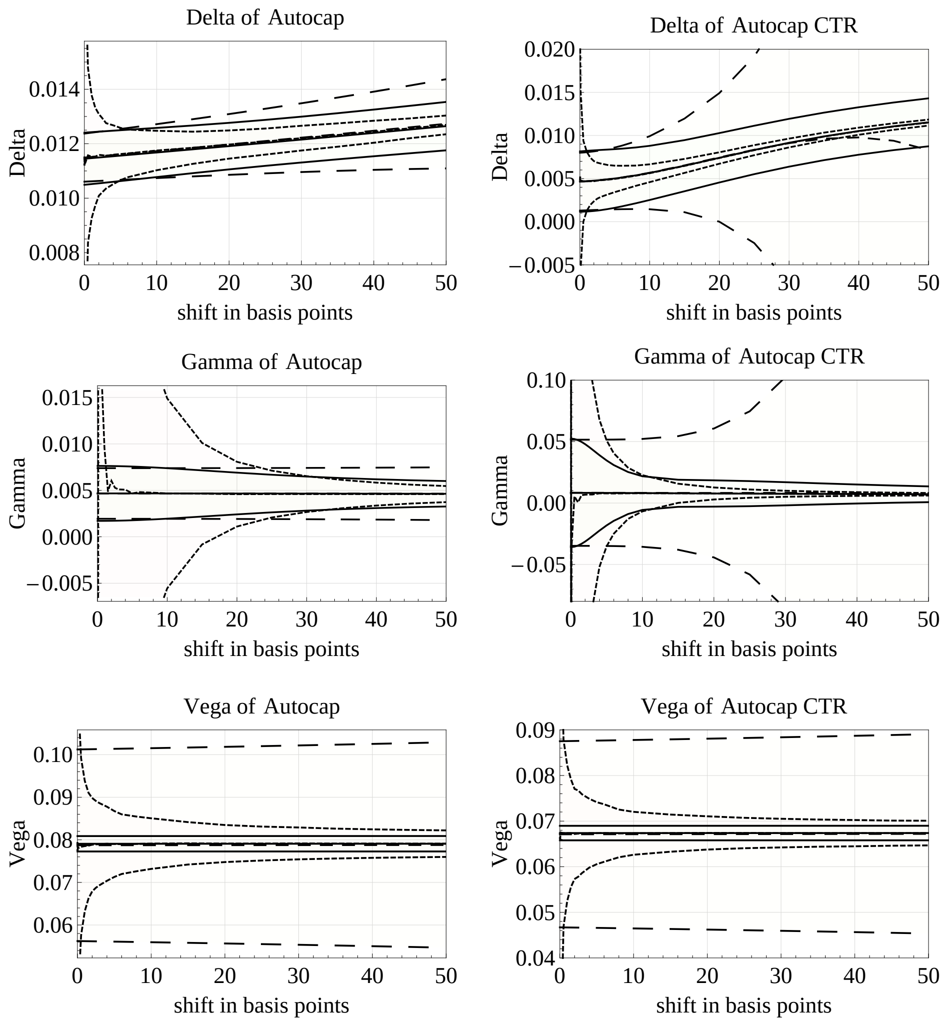

FIGURE 2. Autocap and CTR Autocap: Finite differences are applied to the direct simulation (fine dotted lines), the partial proxy simulation scheme (large dotted lines) and the minimal partial proxy simulation scheme (solid lines). The upper and lower lines represent the 95% confidence interval for the Monte-Carlo Greeks and the middle line represents the mean of the Monte-Carlo Greeks.

that the autocap will terminate once a total of 5 caplets have been triggered and has a strike of 5%. A similar close to trigger reset (CTR) autocap with a short period of 0.02 (approximately one week) to its next reset is also considered (i.e with first fixing on $t = 0.02$ followed by semi-annual fixings with the last fixing on $t = 4.02$). Such a situation may indeed happen during the life-cycle of the autocap and with a short period to next reset, we are effectively approaching the discontinuity in time and space. Under such a situation, the Monte Carlo Greeks calculated using the partial proxy simulation scheme can become very sensitive to the shift size and the variance

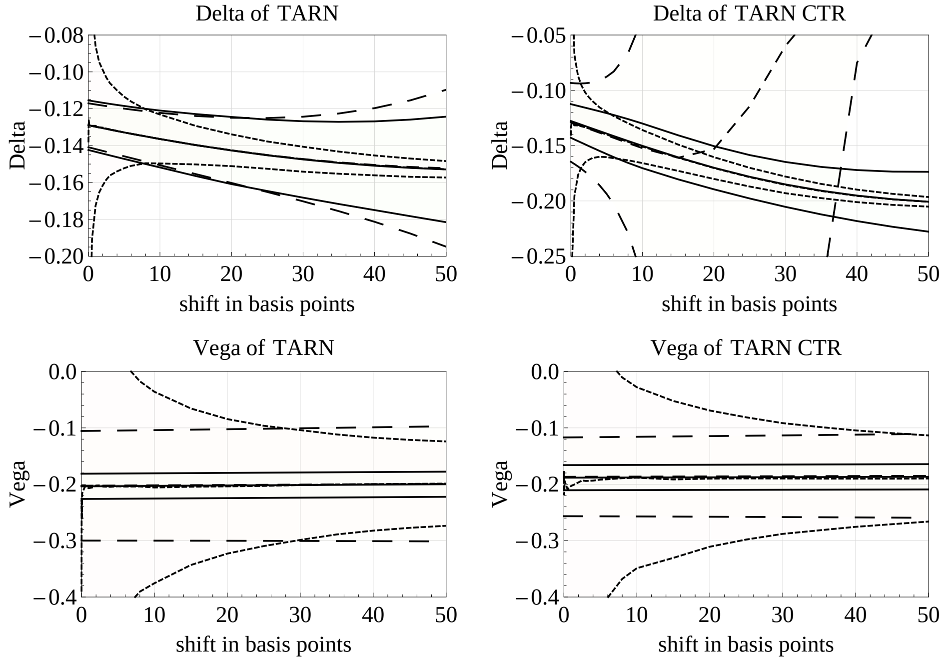

FIGURE 3. TARN and CTR TARN: Finite differences are applied to the direct simulation (fine dotted lines), the partial proxy simulation scheme (large dotted lines) and the minimal partial proxy simulation scheme (solid lines). The upper and lower lines represent the 95% confidence interval for the Monte-Carlo Greeks and the middle line represents the mean of the Monte-Carlo Greeks.

can blow up if a large shift size is applied as we can see from Figure 2. Greeks obtained using the minimal partial proxy simulation scheme remain stable even for cases close to the trigger reset. Once again, standard errors of Vega obtained using the minimal partial proxy simulation scheme are significantly lower than the partial proxy simulation scheme. As we can see from Table 6.2 and Table 6.3, standard errors for Vega and Vega CTR are approximately 13 times lower under the minimal partial proxy simulation scheme.

6.3. **Sensitivities of TARNs.** We consider a 5-year TARN with zero initial coupon and has a target coupon of 9%. We further assume that this TARN pays semi-annual inverse floating coupons of $\text{Max}(10\% - 2f_j, 0)$. We also consider a similar close to trigger reset (CTR) TARN with a short period of 0.02 (approximately one week) to its next reset. Since TARNs are known to have strong payoff discontinuity effects in an upward sloping interest rate environment, we assume that the LIBOR forward rates increase linearly from 2% to 10%.

As we can see from Figure 3, Table 6.4 and Table 6.5, similar conclusions can be made. Based on our numerical tests, standard errors of Vega and Vega CTR calculated using the minimal partial proxy simulation scheme are

	Delta						Vega					
Shift	Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP	
in bp	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
0.010	-13.77%	23.94%	-12.90%	0.61%	-12.89%	0.69%	-38.89%	416.69%	-20.27%	4.96%	-20.37%	1.14%
0.025	-13.07%	14.11%	-12.90%	0.61%	-12.89%	0.69%	-22.81%	188.18%	-20.27%	4.96%	-20.37%	1.14%
0.050	-12.62%	9.68%	-12.90%	0.61%	-12.90%	0.69%	-26.10%	142.96%	-20.27%	4.96%	-20.37%	1.14%
0.100	-12.84%	6.85%	-12.91%	0.61%	-12.90%	0.69%	-21.59%	89.99%	-20.27%	4.96%	-20.37%	1.14%
0.500	-12.92%	3.15%	-12.94%	0.61%	-12.93%	0.69%	-20.35%	39.94%	-20.27%	4.96%	-20.37%	1.14%
1.000	-12.93%	2.21%	-12.98%	0.62%	-12.97%	0.69%	-20.69%	28.33%	-20.27%	4.96%	-20.36%	1.14%
1.500	-13.01%	1.80%	-13.02%	0.62%	-13.01%	0.70%	-20.56%	22.81%	-20.26%	4.97%	-20.36%	1.14%
2.000	-13.03%	1.60%	-13.06%	0.63%	-13.05%	0.70%	-20.23%	19.26%	-20.26%	4.97%	-20.36%	1.14%
2.500	-13.08%	1.41%	-13.10%	0.63%	-13.09%	0.71%	-20.21%	17.18%	-20.26%	4.97%	-20.35%	1.14%
3.000	-13.11%	1.28%	-13.13%	0.64%	-13.13%	0.71%	-20.42%	15.98%	-20.25%	4.97%	-20.35%	1.14%
4.000	-13.20%	1.12%	-13.21%	0.65%	-13.21%	0.72%	-20.46%	13.89%	-20.24%	4.98%	-20.34%	1.14%
5.000	-13.29%	0.99%	-13.29%	0.66%	-13.28%	0.73%	-20.36%	12.09%	-20.24%	4.98%	-20.33%	1.14%
6.000	-13.36%	0.89%	-13.36%	0.67%	-13.36%	0.74%	-20.42%	11.00%	-20.23%	4.98%	-20.33%	1.14%
8.000	-13.49%	0.75%	-13.50%	0.70%	-13.50%	0.76%	-20.39%	9.52%	-20.22%	4.99%	-20.31%	1.14%
10.000	-13.64%	0.68%	-13.64%	0.72%	-13.64%	0.79%	-20.58%	8.66%	-20.20%	5.00%	-20.30%	1.14%
15.000	-13.98%	0.53%	-13.97%	0.80%	-13.97%	0.86%	-20.44%	7.08%	-20.17%	5.02%	-20.26%	1.14%
20.000	-14.25%	0.44%	-14.26%	0.90%	-14.27%	0.93%	-20.37%	6.09%	-20.13%	5.05%	-20.22%	1.14%
25.000	-14.52%	0.38%	-14.51%	1.02%	-14.53%	0.99%	-20.30%	5.44%	-20.10%	5.07%	-20.18%	1.14%
30.000	-14.74%	0.35%	-14.73%	1.17%	-14.75%	1.06%	-20.13%	4.97%	-20.06%	5.10%	-20.15%	1.14%
35.000	-14.92%	0.31%	-14.91%	1.35%	-14.94%	1.14%	-20.03%	4.52%	-20.03%	5.12%	-20.11%	1.14%
40.000	-15.07%	0.27%	-15.05%	1.57%	-15.09%	1.23%	-19.98%	4.21%	-19.99%	5.15%	-20.07%	1.14%
45.000	-15.20%	0.25%	-15.16%	1.84%	-15.21%	1.34%	-19.93%	3.98%	-19.96%	5.18%	-20.03%	1.14%
50.000	-15.28%	0.23%	-15.23%	2.17%	-15.29%	1.46%	-19.89%	3.82%	-19.93%	5.21%	-20.00%	1.14%

TABLE 6.4. The Monte-Carlo Greeks for TARNs. Data corresponding to Figure 3

	Delta						**Vega**					
Shift	Direct Simulation		Partial Proxy		Minimal PP		Direct Simulation		Partial Proxy		Minimal PP	
in bp	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
0.010	-14.02%	23.15%	-12.90%	1.82%	-12.77%	0.78%	-17.82%	264.41%	-18.68%	3.56%	-18.83%	1.14%
0.025	-13.37%	14.20%	-12.90%	1.82%	-12.78%	0.78%	-19.66%	173.12%	-18.68%	3.56%	-18.83%	1.14%
0.050	-13.23%	10.13%	-12.91%	1.82%	-12.78%	0.78%	-21.93%	133.49%	-18.68%	3.56%	-18.83%	1.14%
0.100	-12.97%	7.04%	-12.92%	1.83%	-12.79%	0.78%	-19.39%	90.15%	-18.68%	3.56%	-18.83%	1.14%
0.500	-13.17%	3.26%	-13.01%	1.86%	-12.89%	0.79%	-20.89%	40.52%	-18.68%	3.56%	-18.83%	1.14%
1.000	-13.18%	2.29%	-13.13%	1.91%	-13.00%	0.81%	-20.29%	27.99%	-18.68%	3.57%	-18.83%	1.14%
1.500	-13.25%	1.87%	-13.25%	1.97%	-13.12%	0.83%	-20.01%	22.88%	-18.67%	3.57%	-18.83%	1.14%
2.000	-13.34%	1.61%	-13.36%	2.04%	-13.23%	0.85%	-19.67%	19.82%	-18.67%	3.57%	-18.83%	1.14%
2.500	-13.45%	1.45%	-13.48%	2.12%	-13.34%	0.86%	-19.32%	17.66%	-18.67%	3.57%	-18.82%	1.14%
3.000	-13.55%	1.32%	-13.60%	2.21%	-13.46%	0.88%	-19.42%	16.21%	-18.67%	3.57%	-18.82%	1.14%
4.000	-13.77%	1.14%	-13.83%	2.42%	-13.68%	0.91%	-19.36%	13.98%	-18.67%	3.58%	-18.82%	1.14%
5.000	-13.99%	1.04%	-14.07%	2.70%	-13.91%	0.94%	-19.18%	12.02%	-18.66%	3.58%	-18.82%	1.14%
6.000	-14.23%	0.97%	-14.30%	3.06%	-14.14%	0.97%	-19.11%	10.71%	-18.66%	3.59%	-18.81%	1.14%
8.000	-14.65%	0.85%	-14.77%	4.15%	-14.58%	1.01%	-18.92%	9.11%	-18.65%	3.60%	-18.81%	1.14%
10.000	-15.08%	0.76%	-15.23%	5.97%	-15.02%	1.03%	-18.84%	8.19%	-18.64%	3.60%	-18.80%	1.14%
15.000	-16.09%	0.61%	-16.10%	16.06%	-16.06%	1.02%	-19.14%	7.10%	-18.63%	3.63%	-18.79%	1.14%
20.000	-17.03%	0.50%	-15.31%	32.98%	-17.00%	1.00%	-19.00%	6.16%	-18.61%	3.65%	-18.77%	1.14%
25.000	-17.85%	0.43%	-11.43%	41.63%	-17.82%	1.00%	-18.97%	5.53%	-18.59%	3.67%	-18.76%	1.14%
30.000	-18.55%	0.38%	-5.99%	30.87%	-18.51%	1.04%	-18.98%	5.01%	-18.58%	3.69%	-18.74%	1.14%
35.000	-19.10%	0.33%	-2.11%	13.50%	-19.07%	1.10%	-19.00%	4.68%	-18.56%	3.72%	-18.72%	1.14%
40.000	-19.55%	0.29%	-0.55%	3.55%	-19.52%	1.18%	-19.00%	4.36%	-18.54%	3.74%	-18.71%	1.14%
45.000	-19.86%	0.25%	-0.18%	0.60%	-19.85%	1.27%	-19.01%	4.12%	-18.53%	3.77%	-18.69%	1.14%
50.000	-20.08%	0.22%	-0.11%	0.19%	-20.07%	1.38%	-18.98%	3.90%	-18.51%	3.80%	-18.68%	1.14%

TABLE 6.5. The Monte-Carlo Greeks for TARNs CTR. Data corresponding to Figure 3

at least 3 times lower than the standard errors obtained using the partial proxy simulation scheme. The Delta of TARN obtained using the minimal partial proxy simulation scheme and the partial proxy simulation scheme have similar standard errors. However, for the Delta of a TARN CTR, the minimal partial proxy simulation scheme outperformed the partial proxy simulation scheme. As we can see from Figure 3, under the partial proxy simulation scheme, the Delta of a TARN CTR can have high variance when a large shift size is applied while, under the minimal partial proxy simulation schemes, the Delta of a TARN CTR remains stable .

The minimal partial proxy scheme does not produce stable Gammas for TARNs. The reason is that there are two points of non-smoothness: the trigger level and the fact that the coupons are inverse floating, so they have a first derivative which is discontinuous and a second derivative that is in a distributional sense a delta function. To produce stable Gammas would therefore require smoothing of the inverse floating coupons or an additional proxy constraint. Although, we can potentially extend the idea from the minimal partial proxy simulation scheme to further modify the underlying numerical scheme such that the payoff function of a TARN is $\mathcal{C}^1$, we will leave this for future research.

7. Conclusion

The minimal partial proxy simulation scheme can be applied to popular classes of trigger products including digital caplets, autocaps and target redemption notes. This simulation scheme can be considered as an extension of the partial proxy simulation scheme. The essential difference is that the measure change is performed in an optimal way. While the standard errors of Monte-Carlo Greeks obtained using the partial proxy simulation scheme can blow up in certain cases, they remain stable under the minimal partial proxy simulation scheme. Our numerical results also show that standard errors of Vega calculated using the minimal partial proxy simulation scheme are significantly lower than the partial proxy simulation scheme. Therefore, the minimal partial proxy simulation scheme can be applied effectively to compute the Monte-Carlo Greeks across a large range of cases.

APPENDIX A

Decomposition of Correlation Matrix. In section 4, we have defined A_s to be a pseudo square root such that $a_{s1} = 1$ and $a_{sl} = 0$ for $l = 2, \dots, F$. Here, we present an algorithm to obtain A_s.

Let C denote a $m \times m$ correlation matrix where the entry c_{ij} represents the correlation between random variables, X_i and X_j. We define C' to be the reordered correlation matrix of C where we have interchanged the 1^{st} row and the s^{th} row of C followed immediately by interchanging the 1^{st} column and the s^{th} column of C. Effectively, we have replaced X_1 with X_s and X_s with X_1 in C'. We define A' to be the $m \times F$ pseudo square root matrix of C' satisfying the equation $A'A'^T = C'$ which takes the following form,

$$A' = \left[\begin{array}{c|c} H & J \\ \hline K & L \end{array} \right], \tag{A-1}$$

with

- H is the real number 1.0.
- J is a $F-1$ dimensional row vector equal to zero.
- K is a $m-1$ dimensional column vector.
- L is a $(m-1)\times(F-1)$ matrix.

Using the fact that $A'A'^T = C'$, the entries for the column vector, K, can be easily obtained and they are given by

$$k_{i1} \quad = \quad c'_{i+1,1} \tag{A-2}$$

for $i = 1, 2, 3, \ldots, m-1$ where c'_{ij} denotes the element in the i^{th} row and j^{th} column of C'.

In order to get L, we will define a $(m-1)\times(m-1)$ matrix, $\hat{C}$, representing the correlation between $X_2, \ldots, X_m$ unexplained by K. From $A'A'^T = C'$, we know that the correlation between X_{i+1} and X_{j+1} is equal to

$$\sum_{l=1}^{F} a'_{i+1,l}a'_{j+1,l} \tag{A-3}$$

where a'_{ij} denotes the element in the i^{th} row and j^{th} column of A'. Since we have $a'_{i+1,1} = k_{i1}$ and $a'_{j+1,1} = k_{j1}$, the correlation between X_{i+1} and X_{j+1} explained by K is therefore given by $k_{j1}k_{i1}$. Hence, each entry $\hat{c}_{ij}$ in $\hat{C}$ is obtained by subtracting $k_{j1}k_{i1}$ from the original correlation and this is given by

$$\hat{c}_{ij} = c'_{i+1,j+1} - k_{i,1}k_{j,1}, \tag{A-4}$$

The matrix L must satisfy $LL^T = \hat{C}$ and there are many ways of getting such L, for example, spectral decomposition (see, Joshi 2003a). The matrix A_s is then obtained simply by interchanging the first and the s^{th} rows of A'.

APPENDIX B

B1 The Variance of The Monte-Carlo Weight. In this section we derive the variance of the Monte-Carlo weight. The Monte-Carlo weight for the minimal partial proxy scheme at time t_{i+1} is given by

$$w(t_{i+1}) = \alpha_1(t_i)\exp\left(-\frac{(\alpha_1(t_i)\Delta W_1(t_i) + \beta_1(t_i))^2 - (\Delta W_1(t_i))^2}{2\Delta t_i}\right), \tag{B-1}$$

where

$$\beta_1(t_i) \quad = \quad \frac{\tilde{H}(t_{i+1}) - \tilde{K}_s(t_i) - \mu_s^B(\tilde{K}, t_i)\Delta t_i}{\sigma_s^B(\tilde{K}, t_i)} - \alpha_1(t_i)\Delta W_1^*(t_i). \tag{B-2}$$

Since $\Delta W_1(t_i)$ is a normal random variable with mean of zero and variance of Δt_i, we have

$$\Delta W_1(t_i) = \sqrt{\Delta t_i} Z, \tag{B-3}$$

where Z is a standard normal random variable. Let $\bar{\beta}_1(t_i) = \frac{\beta_1(t_i)}{\sqrt{\Delta t_i}}$, the equation (B-1) can be rewritten as

$$w(t_{i+1}) = \alpha_1(t_i)\exp\left(-\frac{(\alpha_1(t_i)Z + \bar{\beta}_1(t_i))^2 - Z^2}{2}\right). \tag{B-4}$$

The expectation of $w(t_{i+1})$ is given by

$$\begin{aligned} E(w(t_{i+1})) &= \int_{-\infty}^{\infty} \alpha_1(t_i) \exp\left(-\frac{(\alpha_1(t_i)Z + \bar{\beta}_1(t_i))^2 - Z^2}{2}\right) \phi(Z) dZ \\ &= 1, \end{aligned} \tag{B-5}$$

where $\phi(Z)$ is the density function of a standard normal random variable. Similarly, the second moment of $w(t_{i+1})$ is given by

$$\begin{aligned} E(w(t_{i+1})^2) &= \int_{-\infty}^{\infty} \alpha_1(t_i)^2 \exp\left(-(\alpha_1(t_i)Z + \bar{\beta}_1(t_i))^2 + Z^2\right) \phi(Z) dZ \\ &= \frac{\alpha_1(t_i)^2}{\sqrt{2\alpha_1(t_i)^2 - 1}} \exp\left(\frac{\bar{\beta}_1(t_i)^2}{2\alpha_1(T_k)^2 - 1}\right) \\ &= \frac{\alpha_1(t_i)^2}{\sqrt{2\alpha_1(t_i)^2 - 1}} \exp\left(\frac{\beta_1(t_i)^2}{\Delta t_i(2\alpha_1(t_i)^2 - 1)}\right), \end{aligned} \tag{B-6}$$

for $\alpha_1(t_i) > \frac{1}{\sqrt{2}}$. Hence, the variance of the Monte-Carlo weight is

$$\text{Var}(w(t_{i+1})) = \frac{\alpha(t_i)^2}{\sqrt{2\alpha(t_i)^2 - 1}} \exp\left(\frac{\beta_1(t_i)^2}{\Delta t_i(2\alpha(T_k)^2 - 1)}\right) - 1, \tag{B-7}$$

for $\alpha_1(t_i) > \frac{1}{\sqrt{2}}$.

B2 Minimising the Variance of Monte-Carlo weight. In this section, we derive the equation satisfied by $\alpha_1^*(t_i)$ such that $\alpha_1^*(t_i)$ minimises the variance of the Monte-Carlo weight. By differentiating (B-7) with respect to $\alpha_1(t_i)$ and setting the derivative to zero, we can solve for $\alpha_1^*(t_i)$ which minimises the variance of the Monte-Carlo weight. From the equation (B-2), we have

$$\beta_1(t_i) = X - \alpha_1 \Delta W_1^*, \tag{B-8}$$

where $X = \frac{\tilde{H}(t_{i+1}) - \tilde{K}_s(t_i) - \mu_s^B(\tilde{K}, t_i)\Delta t_i}{\sigma_s^B(\tilde{K}, t_i)}$. The partial derivative of $\text{Var}(w(t_{i+1}))$ w.r.t α_1 is given by

$$\begin{aligned} \frac{\partial}{\partial \alpha_1} \text{Var}(w(t_{i+1})) &= \frac{\partial}{\partial \alpha_1}\left(\frac{\alpha_1^2}{\sqrt{2\alpha_1^2 - 1}}\right) \exp\left(\frac{[X - \alpha_1 \Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2 - 1)}\right) \\ &+ \left(\frac{\alpha_1^2}{\sqrt{2\alpha_1^2 - 1}}\right) \frac{\partial}{\partial \alpha_1}\left(\exp\left(\frac{[X - \alpha_1 \Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2 - 1)}\right)\right) \end{aligned} \tag{B-9}$$

and we know that

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\alpha_1^2}{\sqrt{2\alpha_1^2 - 1}}\right) = \frac{2\alpha_1(\alpha_1^2 - 1)}{\sqrt{2\alpha_1^2 - 1}^3} \tag{B-10}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_1}\left(\exp\left(\frac{[X-\alpha_1\Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2-1)}\right)\right)\\
=&\ \frac{1}{\Delta t_i}\frac{\partial}{\partial \alpha_1}\left(\frac{[X-\alpha_1\Delta W_1^*]^2}{(2\alpha_1^2-1)}\right)\exp\left(\frac{[X(t_i)-\alpha_1\Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2-1)}\right)\\
=&\ \left(\frac{2X\Delta W_1^*+4\alpha_1^2X(t_i)\Delta W_1^*-2\alpha_1(\Delta W_1^*)^2-4\alpha_1X^2}{\Delta t_i(2\alpha_1^2-1)^2}\right)\\
&\times\exp\left(\frac{[X-\alpha_1\Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2-1)}\right).
\end{aligned} \tag{B-11}$$

Therefore, by substituting (B-10) and (B-11) in to (B-9), we have

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_1}\mathrm{Var}(w(t_{i+1}))\\
=&\ \frac{2\alpha_1}{\Delta t_i(\sqrt{2\alpha_1^2-1})^2}\exp\left(\frac{[X-\alpha_1\Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2-1)}\right)\\
&\times[2\Delta t_i\alpha_1^4+2X\Delta W_1^*\alpha_1^3-(3\Delta t_i+2X^2+(\Delta W_1^*)^2)\alpha_1^2+X(t_i)\Delta W_1^*\alpha_1+\Delta t_i]\\
=&\ M(\alpha_1)f(\alpha_1),
\end{aligned} \tag{B-12}$$

where

$$M(\alpha_1) = \frac{2\alpha_1}{\Delta t_i(\sqrt{2\alpha_1^2-1})^2}\exp\left(\frac{[X-\alpha_1\Delta W_1^*]^2}{\Delta t_i(2\alpha_1^2-1)}\right), \tag{B-13}$$

$$\begin{aligned}
f(\alpha_1) =&\ 2\Delta t_i\alpha_1^4+2X\Delta W_1^*\alpha_1^3-(3\Delta t_i+2X^2+(\Delta W_1^*)^2)\alpha_1^2\\
&+X\Delta W_1^*\alpha_1+\Delta t_i.
\end{aligned} \tag{B-14}$$

Since $M(\alpha_1) > 0$ for $\alpha_1 > \frac{1}{\sqrt{2}}$, hence α_1^* which minimises the variance of the Monte-Carlo weight must satisfy the equation $f(\alpha_1^*) = 0$ as we require $\frac{\partial}{\partial \alpha_1}\mathrm{Var}(w(t_{i+1})) = 0$.

B3 Proving α_1^* Exists and Minimises The Variance of The Monte-Carlo Weight. In this section, we prove that four distinct real roots always exist for the equation $f(\alpha_1) = 0$ and there is one and only one root which is greater than $\frac{1}{\sqrt{2}}$ and this root is the solution of α_1^* which minimises the variance of the Monte-Carlo weight.

Proof:
we know that

$$\begin{aligned}
\lim_{\alpha_1\to+\infty} f(\alpha_1) &\to +\infty,\\
\lim_{\alpha_1\to-\infty} f(\alpha_1) &\to +\infty,\\
f(0) &= \Delta t_i > 0
\end{aligned}$$

and with some simple manipulations, we have

$$\begin{aligned} f\left(\frac{1}{\sqrt{2}}\right) &= -\left(X-\frac{\Delta W_1^*}{\sqrt{2}}\right)^2 \leqslant 0, \\ f\left(-\frac{1}{\sqrt{2}}\right) &= -\left(X+\frac{\Delta W_1^*}{\sqrt{2}}\right)^2 \leqslant 0. \end{aligned}$$

For cases where $f(\frac{1}{\sqrt{2}})$ and $f(-\frac{1}{\sqrt{2}})$ are less than zero, we can conclude that exactly one real root exists in these intervals $(-\infty,-\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}},0)$, $(0,\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\infty)$ as the sign of $f(\alpha_1)$ changes in these intervals a 4^{th} order polynomial has at most 4 real roots. Therefore, four distinct real roots exist.

For cases where $f(\frac{1}{\sqrt{2}})=0$ and $f(-\frac{1}{\sqrt{2}})=0$, we know that if $f(\frac{1}{\sqrt{2}})=0$, we have $\Delta W_1^*=\sqrt{2}X$ and hence $f'(\frac{1}{\sqrt{2}})=-\sqrt{2}\Delta t_i<0$ and similarly, if $f(-\frac{1}{\sqrt{2}})=0$, we have $\Delta W_1^*=-\sqrt{2}X$ and hence $f'(-\frac{1}{\sqrt{2}})=5\sqrt{2}\Delta t_i>0$ where $f'(\alpha_1)$ is the derivative of $f(\alpha_1)$ given by

$$f'(\alpha_1) = 8\Delta t_i\alpha_1^3+6X\Delta W_1^*\alpha_1^2-2(\Delta t_i+2X^2+(\Delta W_1^*)^2)\alpha_1+X\Delta W_1^*. \tag{B-15}$$

As a result, $f(\alpha_1)$ must cross the x-axis once in the interval $(-\infty,-\frac{1}{\sqrt{2}})$ as we have $\lim_{\alpha_1\to-\infty} f(\alpha_1)\to+\infty$ and $f(-\frac{1}{\sqrt{2}})=0$ with positive slope and similarly, $f(\alpha_1)$ must cross the x-axis once in the interval $(\frac{1}{\sqrt{2}},\infty)$ as $\lim_{\alpha_1\to+\infty} f(\alpha_1)\to+\infty$ and $f(\frac{1}{\sqrt{2}})=0$ with negative slope. Under this scenario, two real roots exist for intervals $(-\infty,-\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\infty)$ and the other two roots are $\pm\frac{1}{\sqrt{2}}$. Similar arguments can be applied to cases where only one of the $f(\frac{1}{\sqrt{2}})$ or $f(-\frac{1}{\sqrt{2}})$ is equal to zero. Therefore, we have shown that four distinct real roots always exist and one and only one root is greater than $\frac{1}{\sqrt{2}}$. To prove that this root, α_1^*, always minimises the variance of the Monte-Carlo weight, we need to show that $\frac{\partial^2}{\partial\alpha_1^2}\mathrm{Var}(w(t_{i+1}))\Big|_{\alpha_1^*}>0$. We have

$$\frac{\partial^2}{\partial\alpha_1^2}\mathrm{Var}(w(t_{i+1})) = \frac{\partial}{\partial\alpha_1}M(\alpha_1)\times f(\alpha_1)+\frac{\partial}{\partial\alpha_1}f(\alpha_1)\times M(\alpha_1) \tag{B-16}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial\alpha_1^2}\mathrm{Var}(w(t_{i+1}))\bigg|_{\alpha_1^*} &= \frac{\partial}{\partial\alpha_1}M(\alpha_1)\bigg|_{\alpha_1^*}\times f(\alpha_1^*)+\frac{\partial}{\partial\alpha_1}f(\alpha_1)\bigg|_{\alpha_1^*}\times M(\alpha_1^*) \\ &= f'(\alpha_1^*)\times M(\alpha_1^*). \end{aligned} \tag{B-17}$$

Therefore, we only have to prove that $f'(\alpha_1^*)>0$ as $M(\alpha_1^*)$ is always greater than zero. Since four distinct real roots exist for $f(\alpha_1)$, there are three turning points between the adjacent pair of roots and therefore $f'(\alpha_1^*)\neq 0$. Given that we have $\lim_{\alpha_1\to+\infty} f(\alpha_1)\to+\infty$ and $f(\frac{1}{\sqrt{2}})\leqslant 0$ (with $f'(\frac{1}{\sqrt{2}})<0$ for the case where $f(\frac{1}{\sqrt{2}})=0$), $f'(\alpha_1^*)$ must be greater than zero as $f(\alpha_1)$ only crosses the x-axis once at α_1^* for $\alpha_1>\frac{1}{\sqrt{2}}$. Else, $f(\alpha_1)$ will have to cross the x-axis three times for $\alpha_1 > \frac{1}{\sqrt{2}}$ in order to have $f'(\alpha_1^*) < 0$. Therefore, we conclude that $\frac{\partial^2}{\partial \alpha_1^2} \mathrm{Var}(w(t_{i+1}))\Big|_{\alpha_1^*} > 0$ which implies α_1^* minimises the variance of the Monte-Carlo weight.

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